Preface

The present book aims at presenting in a systematic, painstaking and rather exhaustive manner the incompressible viscous fluid limits of the system of Vlasov–Maxwell–Boltzmann equations for one or two species. In these regimes, the evolution of the fluid is governed by equations of Navier–Stokes–Fourier type, with some electromagnetic forcing. Depending on the precise scaling, this forcing term takes on various forms: it may be linear or nonlinear, electrostatic or governed by some hyperbolic wave equation, possibly constrained by some relation of Ohm's type.

From the mathematical point of view, these models have very different behaviors; in particular, to establish the existence and stability of solutions require sometimes to work with very weak notions of solutions. The asymptotic analysis, which consists most often in retrieving the structure of the limiting system in the scaled Vlasov–Maxwell–Boltzmann system, uses therefore various mathematical methods with important technical refinements. Thus, in order to make the reading easier, different tools will be presented in separate chapters.

The first part of this work is devoted to the systematic formal analysis of viscous hydrodynamic limits. Chapter 1 introduces the Vlasov–Maxwell–Boltzmann system as well as its formal properties. An important point to be noted is that the a priori bounds coming from these physical laws do not suffice for proving the **existence of global solutions**, even in the renormalized sense of DiPerna and Lions [30], which is a major difficulty for the study of fast relaxation limits. This actually explains the dividing of the three other parts of this book, of increasing difficulty, giving rigorous convergence results in more and more general settings.

Chapter 2 introduces the different scaling parameters arising in the system, and details the formal steps leading to the constraint relations and the evolution equations in each regime. We thus obtain a rather precise **classification of physically relevant models** for viscous incompressible plasmas, some of which actually do not seem to have been previously described in the literature.

Chapter 3 presents a mathematical analysis of these different models. The most singular of them have a behavior that is actually more similar to the incompressible Euler equations than to the Navier–Stokes equations: the lack of weak stability does not allow to prove the existence of global solutions, with the exception of very weak solutions in the spirit of the dissipative solutions introduced by Lions for the Euler equations [59]. This **lack of stability for limiting systems** is the second major difficulty encountered in the study of hydrodynamic limits.

The goal of the second part is to make precise and rigorous the convergence results described formally in the first part. In order to isolate the difficulties which are specific to the asymptotic analysis, we choose here to prove first conditional results, i.e., to consider the convergence of renormalized solutions even though their existence is not known. This of course does not imply the convergence of weaker solutions, which will be studied in the sequel (renormalized solutions with defect measure, and a fortiori solutions with Young measures), but most of the proof will remain unchanged. The important point is that the analysis is based essentially on the uniform estimates coming from the scaled entropy inequality, which holds in all situations.

Furthermore, we will focus exclusively on two typical regimes, namely, leading:

- from the one-species Vlasov–Maxwell–Boltzmann equations to the incompressible quasi-static Navier–Stokes–Fourier–Maxwell–Poisson system (Theorem 4.5);
- from the two-species Vlasov–Maxwell–Boltzmann equations to the two-fluid incompressible Navier–Stokes–Fourier–Maxwell system with Ohm's law in the case of strong interspecies collisions (Theorem 4.7), or to the two-fluid incompressible Navier–Stokes–Fourier–Maxwell system with solenoidal Ohm's law in the case of weak interspecies collisions (Theorem 4.6).

These asymptotic regimes are critical, in the sense that they are the most singular ones among the formal asymptotics mentioned in Chapter 2 and that all remaining regimes can be treated rigorously by similar or even simpler arguments.

We will not detail in this preface the content of all chapters of the second part, but rather insist on the main points requiring a treatment different from the usual hydrodynamic limits [70]. In the case with only one species, the main difference is due to the fact that the transport equation contains force terms involving a derivative with respect to v, which does not allow to transfer equi-integrability from the v-variable to the x-variable as in [37]. This is a major complication. The new idea here consists in getting first some **strong compactness in** v by using regularizing properties of the gain operator [53] and, then, in transferring this strong compactness to the spatial variable by means of **refined hypoelliptic arguments** developed in [7]. The second important difference comes from the fast temporal oscillations which couple **acoustic and electromagnetic modes**. Note that we introduce here a simple method to avoid dealing with non-local projections.

Overall, we are eventually able to establish through weak compactness methods a very general result (Theorem 4.5) asserting the convergence of renormalized solutions of the one-species Vlasov–Maxwell–Boltzmann system towards weak solutions of corresponding macroscopic systems.

In the case of two species, the situation not only requires to exploit the methods for one species, it is considerably more complex:

- First of all, there is **an additional scaling parameter** measuring the strength of interspecies interactions (and, incidentally, the typical size of the electric current, which can be much smaller than the bulk velocities of each of the two species of particles): this implies that the (formal) expansions involve a larger number of terms (for instance, the constraint equations are derived at different orders).
- Secondly, the **linearized collision operator** has a more complicated vectorial structure. The inversion of fluxes and the computation of dissipation terms in the limiting energy inequalities are therefore substantially more technical.

- In the most singular regimes, we get nonlinear constraint equations. This means that **renormalization methods**, **compensated compactness techniques** and **controls on the conservation defects** are already required at this stage of the proof.
- We have no sufficient uniform a priori bound on the electric current to handle nonlinear terms, which prevents us from taking limits in the approximate conservation of momentum law. To avoid this difficulty, we need to introduce a **modified conservation law involving the Poynting vector**.
- Even in this more suitable form, the evolution equations are not stable under weak convergence, and we have no equi-integrability in these singular regimes. We therefore develop an improved modulated entropy method, which allows to consider renormalized solutions without important restrictions on the initial data. Note that this **renormalized modulated entropy method** should also lead to some improvements concerning the convergence of the Boltzmann equation (without any electromagnetic field) to the Navier–Stokes equations for ill-prepared initial data.

The third and fourth parts (which will be published in a second volume) are more technical. They show how to adapt the arguments presented in the conditional case of the second part to take into account the state of the art Cauchy theory for the Vlasov–Maxwell–Boltzmann system.

In the case of long-range microscopic interactions giving rise to a collision cross-section with a singularity for grazing collisions, treated in the third part, we start by proving the existence of renormalized solutions with a defect measure in the spirit of the construction by Alexandre and Villani [1]. This result, which is important independently of the study of hydrodynamic limits, has been addressed in the note [8]. The study of hydrodynamic limits follows then essentially the lines of [4] (combined with the results of the conditional part). We would like however to mention some important novelties:

- The first one concerns the **estimate of the defect measure**. A refined analysis of the convergence of approximate solutions to the Vlasov–Maxwell–Boltzmann system shows that the defect measure can be controlled by the entropy dissipation. This remark allows for a simplification of the proofs from [4], especially the passage to the limit in the kinetic equation leading to the characterization of the limiting form of the dissipation, and the control of conservation defects.
- The other simplification is related to the **renormalization process**. Here we choose a decomposition of the renormalized collision operator which allows both to control the singularity due to the collision cross-section, and to preserve the good scalings for the fluctuation. In particular, the same decomposition can be used for the control of the transport and of the conservation defects (with a loop estimate).

In the case of general microscopic interactions (including for instance the case of hard spheres), it is not known how to prove the convergence of approximation schemes of the Vlasov–Maxwell–Boltzmann system, due to a lack of compactness produced by the electromagnetic interaction. The existence of renormalized solutions is therefore still an open problem. Nevertheless, Lions [55] has defined a very weak notion of solution – the measure-valued renormalized solution – defined as limit of approximate solutions: the equation to be satisfied involves indeed Young measures.

In the fourth part, we begin by refining the control of Young measures for such solutions by the entropy inequality. We then proceed by showing that the estimates obtained in the second part are very stable, so that they can be generalized with Young measures. By using convexity properties and Jensen inequalities, we can extend all the arguments, and operate both the moment method and the entropy method in more singular regimes. This extension to solutions of the Vlasov–Maxwell–Boltzmann system defined in a very weak sense shows that the methods based on the entropy inequality are extremely robust, and that the convergence is essentially determined by the limiting system.

These good asymptotic properties seem to further indicate that the measure-valued solutions defined by Lions (which have never been really studied from the qualitative point of view) are relevant in some sense.

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