For Betina

Preface

"Random matrices are cool," according to Alan Edelman. One appealing feature of theirs is the connection to most of modern mathematics. This has the effect that one can be drawn into random matrices from very different directions. In my case, it was operator algebras and free probability theory, where I started my career and, where I could, for a while, ignore the relation of those subjects with random matrices (which had been discovered by Dan Voiculescu around 1990). But finally, when I moved from Heidelberg to Queen's in 2000, I was ready to go into a new direction, and my decade at Queen's can, with a grain of salt, be described as the time when I got started and drawn into random matrices. Together with Jamie Mingo, I learned the basics of the subject and became fascinated by its depth, usefulness and beauty. Also, coming from a quite pure subject, operator algebras, it was revealing and exciting that suddenly one can "see" what one has proved abstractly. My sincere thanks go in particular to Alan Edelman and (then his PhD-student) Raj Rao Nadakuditi, who introduced us not only to MATLAB and the revelation of how easy and satisfying it is to draw histograms of eigenvalues – but to the cool random matrix world in general. This time marks the transition in the title of my talks, and even books, from "Free Probability" to "Free Probability and Random Matrices" and eventually I was ready to give lectures on random matrix theory itself, independent of any free probability or operator algebra connection.

That's, where this book arose from. In the summer term 2018 and the winter term 2019/2020 I gave lectures, at Saarland University, for graduate and advanced undergraduate students on random matrices, and the present book is essentially a primped up version of the corresponding lecture notes. I tried to keep the flavour of the lectures and the lecture notes. In particular, I resisted the temptation to add more details and extra material, so that it should still be possible to cover the material of the book in 24 lectures of 90 minutes each. If you do not believe it, you can have a look at the recordings of the 2019/2020 lectures on youTube.

Random matrices is such a vast field (just have a look at [1]) that each presentation of the subject is biased by personal preference and background. In these lectures I will cover some of the generally agreed upon foundational results of the subject, but surely there are even more which are touched upon only lightly or maybe not at all. In particular, Wishart matrices do not show up (apart from one exercise on them), and from the two main big classes of random matrices – namely, Wigner matrices and unitarily invariant matrices – the latter are mostly ignored. Actually, our main actor is the Gaussian ensemble, the only one which is both Wigner and unitarily invariant. My personal bias and history, coming from the combinatorial description of free probability theory, also puts quite a bit of emphasis on the more combinatorial moment

methods; though, the analytic approach via Stieltjes transforms is also covered and appreciated. Finally, I could not resist mentioning free probability theory at least in the last chapter – actually, from a random matrix perspective, free probability can be seen as a theory dealing with asymptotic "eigenvalue distributions", not just for one but for several, typically non-commuting, random matrices.

Thanks are due to many people who were involved either in the actual classes running at Saarland University or in finding typos, or worse, in preliminary versions of the lecture notes. In particular, for the 2018 version of the class, where all the material was organised and put together for the first time, I especially want to mention Marwa Banna, who was co-teaching the class with me, and Ricardo Schnur, who was in charge of organizing the exercise sessions and who, in particular, also typed the first version of the lecture notes. I also want to mention that for this first version I took a lot of inspiration from the lecture notes [24] of Todd Kemp, who had taught a random matrix class at UCSD a couple of years earlier.