

Abstract

This work is devoted to the class of sets in the complex plane which nowadays are known as Carathéodory sets, more precisely speaking, as Carathéodory domains and Carathéodory compact sets. These sets naturally arose many times in various research areas in Real, Complex and Functional Analysis and in the Theory of Partial Differential Equations. For instance, the concept of a Carathéodory set plays a significant role in such topical themes as approximation in the complex plane, the theory of conformal mappings, boundary value problems for elliptic partial differential equations, etc. The first appearance of Carathéodory domains in the mathematical literature (of course, without the special name at that moment) was at the beginning of the 20th century, when C. Carathéodory published his famous series of papers about boundary behavior of conformal mappings. The next breakthrough result which was obtained with the essential help of this concept is the Walsh–Lebesgue criterion for uniform approximation of functions by harmonic polynomials on plane compacta (1929). Up to now the studies of Carathéodory domains and Carathéodory compact sets remains a topical field of contemporary analysis and a number of important results were recently obtained in this direction. Among them one ought to mention the results about polyanalytic polynomial approximation, where the class of Carathéodory compact sets was one of the crucial tools, and the results about boundary behavior of conformal mappings from the unit disk onto Carathéodory domains. Our aim in the present memoir is to give a survey on known results related with Carathéodory sets and to present several new results concerning the matter. Starting with the classical works of Carathéodory, Farrell, Walsh, and passing through the history of Complex Analysis of the 20th century, we come to recently obtained results, and to our contribution to the theory.

Keywords. Carathéodory domain, Carathéodory compact set, conformal mapping, uniform approximation, pointwise approximation, L^p -approximation, complex polynomials, complex rational functions, harmonic functions, polyanalytic functions

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