# Preface

In this work we are dealing with the class of sets in the complex plane that nowadays are called Carathéodory sets. We recall that a bounded domain in the complex plane is called a Carathéodory domain, if its boundary coincides with the boundary of the unbounded connected component of the complement to its closure. A compact set is called a Carathéodory compact set, if its boundary coincides with the boundary of its polynomial convex hull, that is the union of this compact set and all bounded connected components of its complement.

Our aim is to give a survey on results related with the class of Carathéodory sets, as well as to establish some new properties of these sets. The concept of Carathéodory set turned out to be closely related to many other topics in Complex Analysis. Among them it is worth emphasizing certain topics in planar topology, conformal mappings, theory of Hardy and Bergman spaces, orthogonal polynomials, pointwise approximation by polynomials in the complex variable, uniform approximation by holomorphic and harmonic functions, Szegő's theorem and so on. Then, to make the memoir more self-contained, we have included some concepts and results from these related areas. Many of them are certainly known to the specialist in the corresponding areas but not to readers working in other areas of analysis. We have tried to write this survey in such a way that it will be accessible to a wide mathematical audience. Of course we targeted mostly on complex analysts, but we hope that mathematicians working in different areas will find in our survey some interesting topics.

An overlapping between the theory of Carathéodory sets with different branches of real and complex analysis makes it difficult to fix a solid, consistent and reasonable system of notation. Thus, the notation that comes from the theory of conformal maps is different from the corresponding notation coming from the theory of spaces of analytic functions (Hardy spaces, Bergman spaces, etc.), or from the theory of orthogonal polynomials. Even in the research papers in one topic the notation is changing. Through the survey we have tried to use our own unified system of notation which seems to us adequate to the topic under consideration. However, occasionally we have used the same notation as the original sources, in order to help the reader to compare both expositions.

In this survey we have included all results that we know, where the notion of Carathéodory set has some relation, perhaps a very small one. In order to be self-contained we have included complete proofs or their sketched versions for the majority of results, even if they are regarded as classical and well-known. If a result is included in the text without proof (neither completed nor sketched), we not only give a corresponding reference, but also explain the principal ideas underlying its proof. Moreover, a number of specific references are given in order that the interested reader can delve into the study of each of the topics covered. Also we have included several new results, and in such a case the corresponding proofs are provided with all necessary details. Occasionally we have included new variations of background results with the aim of contributing to a better understanding of the theory. Moreover, we have paid attention to constructing, to mentioning or to referring to various examples, so that the geometric behavior of sets under consideration becomes more clear. In this subject there are still many open questions, and we will mention some of them.

We have tried to follow the historical order of exposition in each section, and we have included some historical notes. This can be useful to understand the theory, to compare chronologically the different lines of research and to realize that occasionally someone has made contributions to the theory without knowing previous results closely related to theirs. We believe that a survey of this type will be useful as a first step to clarify and unify the world of Carathéodory sets.

Let us present a brief overview of the history of the investigations of Carathéodory sets. As far as we know, the first Carathéodory sets to be introduced were the Carathéodory domains. At [20, page 136] in 1912 the first nontrivial example of such domains was presented. Later domains of this kind were used at the end of the 1920s in the work [132] by J. Walsh on approximation of functions by harmonic polynomials. However, the special name for this class of domains was not assigned at that time, the name "Carathéodory domains" for this class was given much later. Then, Walsh encouraged his student O. J. Farrell to continue the line of research related to harmonic approximation and properties of conformal maps of planar domains, and Farrell in the 1930s proved several new results about Carathéodory domains (see [44–46] and the corresponding discussion in what follows).

It seems that Farrell's works and results were forgotten for thirty years in the occidental school until the moment of publication of [112] about the pointwise convergence of polynomials in a complex variable. However, it seems, that some ideas about Carathéodory sets were presented in the Russian school of complex analysis during the 1930s and 1940s. A remarkable contribution was made by A. I. Markushevich in his thesis [84], where he considered polynomial approximations in the space of square integrable holomorphic functions in Carathéodory domains. Let us note that Markushevich at that time also did not use the special name for such domains. Perhaps the first (as far as we know) occurrence of the term "Carathéodory domain" in the mathematical literature was in 1939 in the paper [71] by M. V. Keldysh, which also studied polynomial approximation in spaces of square integrable holomorphic functions on compact sets in the complex plane.

With a great deal of certainty, one can assume that the origins of this name are related to the fundamental works [20–22] by C. Carathéodory on conformal maps, which were published in 1912 and 1913 and in which domains such as a cornucopia (see the domain $G_1$ in Figure 2 below) appeared for the first time in mathematical literature. We will discuss this picture in the chapter on the properties of the conformal maps of Carathéodory domains. It will become clear that the term "Carathéodory domain" is fairly appropriate and adequate.

One ought to make here the following remark. Taking into account the historical circumstances and given the contribution of the aforementioned mathematicians it would be more fair to use the name "Walsh–Farrell–Carathéodory domains" for the class of such domains. But nowadays it makes no sense to adopt this term, because the current terminology is already fixed in the literature. However, it is interesting to have this in mind. In this connection it is worth noticing that neither Farrell in his later paper [48], nor Walsh in his book [134] used any special name for this class of domains.

Later on the class of Carathéodory domains, already with its specific name, may be found, for instance, in the following papers [64, 112, 122, 123] and books [34, 85]. The book [34] is the unique source that we are aware of, where a Carathéodory domain is not assumed to be bounded. The reader interested in the topics on holomorphic and harmonic approximation on unbounded sets may refer, for example, to [5] and [57]. The concept of a $K$-set defined in [5] may seem interesting concerning our context.

Several topics, where the concept of a Carathéodory domain plays a crucial role were intensively developed in the 1960s–1980s. Let us mention, for instance, the studies of generators for algebras of functions. As was shown in the work [120] by D. Sarason, and in several subsequent papers, the concept of a Carathéodory domain turned out to be closely related to such topics.

Besides, in the 1950s–1960s, the concept of a Carathéodory compact set was introduced and substantially used in a series of works about approximation of functions by rational functions and polynomials in a complex variable. The first occurrence of the term "Carathéodory compact set" itself was, as far as we know, in the work [123] by S. O. Sinanyan who obtained several results about approximation by holomorphic and harmonic polynomials on Carathéodory compact sets in the $L^p$-norm, $1 \leqslant p < +\infty$. These results of Sinanjan generalize the previous results by Farrell and Markushevich about similar approximation on Carathéodory domains. However, the concept of a Carathéodory compact set was used previously by E. Bishop (see [13–15]) in his studies of measures orthogonal to algebras of polynomials on such compact sets; he called such compact sets "balanced compact sets". For some unclear reasons these important papers of Bishop have been rarely mentioned thereafter, so the name of balanced sets (in both cases of open and compact sets) has been only occasionally used in what follows. Moreover, certain of Bishop's results were rediscovered later by different authors (highly likely completely independently) in similar or slightly different settings.

For the reader's convenience, and mostly for didactic purposes, we will include in the exposition some general results from complex analysis, approximation theory and planar topology. We will use the special symbol ¶ (and write, for instance, Theorem¶)

to highlight the results of the following three kinds: new results about Carathéodory sets which are obtained here for the first time, recent results by the authors concerning the matter, and results which may be regarded as valuable modifications or refinements of certain known results about Carathéodory sets.

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