Chapter 1

Definitions and topological properties of Carathéodory sets

In this chapter, we define the classes of Carathéodory sets which we are dealing with, and explore topological properties of such sets.

1.1 Definitions and first examples

Take a compact set $K \subset \mathbb{C}$. The open set $\mathbb{C} \setminus K$ has at most a countable number of bounded open connected components $\Omega_j = \Omega_j(K)$, $j \in I$, where $I = I(K)$ is some set of indices, and one component $\Omega_\infty(K)$ which is unbounded. The domain $\Omega_\infty(K)$ is called the *outer domain* of K. It will be convenient to put $\Omega'_\infty(K) := \Omega_\infty(K) \cup \{\infty\}$.

The set $\partial_{\text{ext}} K := \partial\Omega_\infty(K)$ is traditionally called the *external boundary* of K. The set

$$\partial_{\text{int}} K := \partial K \Big\backslash \Big(\partial\Omega_\infty(K) \cup \bigcup_{j\in I} \partial\Omega_j\Big)$$

is called the *inner boundary* of K. Thus,

$$\partial K = \partial_{\text{int}} K \cup \partial_{\text{ext}} K \cup \bigcup_{j\in I} \partial\Omega_j(K).$$

Accordingly, for an open set $U \subset \mathbb{C}$ we set

$$U_\infty := \Omega_\infty(\bar{U}), \quad U'_\infty := \Omega_\infty(\bar{U}) \cup \{\infty\}. \tag{1.1}$$

One of the most significant entities for our further considerations will be the concept of the *polynomial convex hull* of a set. Let us recall that for a bounded set $E \subset \mathbb{C}$ its polynomial convex hull, denoted by $\widehat{E}$ or $E^\wedge$, is defined as follows:

$$\widehat{E} = \Big\{z \in \mathbb{C} : |p(z)| \leqslant \sup_{w\in E} |p(w)|,\ p \in \mathcal{P}\Big\}.$$

A bounded set E is called *polynomially convex* if $E = \widehat{E}$.

Observe that always $E \subset \widehat{E} = (\widehat{E})^\wedge$ and the set $\widehat{E}$ is closed. Moreover, it can be easily verified that

$$(\bar{E})^\wedge = \widehat{E}$$

for any bounded set $E \subset \mathbb{C}$. This equality will be frequently and implicitly used in what follows.

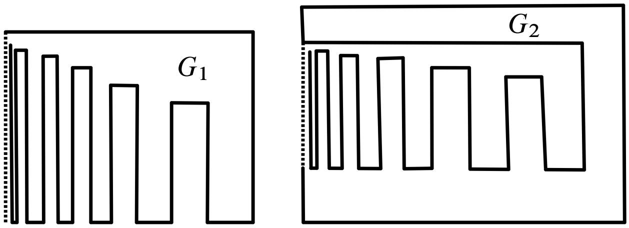

Figure 1. G_1 is a Carathédory domain, while G_2 is not.

If K is a compact subset of $\mathbb{C}$ then the maximum modulus principle for holomorphic functions and the classical Runge approximation theorem yield

$$\widehat{K} = K \cup \bigcup_{j \in I(K)} \Omega_j(K),$$

and $\partial \widehat{K} = \partial_{\text{ext}} K$. Thus, $\widehat{K}$ is the union of K and all bounded components of $\mathbb{C} \setminus K$.

In what follows we will be dealing with several kinds of Carathéodory sets, namely, with Carathéodory open sets (in particular, with Carathéodory domains), and Carathéodory compact sets. Despite the fact, that the principal ideas underlying these concepts are the same, it is convenient to define them separately.

Definition 1.1. A set $G \subset \mathbb{C}$ is called a Carathéodory open set if it satisfies the following conditions:

(1) G is nonempty, open and bounded;

(2) $\partial G = \partial_{\text{ext}}(\partial G)$.

A connected open Carathéodory set G is called a Carathéodory domain.

Since $G_\infty = \Omega_\infty(\partial G)$, condition (2) in Definition 1.1 also means that $\partial G = \partial G_\infty$.

A very simple example of a Carathéodory domain is provided by any Jordan domain, that is a domain of the form $D(\Gamma)$ for an arbitrary Jordan curve Γ. It follows directly from Definition 1.1 that the domains $D = \mathbb{D} \setminus [0, 1)$, $D_2 = \mathbb{D} \setminus \overline{D}(\frac{1}{2}, \frac{1}{2})$, and $D_3 = \mathbb{D} \setminus [-\frac{1}{2}, \frac{1}{2}]$ are not Carathéodory domains. In the picture in Figure 1 one can see two more complicated examples.

Notice, that for a Carathéodory domain G the set $\mathbb{C} \setminus \overline{G}$ may be disconnected. The "outer cornucopia", which is a ribbon which winds around $\overline{\mathbb{D}}$ and accumulates to $\mathbb{T}$, see the domain G_1 at Figure 2 below, gives an example of such behavior. Observe that the domain G_2 in Figure 2 is not a Carathéodory domain. This domain will be useful for certain further considerations.

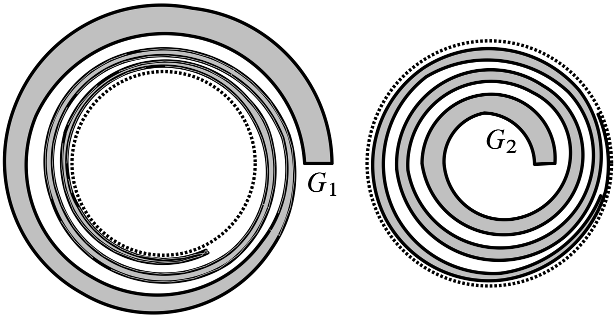

Figure 2. A cornucopia G_1 and an "inner snake" domain G_2.

In fact, the set $\mathbb{C} \setminus \overline{G}$ (for a Carathéodory domain G) can be even infinitely connected as shown by the infinite cornucopia given in Figure 3.

Definition 1.2. A nonempty compact set $K \subset \mathbb{C}$ is said to be a Carathéodory compact set, if $\partial K = \partial \widehat{K}$.

1.2 Properties of connectivity

We will explore in this section certain properties of connectivity of Carathéodory sets. These properties are not only of interest in their own right in the general context of the theory of Carathéodory sets, but they will be used repeatedly (but sometimes implicitly) in what follows.

We recall that an open set U is simply connected if and only if the set $\mathbb{C}_\infty \setminus U$ is connected. The following result is easy to prove.

Proposition 1.3. *Let K be a Carathéodory compact set, and let U be a Carathéodory open set. Then, the following hold.*

(a) *If $K^\circ = \emptyset$, then any compact subset $Y \subset K$ is also a Carathéodory compact set;*

(b) *If a compact set Y is the union of some components of K, then Y is a Carathéodory compact set;*

(c) *If Ω is the union of some components of U, then Ω is a Carathédoroy open set.*

For a bounded open set U let us introduce the concept of its Carathéodory hull, which is yet another variety of the concept of a Carathéodory set.

Definition 1.4. Let $U \neq \emptyset$ be a bounded open set. The set $U^* := \mathrm{Int}(\widehat{U})$ is called the Carathéodory hull of U.

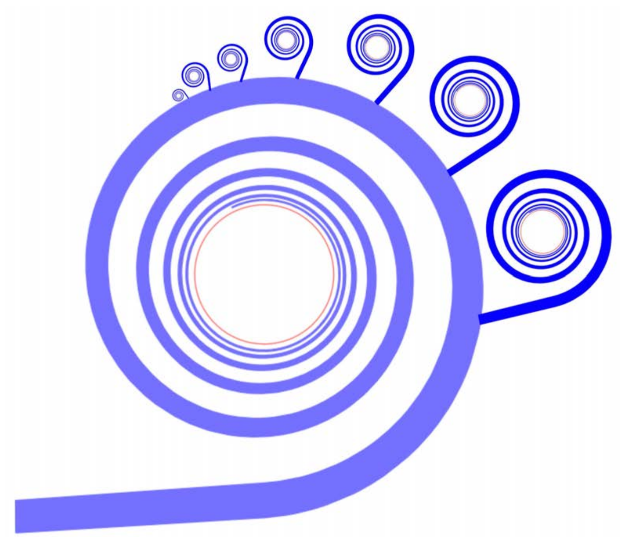

Figure 3. An infinite cornucopia.

For example, in Figure 3 the set G^* is the union of the cornucopia domain G itself (the domain shown in blue in this picture) and all disks, where this cornucopia accumulates. For the inner snake at the right-hand side in Figure 2, the set G_2^* is the small open disc, where the cornucopia is included.

Proposition¶ 1.5. *For Carathéodory open sets, the following holds.*

(1) *Every Carathéodory open set U is simply connected.*

(2) *If G is a Carathéodory domain, then G is a component of G^*. Conversely, for any bounded open set B, each component of B^* is a Carathéodory domain.*

Proof. Take a Carathéodory open set U and assume that it is not simply connected. Then, there exists a component V of U which is not simply connected. In such a case the set $\mathbb{C}_\infty \setminus V$ is not connected. Then, there exist two closed sets $X, Y \subset \mathbb{C}_\infty \setminus V$ such that

$$X \cap Y = \emptyset, \quad X \neq \emptyset, \quad Y \neq \emptyset, \quad \mathbb{C}_\infty \setminus V = X \cup Y. \tag{1.2}$$

Assume that $\infty \in Y$, then X is a compact subset of $\mathbb{C}$. Then,

$$\Omega_\infty(\bar{V}) \subset \mathbb{C}_\infty \setminus \bar{U} \subset X \cup Y.$$

Since $\Omega_\infty(\bar{V})$ is a connected set, then (1.2) yields

$$\Omega_\infty(\bar{V}) \subset Y \Rightarrow \partial\Omega_\infty(\bar{V}) \subset Y. \tag{1.3}$$

It holds, moreover, that $\partial X \subset \partial V$. Indeed, for any $w_0 \in \partial X$ and $\delta > 0$ with $\delta < \operatorname{dist}(X, Y)$, one has

$$D(w_0, \delta) \cap X \neq \emptyset, \quad \emptyset \neq D(w_0, \delta) \cap X^{\complement} = D(w_0, \delta) \cap (Y \cup V) = D(w_0, \delta) \cap V.$$

So, $w_0 \in \partial V$. Since V is a Carathéodory domain, the properties (1.2) and (1.3) imply

$$\emptyset \neq \partial X \subset \partial U \cap \partial X = \partial \Omega_\infty(\overline{V}) \subset Y \cap X = \emptyset,$$

which gives a contradiction. Thus, any Carathéodory open set U is simply connected as it is claimed.

Let now $K = \widehat{G}$. Since $G \subset \operatorname{Int}(\widehat{K})$, let V be the component of $\operatorname{Int}(\widehat{K})$ such that $G \subset V$. It needs to be shown that $G = V$. Assume that there is a point $z_1 \in V \setminus G$. Let $z_0 \in G$ and take a polygonal line $L \subset V$ such that $z_0, z_1 \in L$ (it is possible since V is a domain). Then, there exists a point w such that

$$w \in \partial G \cap L \subset \partial G \cap V \subset \partial \widehat{G} \cap \operatorname{Int}(\widehat{G}) = \emptyset,$$

which is a contradiction. ■

Proposition¶ 1.6. *The following properties are satisfied.*

(1) *If U is a Carathéodory open set, then $U = \operatorname{Int}(\overline{U})$.*

(2) *If G is a simply connected domain such that the set $\overline{G}$ does not separate the plane, then G is a Carathéodory domains if and only if $G = \operatorname{Int}(\overline{G})$.*

Proof. It is clear that $U \subset \operatorname{Int}(\overline{U})$. Assume that there exists a point $z \in \operatorname{Int}(\overline{U}) \setminus U$. Then, for some $\delta > 0$, it holds that $D(z, \delta) \subset \overline{U}$, and hence $z \in \partial U = \partial \Omega_\infty(\overline{U})$. So, $\emptyset \neq D(z, \delta) \cap (\mathbb{C} \setminus \overline{U}) \subset \overline{U} \cap (\mathbb{C} \setminus \overline{U}) = \emptyset$, which is a contradiction. Thus, (1) is proved.

We are going now to prove the second part of the proposition. Since $\overline{G}$ does not separate the plane, one has that $\Omega_\infty(\overline{G}) = \mathbb{C} \setminus \overline{G}$. Then,

$$\partial \Omega_\infty(\overline{G}) = \partial(\mathbb{C} \setminus \overline{G}) = \partial \overline{G} = \overline{\overline{G}} \setminus \operatorname{Int}(\overline{G}) = \overline{G} \setminus G = \partial G,$$

and the proof is completed. ■

Note that if K is a Carathéodory compact set, then K may be different from $\overline{\operatorname{Int}(K)}$.

The domain G_2 at Figure 2 is an "inner snake" which is not a Carathéodory domain. It gives an example showing that the converse assertion to the part (1) of Proposition 1.6 is not true, and also that the hypothesis in the part (2) of this Proposition that G does not separate the plane is essential.

The concept of a Carathéodory domain is not topologically invariant. For example one can consider the domain $f(G_2)$, where G_2 is presented in the picture in Figure 2 and $f(z) = 1/(z-a)$ with a being the center of the center of the disk in which G_2 accumulates. In order for the Carathéodory property for a given domain to be preserved by a homeomorphism of G, some additional assumptions on this homeomorphism are required. The next result generalizes [39, Theorem 2], where the hypotheses that the set $\overline{G}$ does not separate the plane is additionally assumed.

Theorem¶ 1.7. *Let G be a Carathéodory domain. Assume that $f\colon \widehat{G} \to \mathbb{C}$ is a continuous injection. Then, $f(G)$ is a Carathéodory domain. If $\overline{G}$ does not separate the plane then $\overline{f(G)}$ also does not separate the plane.*

Proof. For a subset $A \subset \widehat{G}$ the continuity of f and compactness of $f(\overline{A})$ imply that $f(\overline{A}) = \overline{f(A)}$. Let now $Y := f(\overline{G}) = \overline{f(G)}$.

In view of the theorem on the invariance of open sets (see, for instance [94, page 122] or [78, page 475]) the function f maps open sets in $\mathbb{C}$ to open sets in $\mathbb{C}$, in particular the set $f(G)$ is a domain.

Assume now that G is such that $\widehat{G} \neq \overline{G}$. Then, for any bounded component B of $\mathbb{C} \setminus \overline{G}$ its image $f(B)$ coincides with some bounded component Ω of $\mathbb{C} \setminus Y$.

Let us prove this claim. Take such B. Then, the set $f(B)$ is a domain and $f(B) \cap Y = f(B \cap \overline{G}) = \emptyset$ since f is injective. So, $f(B)$ has to be included into some component of $\mathbb{C} \setminus Y$. Assume that $f(B) \subset \Omega_\infty(Y)$. In this case, one can take a point $a \in B$ and find some infinite polygonal line $L \subset \Omega_\infty(Y)$ joining $f(a)$ with ∞. Clearly $L \cap f(B) \neq \emptyset$ and $L \cap f(B)^{\complement} \neq \emptyset$, since $f(B)$ is a bounded set. Then, there exists a point $b \in L \cap \partial f(B) \subset \Omega_\infty(Y) \cap Y = \emptyset$, which gives a contradiction. Therefore, there is a bounded component Ω of $\mathbb{C} \setminus Y$ such that $f(B) \subset \Omega$. Thus, $f(\overline{B}) \subset \overline{\Omega}$.

Now, let us assume, that $f(B) \neq \Omega$. In such a case one can take a point $a' \in \Omega \setminus f(B)$ and a point $b' \in f(B)$. If $\Lambda \subset \Omega$ is a polygonal line joining a' with b', one has $\emptyset \neq \Lambda \cap \partial f(B) \subset \Omega \cap Y = \emptyset$, which, again, is a contradiction. Therefore, $f(B) = \Omega$ and, hence, $f(\overline{B}) = \overline{\Omega}$, as it was claimed.

Going further, let Ω be a bounded component of the set $\mathbb{C} \setminus Y$. Then, $\partial\Omega \subset \partial Y$. Put $F := f^{-1}(\partial\Omega)$ so that F is a compact subset of ∂G. Thus, $\widehat{F} \subset \widehat{G}$ and $f(\widehat{F}) = \overline{\Omega}$ by the previous arguments.

Now, we have to prove that

$$f(G) = \operatorname{Int}(\overline{f(G)}). \tag{1.4}$$

By part (1) of Proposition 1.6 we obtain

$$f(G) = f(\operatorname{Int}(\overline{G})) \subset f(\overline{G}) = \overline{f(G)}.$$

But $f(\operatorname{Int}(\overline{G}))$ is an open set, so $f(G) \subset \operatorname{Int}(\overline{f(G)})$. In order to verify the opposite inclusion it suffices to observe that

$$f^{-1}(\operatorname{Int}(\overline{f(G)})) = f^{-1}(\operatorname{Int}(f(\overline{G}))) \subset \operatorname{Int}(f^{-1}(f(\overline{G}))) = \operatorname{Int}(\overline{G}) = G.$$

Thereafter, using (1.4) we have

$$\partial\overline{f(G)} = \overline{\overline{f(G)}} \setminus \operatorname{Int}(\overline{f(G)}) = \overline{f(G)} \setminus f(G) = \partial f(G). \tag{1.5}$$

Assume now that $f(G)$ is not a Carathéodory domain. Using (1.5) again, one has

$$\partial f(G) = \partial\overline{f(G)} = \partial_{\text{int}} Y \cup \partial_{\text{ext}} Y \cup \bigcup \partial\Omega_j(Y) \neq \partial_{\text{ext}} Y,$$

where $\{\Omega_j = \Omega_j(Y)\}$ is the collection (nonempty in the case under consideration) of components of the set $\mathbb{C} \setminus Y$.

Therefore, for every bounded component B of the set $\mathbb{C} \setminus \overline{G}$ there exists a point $z \notin \partial_{\text{ext}} B$ but $z \in \partial_{\text{int}} B \cup \bigcup \partial\Omega_j$. This implies that there exists a point $z_1 \in \partial\Omega_j$ for some index j such that $z_1 \notin \partial_{\text{ext}} B$. Put now $M := \partial B$ and $K := \partial\Omega_j$, so that K is a component of M. By Zoretti's theorem [136, page 109] there exists a Jordan curve Υ that encloses $\partial\Omega_j$ and such that $M \cap \Upsilon = \emptyset$ while $d(\Upsilon, K) < \varepsilon$ for some sufficiently small ε. Then, $\Upsilon \subset f(G)$, $f^{-1}(\Omega_j) = G_j$ and $\Upsilon_1 = f^{-1}(\Upsilon)$ is a Jordan curve on G such that Υ_1 encloses ∂G_j. But it is a contradiction because Υ_1 separates points of $\Omega_\infty(\overline{G})$, which are in the bounded component of $\mathbb{C} \setminus \Upsilon_1$, from ∞.

If $\widehat{G} = \overline{G}$ then, by the theorem of invariance by homeomorphisms on $\mathbb{C}_\infty$ [78, page 550], the set $\mathbb{C} \setminus Y$ is also connected. In this case, the proof may be completed using Proposition 1.6 (part (2)) and (1.4). ∎

Proposition¶ 1.8. *Let K be a compact subset of $\mathbb{C}$. Then, K is a Carathéodory compact set if and only if*

$$\operatorname{Int}(\widehat{K}) = \operatorname{Int}(K) \cup \bigcup G_j,$$

where $\{G_j\}$ is the collection of all bounded components of the set $\mathbb{C} \setminus K$.

Proof. In the case that $K = \widehat{K}$ (that is for compact sets which do not separate the plane) there is nothing to prove. Assume that K is a now a general Carathéodory compact set. Since $\operatorname{Int}(K) \cup \bigcup G_j \subset \widehat{K}$, then

$$\operatorname{Int}(K) \cup \bigcup G_j \subset \operatorname{Int}(\widehat{K}).$$

Let $w \in \operatorname{Int}(\widehat{K})$ and take $\varepsilon > 0$ such that $D(w, \varepsilon) \cap \Omega_\infty(K) = \emptyset$. Then, $D(w, \varepsilon) \cap \partial K = \emptyset$. This means that $D(w, \varepsilon) \subset K$ or $D(w, \varepsilon) \subset K^{\mathsf{C}} \setminus \Omega_\infty(K) = \bigcup G_j$. Then, $w \in \operatorname{Int}(K)$ or $w \in G_k$ for some index k.

Conversely, assume that $\operatorname{Int}(\widehat{K}) = \operatorname{Int}(K) \cup \bigcup G_j$, then

$$\partial\widehat{K} = \widehat{K} \cap \Big(\operatorname{Int}(K) \cup \bigcup G_j\Big)^{\mathsf{C}} = \widehat{K} \cap \Big(\bigcup G_j\Big)^{\mathsf{C}} \cap \operatorname{Int}(K)^{\mathsf{C}} = K \cap \operatorname{Int}(K)^{\mathsf{C}} = \partial K.$$

Therefore, K is a Carathéodory compact set and the proof is completed. ∎

Ending this section let us provide yet other clear relations among Carathéodory sets.

Proposition¶ 1.9. *The following statements hold.*

(1) *If K is a Carathéodory compact set with $K^\circ \neq \emptyset$, then K° is a Carathéodory open set.*

(2) *If U is a Carathéodory open set, then $\bar{U}$ is a Carathéodory compact set.*

1.3 Accessible boundary points

In this section we recall the concept of an accessible boundary point and present certain properties of accessible points on boundaries of Carathéodory domains.

Definition 1.10. Let U be an open set in $\mathbb{C}$, and let $a, b \in \partial U$.

(1) An arc $\mathcal{E}$ beginning at some point $w \in U$, ending at a, and such that $\mathcal{E} \setminus \{a\} \subset U$ is called *an end-cut* of U (or in U).

(2) An arc $\mathcal{C}$ beginning at a, ending at b, and such that $\mathcal{C} \setminus \{a, b\} \subset U$ is called *a cross-cut* of U (or in U).

The following fact may be found in [94, page 145].

Theorem 1.11. *Let G be a domain in $\mathbb{C}$.*

(1) *If G is simply connected and if $\mathcal{C}$ is a cross-cut in G, then $G \setminus \mathcal{C} = G_1 \cup G_2$, where G_1 and G_2 are disjoint simply connected domains.*

(2) *If for each cross-cut $\mathcal{C}$ in G the set $G \setminus \mathcal{C}$ is not connected, then G is a simply connected domain.*

The next definition is a small refinement of the definition given in [136, page 111].

Definition 1.12. (1) Let X be a subset of $\mathbb{C}$. A point $a \in \partial X$ is said to be accessible from X, if there exists some end-cut $\mathcal{E}$ of X ending at a.

(2) Let G be a simply connected domain in $\mathbb{C}$. A point $z \in \partial G$ is accessible from, at least, two sides of G, if there exists a cross-cut $\mathcal{C}$ in G with endpoints $a, b \in \partial G$, such that $z \notin \{a, b\}$, $z \in \partial G_1$, $z \in \partial G_2$, where $G \setminus \mathcal{C} = G_1 \cup G_2$, and the point z is accessible both from G_1 and from G_2.

For a simply connected domain G in $\mathbb{C}$ we put

$$\partial_a G := \big\{z \in \partial G : z \text{ is accessible from } G\big\}.$$

It is natural to call the set $\partial_a G$ the accessible part of the boundary of G. The set $\partial_a G$ is always dense in ∂G. This follows from the fact, easy to prove, that the set of points which are accessible by segments (as end-cuts) is dense. In [89] it was proved the important fact that $\partial_a G$ is a Borel set for every domain G.

Definition 1.13. Let M be a connected set and $w \in M$. The point w is called a cut point of M if the set $M \setminus \{w\}$ is not connected. The point w is called an end point of M, if there exists a sequence (U_n) of (circular) neighborhoods of w such that $\operatorname{diam}(U_n) \to 0$, as $n \to \infty$, and the set $\partial U_n \cap M$ consists of a single point for each n.

Proposition¶ 1.14. *If G is a Carathéodory domain, then ∂G does not have points which are accessible from both sides of G. Moreover, ∂G has neither cut points nor end points.*

Proof. Assume that a point z is an accessible point from both sides of G and let $\mathcal{C}$ be a cross-cut of G with endpoints a and b such that $z \neq a$, $z \neq b$, satisfying all requirements of Definition 1.12. For $s = 1, 2$, let $\mathcal{E}_s$ be two end-cuts in G_s starting at some points $z_s \in G_s$ and ending at the point z. Since G is a domain, let $L \subset G$ be a polygonal line joining the points z_1 and z_2 such that $L \cap \mathcal{E}_s \subset \{z_1, z_2\}$ for each $s = 1, 2$. Then, $\Upsilon := \mathcal{E}_1 \cup \mathcal{E}_2 \cup L$ is a Jordan curve that separates a and b. If Ω_1 and Ω_2 are the components of $\mathbb{C} \setminus \Upsilon$ we may assume that $a \in \Omega_1$, $b \in \Omega_2$. Since G is a Carathéodory domain then $\Omega_s \cap \Omega_\infty(\overline{G}) \neq \emptyset$ for each s. So,

$$\emptyset \neq \Upsilon \cap \Omega_\infty(\overline{G}) \subset (G \cup \{z\}) \cap (\mathbb{C} \setminus \overline{G}) = \emptyset,$$

which gives the desired result.

Assume now that $w \in \partial G$ is a cut point. Then, $\partial G = M_1 \cup \{w\} \cup M_2$ with $\overline{M_1} \cap M_2 = \overline{M_2} \cap M_1 = \emptyset$. By the separation theorem (see [136, page 108]) applied to the sets $A = \overline{M_1}$ and $B = \overline{M_2}$ there exists a Jordan curve $\Upsilon \subset G \cup \{w\}$ that separates M_1 and M_2. After that the proof may be completed as it was for accessible points.

Finally, if $w \in \partial G$ is an end point then, by its definition, w is the limit of some sequence (ζ_n) of points which are cut points of ∂G. However, this sequence cannot exist, therefore such a point w does not exist. ∎

The next result was obtained in [26] but here we prove it in a more simple manner.

Proposition¶ 1.15. *Let G be a Carathéodory domain and let B be a bounded component of $\mathbb{C} \setminus \overline{G}$. Then, the set $\partial_a G \cap \partial B$ consists of at most one point.*

Proof. Assume the opposite, which means, that there exists a cross-cut $\mathcal{C} \subset B \cup \{\zeta_1, \zeta_2\}$, $\zeta_1 \neq \zeta_2$, $\zeta_1, \zeta_2 \in \partial_a G \cap \partial B$. Let G_1 and G_2 be two simply connected domains such that $G \setminus \mathcal{C} = G_1 \cup G_2$. Then, take a line R, orthogonal to the segment $[\zeta_1, \zeta_2]$ and passing through the middle point of this segment. Denote by $R^\pm$, respectively, two rays of R starting at the middle point of $[\zeta_1, \zeta_2]$. Then, take the last point $\zeta_3 \in R \cap \partial B$ in the direction of the ray R^+, and the last point $\zeta_4 \in R \cap \partial B$ in the direction of the ray R^-. All points ζ_j, $j = 1, 2, 3, 4$ are different. Put $C_0 = \mathcal{C}$, $C_1 = \overline{G}$, then $C_0 \cap C_1 = \{\zeta_1, \zeta_2\}$ is not connected, then, by the second theorem of Janiszewski (see [78, page 506]), the continuum $C_0 \cup C_1$ separates the plane. So, $\mathbb{C} \setminus (C_0 \cup C_1) = U_1 \cup U_2$, where U_1 and U_2 are two open sets, with $G_j \subset U_j$ for $j = 1, 2$. In each of two small discs $D(\zeta_k, \delta)$, for $k = 3, 4$, there exists a point $z_j \in \Omega_\infty(\overline{G}) \cap U_j \neq \emptyset$ for $j = 1, 2$. These facts together with $\Omega_\infty(\overline{G}) = (\Omega_\infty(\overline{G}) \cap U_1) \cup (\Omega_\infty(\overline{G}) \cap U_2)$ give a contradiction. ∎

Example 2 in [26] shows a Carathéodory domain G such that the set $\partial_a G \cap \partial B$ is a singleton. A more informative example is given in Example 2.20 in Chapter 2, see Figure 6 below.

We mention here that the authors of [26] were unaware at that moment of the result proved in [38, page 172]. The aforementioned result says, in the notations of Proposition 1.15, that $\partial_a G \cap \partial_a B$ is either empty or consists of a single point. The difference of considering $\partial_a B$ in place of ∂B allows the author of [38] to argue more directly. But this difference is essential, because of Example 2.20. Let us also refer Proposition 2.19, where additional information is presented concerning the matter.

Corollary¶ 1.16. *If G is a Carathéodory domain such that $\partial G = \partial_a G$, then the set $\mathbb{C} \setminus \overline{G}$ is connected.*

Corollary¶ 1.17. *If W_1 and W_2 are two different components of a Carathéodory open set U, then $\partial_a W_1 \cap \partial_a W_2$ consists of at most one point.*

We end this section with the next result, which will be used several times in what follows.

Proposition¶ 1.18. *For every Carathéodory compact set X there exists a Carathéodory continuum Y such that $X \subseteq Y$ and $X^\circ = Y^\circ$.*

Proof. In order to prove this assertion we consider for each integer $k \geqslant 1$ the family $\mathcal{D}_k$ of the dyadic squares of the generation k, i.e.,

$$\mathcal{D}_k = \left\{ Q = \left[\frac{j_1}{2^k}, \frac{j_1+1}{2^k}\right) \times \left[\frac{j_2}{2^k}, \frac{j_2+1}{2^k}\right) : j_1, j_2 \in \mathbb{Z} \right\}.$$

Define the subfamily $\mathcal{D}_k(X)$ consisting of all squares $Q \in \mathcal{D}_k$ such that $X \cap \overline{Q} \neq \emptyset$, put $F_k := \bigcup_{Q \in \mathcal{D}_k(X)} Q$ and suppose $F_{k,1}, \ldots, F_{k,r_k}$ to be the closures of the polynomial hulls of the components of F_k. In such a case one has that $X \subset F_{1,1}^\circ \cup \cdots \cup F_{1,r_1}^\circ$. For each k and $j = 1, \ldots, r_k$ we choose a point $z_{k,j} \in \partial F_{k,j}$. Set $F_k^* := \bigcup_{j=1}^{r_k} F_{k,j}$. Denote by $I_{k+1,j}$ the set of indexes $s = 1, \ldots, r_k$ such that $F_{k+1,s} \subset F_{k,j}$ and set $F_{k+1,j}^* := \bigcup_{s \in I_{k+1,j}} F_{k+1,s}$.

In what follows by a tree we mean a connected polygonal line T such that $\mathbb{C} \setminus T$ is connected.

Let us construct a sequence of trees (T_k) with $T_{k-1} \subset T_k$ by induction. Take a point $z \notin X$ and choose a tree T_1 such that T_1 connects z with all points $z_{1,j}$, $j = 1, \ldots, r_1$ and such that the set $\mathbb{C} \setminus (F_1^* \cup T_1)$ is connected. Suppose now that the trees $T_1, \ldots, T_k$ are already constructed. Let us show how to construct the tree T_{k+1}. Since

$F_{k,j}$ for $j = 1, \dots, r_k$ contains a finite number of $\{F_{k+1,s}\}$ (where $s = 1, \dots, r_{k+1}$), we can choose a new tree $T_{k,k+1,j}$ that connects $z_{k,j}$ with all $z_{k+1,s}$ for $s \in I_{k+1,j}$ such that the domain $G_k := \mathbb{C} \setminus (T_k \cup Y_k)$, where $Y_k = \bigcup_{j=1}^{r_k}(F^*_{k+1,j} \cup T_{k,k+1,j})$ is simply connected. Now, we put $T_{k+1} = T_k \cup (\bigcup_{j=1}^{r_k} T_{k,k+1,j})$.

Finally, we take $T = \overline{\bigcup_{k=1}^{\infty} T_k}$ and let $Y = X \cup T$. Then, Y is a compact set such that $X^\circ = Y^\circ$. Since all G_k are simply connected domains and $\mathbb{C} \setminus Y = \bigcup_k G_k$, then Y is connected and finally, Y is a Carathéodory compact set because of the fact that $\partial Y = \partial X \cup T$. ∎