Chapter 2

Carathéodory sets and conformal maps

2.1 Some background on conformal maps

Let B and G be domains in $\mathbb{C}$. One says that a function f maps B conformally onto G (respectively, into G) if f is holomorphic and injective in B, and $f(B) = G$ (respectively, $f(B) \subset G$). The Riemann mapping theorem is the starting point of all studies of conformal maps. Let us recall some historical remarks concerning the Riemann theorem since they are important for better understanding the role of Carathéodory's ideas and results. B. Riemann enunciated his outstanding theorem on conformal maps in his dissertation in 1851. The Riemann theorem says that, if G is a simply connected domain such that $G \neq \mathbb{C}$ and $G \neq \emptyset$, then there exists a conformal map f from $\mathbb{D}$ onto G. If $a \in G$ and $\vartheta \in [0, 2\pi)$ are fixed, then there exists a unique conformal map f satisfying the normalization conditions $f(0) = a$ and $\arg f'(0) = \vartheta$. If $\vartheta = 0$, the corresponding f is called the *Riemann mapping function* (with respect to a). Notice, that the proof given by Riemann contained a gap which was eliminated later on by D. Hilbert and other authors. The standard modern proof was developed by R. Riesz and L. Fejér and was published by T. Radó in 1923. It may be found, for instance, in [33, Chapter vii] and in [61, page 30]. Montel's theory of normal families of holomorphic functions plays a crucial role in this proof. Also we refer to [19, page 298], where one can find a constructive proof of the Riemann mapping theorem made by P. Koebe and C. Carathéodory.

If f is the Riemann map from $\mathbb{D}$ onto G and $g = f^{-1}$, the number $R = 1/g'(a)$ is called the *conformal radius* of G we respect to a. The function $g_0 := g/g'(a)$ defined on G maps G conformally onto $D(0, R)$ and satisfies the normalization conditions $g_0(a) = 0$ and $g_0'(a) = 1$. Sometimes this function is more easily handled than the Riemann map. For instance, the function g_0 possesses several minimality properties, one of which is given by the following proposition.

Proposition 2.1. *Let $G \neq \mathbb{C}$ be a simply connected domain and let $a \in G$. Then, the function g_0 defined above is the unique solution to the extremal problem*

$$\int_G |g_0'|^2 \, dA = \inf\left\{\int_G |h'|^2 \, dA : h \in H(G),\ h(a) = 0,\ h'(a) = 1\right\} = \pi R^2.$$

The details of the proof may be found in [61, page 55]. It is appropriate to recall that the standard area formula (see, e.g., [43, page 96]) yields that for any measurable set $E \subset G$, for each function $h \in H(G)$ (not necessarily univalent) and for every real nonnegative measurable function F defined on $h(E)$ it holds

$$\int_{h(E)} F(z)\, n(h,z)\, dA(z) = \int_E F(h(w))\, |h'(w)|^2\, dA(w), \tag{2.1}$$

where $n(h,a)$ stands for the number of points of $h^{-1}(\{a\})$, for each $a \in \mathbb{C}$. The above formula also holds for complex measurable functions F, provided that one of its two entries is well defined.

One of the central questions in the theory of conformal maps which is of high importance for our considerations is the study of the behavior of a conformal map $f\colon \mathbb{D} \to G$ near a point $\zeta \in \partial\mathbb{D}$. In a general sense this behavior is given by the concept of a *prime end*. We denote by $\operatorname{diam}^{\#}$ the diameter of sets in the spherical metric in $\mathbb{C}_\infty$ (see [104, page 1]).

Definition 2.2. Let G be a simply connected domain. We call a sequence of cross-cuts $(\mathcal{C}_n)$ a *null-chain* of G if

(i) $\overline{\mathcal{C}}_n \cap \overline{\mathcal{C}}_{n+1} = \emptyset$ for each $n = 0, 1, 2, \ldots$;

(ii) $\mathcal{C}_n$ separates $\mathcal{C}_0$ and $\mathcal{C}_{n+1}$ for each $n = 1, 2, 3, \ldots$;

(iii) $\operatorname{diam}^{\#}(\mathcal{C}_n) \to 0$ as $n \to \infty$.

If ∂G is bounded one can replace $\operatorname{diam}^{\#}$ with the Euclidean diameter.

Let us recall the notion of equivalence of null-chains. We say that two null-chains $(\mathcal{C}_n)$ and $(\mathcal{C}'_n)$ are equivalent if, for every large number m, there exists a number n such that $\mathcal{C}'_m$ separates $\mathcal{C}_n$ from $\mathcal{C}_0$, and $\mathcal{C}_m$ separates $\mathcal{C}'_n$ from $\mathcal{C}'_0$. The equivalence classes of null-chains with respect to this relation are called the *prime ends* of G. Let us denote the set of all prime ends of G by $\mathfrak{Pr}(G)$. It is possible to define a topology on the set $\mathfrak{Pr}(G)$ such that $G \cup \mathfrak{Pr}(G)$ become compact. The next result is one of the keystones in the conformal mapping theory, it is known as *Carathéodory prime ends theorem.*

Theorem 2.3. *Let f maps $\mathbb{D}$ conformally onto a bounded simply connected domain G. There exists a homeomorphism*

$$\hat{f}\colon \overline{\mathbb{D}} \to G \cup \mathfrak{Pr}(G)$$

which extends f (that is $f(z) = \hat{f}(z)$ for $z \in \mathbb{D}$) and for any $\zeta \in \mathbb{T}$ and for any null-chain $(\mathcal{C}_n)$ representing the prime end $\hat{f}(\zeta)$ the sequence $f^{-1}(\mathcal{C}_n)$, for sufficiently large n, forms a null-chain in $\mathbb{D}$ separating 0 from ζ.

In the simplest case that $G = \mathbb{D}$, the set $\mathfrak{Pr}(\mathbb{D})$ is homeomorphic to $\mathbb{T}$. Going further we need to recall some notation whose detailed account may be found in [104, Section 2.5] and [32, Chapter 9]. The *impression* of the prime end $\hat{f}(\zeta)$ (which also is the cluster set of f at the point ζ) is the set

$$I(\hat{f}(\zeta)) = I(f,\zeta) = C(f,\zeta) = \bigcap_{r>0} \overline{f(D(\zeta,r)\cap\mathbb{D})},$$

while the set of principal points of $\hat{f}(\zeta)$ is the set

$$\Pi(\hat{f}(\zeta)) = \Pi(f,\zeta) = C_{[0,\zeta)}(f,\zeta) = \bigcap_{0<r<1} \overline{f([r\zeta,\zeta))}.$$

The global cluster set of f is defined as

$$C(f) = \bigcap_{n\geqslant 2} \overline{f(\{z\in\mathbb{D} : |z| > (n-1)/n\})}.$$

In terms of these sets the prime ends of f are classified as follows:

First kind:	$\Pi(f,\zeta)$ is a singleton and $\Pi(f,\zeta) = I(f,\zeta)$;
Second kind:	$\Pi(f,\zeta)$ is a singleton, but $\Pi(f,\zeta) \neq I(f,\zeta)$;
Third kind:	$\Pi(f,\zeta)$ is not a singleton and $\Pi(f,\zeta) = I(f,\zeta)$;
Fourth kind:	$\Pi(f,\zeta)$ is not a singleton, but $\Pi(f,\zeta) \neq I(f,\zeta)$.

In the case that ∂G is a Jordan curve we have the following result, which is often called *Carathéodory–Osgood–Taylor theorem* (in several textbooks this theorem is also called *Carathéodory extension theorem for Jordan domains*).

Theorem 2.4. *Let f map $\mathbb{D}$ conformally onto a bounded domain G. The following conditions are equivalent:*

(i) *f has a continuous injective extension to a mapping from $\overline{\mathbb{D}}$ onto $\overline{G}$;*

(ii) *∂G is a Jordan curve;*

(iii) *∂G is locally connected and has no cut points.*

We refer the reader, depending on his expertise, to [19, page 309], [77, Chapter II], or [104, Chapter 2], where several proofs of this theorem with different levels of details may be found. The question when f has a continuous extension to $\overline{\mathbb{D}}$, perhaps without injectivity, was also solved by Carathéodory. The next result is referred as *Carathéodory continuity theorem.*

Theorem 2.5. *Let f map $\mathbb{D}$ conformally onto a bounded domain G in the complex plane. Then, the following four conditions are equivalent:*

(i) *f has a continuous extension to a mapping from $\overline{\mathbb{D}}$ onto $\overline{G}$;*

(ii) *there exists a continuous map ψ on $\mathbb{T}$ such that $\psi(\mathbb{T}) = \partial G$;*

(iii) *the set ∂G is locally connected;*

(iv) *the set $\mathbb{C} \setminus G$ is locally connected.*

The implication (iii)⇒(ii) in Theorem 2.5 is a special case of the Hahn–Mazurkiewicz theorem (see [94, page 59]). The implication (ii)⇒(iii) is a general fact on continuous images of locally connected compacta. A complete proof of the other equivalences may be found, for instance, in [104] or in [94].

The next question which is natural to pose for an arbitrary conformal map f is the question whether the boundary values f exist on the boundary of the domain, where f is defined except, may be, some "relatively small" set. This question may be solved in different manners, depending on the tools used. We will need the following result concerning the matter. Let us recall the definition of Hardy spaces in $\mathbb{D}$. For $p > 0$ the space $H^p = H^p(\mathbb{D})$ consists of all functions $f \in H(\mathbb{D})$ such that $M_p(f) < \infty$, where

$$M_p(f) = \sup_{r \to 1} \int_{\mathbb{T}} |f(r\zeta)|^p \, dm_{\mathbb{T}}(\zeta).$$

For all $p > q > 0$ the inclusions $H^\infty \subset H^p \subset H^q \subset N$ hold, where $N = N(\mathbb{D})$ is the Nevanlinna class in $\mathbb{D}$. We recall that any function $f \in N(\mathbb{D})$ has the form $f = f_1/f_2$, where $f_1, f_2 \in H^\infty(\mathbb{D})$.

Given a point $\zeta \in \mathbb{T}$ and β with $0 < \beta < \pi/2$, then the Stolz angle $S_\zeta(\beta)$ is the set

$$\{z \in \mathbb{D} : |\arg(1 - \overline{\zeta} z)| < \beta, |z - \zeta| < 2\cos\beta\}.$$

Let now h be a function from $\mathbb{D}$ to $\mathbb{C}_\infty$. One says that h has angular limit (or, in other words, boundary value) at the point $\zeta \in \mathbb{T}$, if for each $\beta \in (0, \pi/2)$ the limit

$$\lim_{S_\zeta(\beta) \ni z \to \zeta} h(z)$$

exists and is independent on β. This common value is denoted by $h(\zeta)$.

Theorem 2.6. *The following statements hold.*

(1) *Let $h \in N(\mathbb{D})$. Then, the angular limit $h(\zeta) \neq \infty$ exists for $m_{\mathbb{T}}$-a.a. $\zeta \in \mathbb{T}$.*

(2) *If f maps $\mathbb{D}$ conformally into $\mathbb{C}$, then $f \in H^p$ for every $p < 1/2$, and therefore $f(\zeta) \neq \infty$ exists for a.a. $\zeta \in \mathbb{T}$.*

(3) *Moreover, if f maps $\mathbb{D}$ conformally into $\mathbb{C}$, then the boundary values $f(\zeta)$ exist for all $\zeta \in \mathbb{T}$, except a set of logarithmic capacity zero.*

We are not providing any special reference for these results, the interested reader can follow, for instance, [104, Theorems 1.7, 8.2, 9.19, and Corollary 2.17], as well as the explanation given in [105, Chapter ii, Sections 1 and 2]. In fact, one important ingredient here is the classical Fatou's theorem that says that any function $f \in H^\infty$ has a.e. on $\mathbb{T}$ finite angular limits.

Definition 2.7. For a function $f \in H^\infty$ let $F(f)$ be the set of all points $\zeta \in \mathbb{T}$ for which the boundary value $f(\zeta)$ exists. This set is called a *Fatou set* of f.

Carathéodory in [20] considered sequences (G_n) of simply connected domains and studied when the sequence of the corresponding Riemann maps converges in some sense. We ought to recall some results from this subject.

Definition 2.8. Let (G_n) be a sequence of domains (not necessarily simply connected) and assume that there exists $a \in \bigcap_{n=1}^{\infty} G_n$. One says that G_n converges to a set G in the sense of the kernel convergence with respect to a, and G is the kernel of this sequence, if one of the two following conditions holds.

(1) If there exists $\rho > 0$ such that $D(a, \rho) \subset G_n$ for all sufficiently large n, then G must be a domain, $a \in G$, $G \neq \mathbb{C}$, and the following two conditions must be satisfied:

(1a) if $w \in G$ then there exists $\varepsilon > 0$ such that $D(w, \varepsilon) \subset G_n$ for large n,

(1b) if $w \in \partial G$, then $w = \lim w_n$ for some sequence of points (w_n) such that $w_n \in \partial G_n$ for each n.

(2) If the previous condition (1) is not satisfied then $G = \{a\}$.

This convergence is well defined, but it clearly depends on the choice of the given point a. In the case that G_n converges to G with respect to a in the sense of kernel convergence we will write $G_n \to G$ with respect to a. If it is clear from the context what a we are dealing with we will simply write $G_n \to G$.

The notion of kernel convergence has several surprising properties, for instance it underlies several deep results about convergence of sequences of conformal maps. The following Carathéodory kernel convergence theorem shows the relations between concepts of kernel convergence and locally uniform convergence of the corresponding conformal maps in the case of simply connected domains.

Theorem 2.9. *Let (G_n) be a sequence of simply connected domains, $G_n \neq \mathbb{C}$, and let a be a point such that $a \in G_n$ for each n. Let f_n be a conformal map from $\mathbb{D}$ onto G_n such that $f_n(0) = a$, $f_n'(0) > 0$. Then,*

$$f_n \rightrightarrows f \text{ locally in } \mathbb{D} \quad \text{if and only if} \quad G_n \to G \text{ with respect to } a, \tag{2.2}$$

where f and G are defined as follows: if $G = \{a\}$ then f is the constant function, so that $f(z) = a$ for all z; while in the case that $G \neq \{a\}$, so that the domain G must be simply connected and $G \neq \mathbb{C}$, the function f is the conformal map from $\mathbb{D}$ onto G with the normalization $f(0) = a$ and $f'(0) > 0$.

Moreover, in the case that G is a simply connected domain and $G_n \to G$ with respect to a, it holds that $f_n^{-1} \rightrightarrows f^{-1}$ locally in G.

In the proof of this theorem several important tools of the theory of conformal maps are used, let us notice, for instance Hurwitz's and Montel's theorems, Koebe's distortion theorem, etc. The proof of Carathéodory kernel convergence theorem may be found in many sources, see for example [61, page 54] or [104, page 14].

Let now (G_n) be a sequence of domains which converges to a Jordan domain G in the sense of kernel convergence with respect to some point a. In this case, Walsh (see [129, 130] as well as [134, pages 32–34]) was able to obtain a more strong result. Notice that this theorem is of high importance, but nowadays it seems to be almost forgotten and did not appear in the mathematical literature during many decades.

Theorem 2.10. *Let G be a Jordan domain, $a \in G$ and let (G_n) be a sequence of simply connected domains satisfying $\overline{G} \subset G_n$ such that $\overline{G_{n+1}} \subset G_n$ for all n and $G_n \to G$ with respect to a. Let ψ_n be the conformal map from G_n onto G normalized by the conditions $\psi_n(a) = a$ and $\psi_n'(a) > 0$. Then, $\psi_n(z) \rightrightarrows z$ on $\overline{G}$.*

It is not clear, whether it is possible to extend this theorem for more wide class of domains. The next question looks quite reasonable.

Question I. Will the statement of Theorem 2.10 hold in the case that G is a Carathéodory domain with accessible boundary, that is $\partial G = \partial_a G$?

2.2 Carathéodory domains and conformal maps

The reason that Carathéodory paid attention to the domains which nowadays are called by his name is shown in the next result. As far as we know, the paper [20] contains the first occurrence of the cornucopia (see, for instance, the domain G_1 in Figure 2), in the mathematical literature.

Theorem 2.11. *Let $G \neq \emptyset$ be a bounded simply connected domain. Then, G is a Carathéodory domain if and only if there exists a sequence (Γ_n) of Jordan curves such that*

$$\Gamma_n \subset \Omega_\infty(\overline{G}), \quad \overline{D(\Gamma_{n+1})} \subset D(\Gamma_n)$$

for each n, and $D(\Gamma_n) \to G$ as $n \to \infty$ with respect to any fixed point $a \in G$. This equivalence does not depend on the choice of a.

If G is a Carathéodory domain, let g_n be the conformal map from $D(\Gamma_n)$ onto $\mathbb{D}$ with the normalization $g_n(a) = 0$ and $g_n'(a) > 0$, and let g be the conformal map from G onto $\mathbb{D}$ with the same normalization. Then, $g_n \rightrightarrows g$ locally in G as $n \to \infty$. In fact,

$$\overline{G} \subset W := \bigcap_{n=1}^{\infty} D(\Gamma_n), \tag{2.3}$$

but it can happen that $\overline{G} \neq W$.

Proof. Let us take a nonempty bounded simply connected domain G satisfying all conditions of the theorem. In order to prove that G is a Carathéodory domain, let us take an arbitrary point $w \in \partial G$. By condition (1b) of Definition 2.8 there exist a sequence of points (w_n) such that $w_n \in \partial D(\Gamma_n) = \Gamma_n \subset \Omega_\infty(\overline{G})$, and $w_n \to w$ as $n \to \infty$. Also there exists another sequence (w_n') such that $w_n' \in G \subset \mathbb{C} \setminus G_\infty(\overline{G})$ and $w_n' \to w$ as $n \to \infty$. So, $w \in \partial\Omega_\infty(\overline{G})$. Since G is simply connected, then G is a Carathéodory domain by definition.

Assume now that G is a Carathéodory domain. The domain $G_\infty' = \Omega_\infty(\overline{G}) \cup \{\infty\}$ is simply connected in $\mathbb{C}_\infty$. So, one can take a conformal map h from $\mathbb{D}$ onto G_∞' with the normalization $h(0) = \infty$. Let us now define $\Gamma_n := h(\{t : |t| = n/(n+1)\})$. Then, each Γ_n is a Jordan curve such that $\overline{G} \subset \overline{D(\Gamma_{n+1})} \subset D(\Gamma_n)$. Since $(D(\Gamma_n))$ is a decreasing sequence of domains it converges to the component of $\bigcap_{n=1}^{\infty} D(\Gamma_n)$ that contains a, which is G. The remaining conclusions follow from Theorem 2.9. ■

Let now G be a Carathéodory domain, and let f be some conformal map from $\mathbb{D}$ onto G. Take a point $w \in \partial G$. According to [104, Proposition 2.14] one has $w \in \partial_a G$ if and only if there exists a curve $\gamma\colon [0,1] \to \overline{\mathbb{D}}$ having the properties $\gamma(s) \in \mathbb{D}$ for $s \in [0,1)$ and $\gamma(1) = t$ for some $t \in \partial\mathbb{D}$, such that $\lim_{s\to 1^-} f(\gamma(s)) = w$. Moreover, it follows that $t \in F(f)$ and $f(t) = w$.

Proposition¶ 2.12. *Let G be a Carathéodory domain, and $w \in \partial_a G$. Then, there exists a unique point $t \in F(f)$ such that $f(t) = w$.*

Proof. The existence of two points t and $t' \neq t$ such that $\varphi(t) = \varphi(t') = w$ would imply that the point w is accessible from both sides of G. But Proposition 1.14 says that the boundary of the (Carathéodory domain) G does not have points which are accessible points from both sides of G. ■

Corollary¶ 2.13. *Let G be a Carathéodory domain. Then, ∂G is locally connected if and only if ∂G is a Jordan curve. In particular, if ∂G is rectifiable then ∂G is a Jordan curve.*

Proof. Assume that ∂G is locally connected and take a conformal map f from $\mathbb{D}$ onto G. By Theorem 2.5, f has a continuous extension to $\overline{\mathbb{D}}$. Let $f_1\colon \partial\mathbb{D} \to \partial G$ be the restriction of such extension. By Proposition 2.12, f_1 is injective, and since it is defined in a compact set, then f_1 is a homeomorphism from $\partial\mathbb{D}$ onto ∂G, so that ∂G is a Jordan curve. The second assertion is a consequence of the general fact that a continuum with finite length is locally connected. ■

A Carathéodory domain may have prime ends of all four kinds, as it can be seen at Figure 4, where the domain G_1 gives the desired example.

The class of non-degenerate continua E possessing the property that there exists a bounded univalent function f in $\mathbb{D}$ and a point $\zeta \in \mathbb{T}$ such that $C(f,\zeta) = E$, was studied and characterized in details, see for instance, [30, Proposition 5]. Next result establishes some restriction to the size of $C(f,\zeta)$ when f is the Riemann map onto some Carathéodory domain.

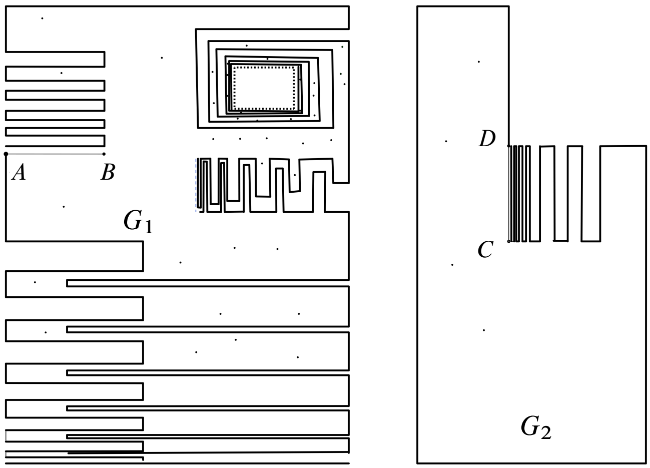

Figure 4. The Carathéodory domain G_1 has prime ends of all four kinds.

Proposition¶ 2.14. *The following properties are satisfied.*

(1) *If G is a Carathéodory domain and f is a conformal map from $\mathbb{D}$ onto G, then the set $C(f, \zeta)$ is a Carathéodory continuum for each point $\zeta \in \partial\mathbb{D}$.*

(2) *Conversely, if K is a Carathéodory continuum, then there exists a Carathéodory domain G and a conformal map $f : \mathbb{D} \to G$ such that f has a continuous extension to $\overline{\mathbb{D}} \setminus \{1\}$ and $C(f, 1) = \Pi(f, 1) = \partial K$.*

Proof. (1) Fix a point $\zeta \in \partial\mathbb{D}$ and let $z \in K := C(f, \zeta)$. Since $K \subset \partial G = \partial G_\infty$, then there exists a sequence (z_n) such that $z_n \in G_\infty$ for all n, and $z = \lim_{n\to\infty} z_n$. Each point z_n can be joined to ∞ by an infinite polygonal line L such that $L \subset \mathbb{C} \setminus G \subset \mathbb{C} \setminus K$, so $z_n \in \Omega_\infty(K)$. Therefore, $\partial K = K \subset \partial\Omega_\infty(K) \subset K$.

(2) Let K be such that $\partial\Omega_\infty(K) = \partial K$, then $\Omega' = \Omega_\infty(K) \cup \{\infty\}$ is a simply connected domain in $\mathbb{C}_\infty$. Let $h : \mathbb{D} \to \Omega'$ be a conformal map such that $h(0) = \infty$. Going further let us take an open ribbon $S \subset \mathbb{D}$ which spirals to $\mathbb{T}$ and such that $0 \notin \overline{S}$. Let $\psi : \mathbb{D} \to S$ be a conformal map such that $C(\psi, 1) = \Pi(\psi, 1) = \mathbb{T}$. Then, $G = h \circ \psi(\mathbb{D})$ is the desired domain. In fact, it is clear that $f = h \circ \psi$ has a continuous extension to $\overline{\mathbb{D}} \setminus \{1\}$. Moreover, take $w \in \partial_a\Omega' \subset \partial K$, and let $\mathcal{E}$ be an end-cut ending at w. Then, $h^{-1}(\mathcal{E})$ is an end-cut that ends at some point of ∂G. Then, $h^{-1}(\mathcal{E})$ cuts infinitely many points of S and of $\mathbb{D} \setminus S$. Then, $w \in \partial G_\infty \cap \partial G$. This situation holds for each point of a dense set, then $\partial K \subset \partial G_\infty \cap \partial G$. Thus,

$$\partial G_\infty = (\partial G \cap \Omega') \cup \partial K \subset \partial G \cup \partial K,$$

which means that G is a Carathéodory domain. ∎

Let us recall that a continuum K is said to be *indecomposable* if it cannot be written in the form $K = M \cup N$, where M and N are proper subcontinua of K (for more information about this notion see [78, Chapter V]) and [68, Section 3.8]. A Carathéodory continuum can be indecomposable. One of the simplest example of such continua is the Knaster buckethandle, see [78, Example 1, page 204]. Let us denote this continuum by K_b. Applying Proposition 2.14 we can see that there exists a Carathédory domain G and a conformal map $f : \mathbb{D} \to G$ such that f has a continuous extension to $\overline{\mathbb{D}} \setminus \{1\}$ and $\Pi(f, 1) = C(f, 1) = K_b$ is an indecomposable continuum. A related example is given in [29, Proposition 4] but therein the set $\Pi(f, 1)$ is a singleton, $C(f, 1) = \partial G$ and ∂G is an indecomposable continuum however G is not Carathéodory domain. Thus, to obtain a more involved example it is necessary to have some free space between G and $\Omega_\infty(\overline{G})$. This can be done using the construction of the Lakes of Wada, see [68, Section 3.8]. We need to make some modification of this construction for further considerations.

Example 2.15. Consider the compact set

$$X_0 = \left\{z : -2 \leqslant \operatorname{Re} z \leqslant 4, |\operatorname{Im} z| \leqslant \frac{3}{2}\right\} \Big\backslash \left(D\left(-1, \frac{1}{2}\right) \cup D\left(1, \frac{1}{2}\right) \cup D\left(3, \frac{1}{2}\right)\right).$$

To preserve the poetic flavor of the original example, we will imagine that X_0 is an island in the ocean and the small discs are three lakes, the first one having blue water, the second one green, while the third one red. Let us dig a system of canals in X_0 following the next procedure. For $k \in \mathbb{N}$ define the system of time moments $t_k = (k-1)/k$, and the sequence of distances $d_k = 1/k$, so that $t_k \to 1$ and $d_k \to 0$ as $k \to \infty$. Let V_1 be the canal (considering as an open set) that brings water from the ocean to every point of the land within distance d_1 of every point of X_0, and let $X_1 = X_0 \setminus V_1$. At the time moment t_2 let V_2 be the canal that brings water from the blue lake to every dry point within distance d_2 of every point of X_1. The first steps of this construction is illustrated by Figure 5. For time moments t_3 and t_4 let us do the same, but using water from the green lake and from the red lake, respectively. Thereafter let us repeat this cycle of construction of canals infinitely many times until we arrive to the time $t = 1$. It is possible to make this construction in such a way that the entrances to the canals in the blue lake are two sequence of open intervals on $\partial D(-1, \frac{1}{2})$ which are mutually disjoint and accumulate only at the points $-\frac{1}{2}$ and $-\frac{3}{2}$. Then, take W_{blue}, a simply connected domain formed by the blue lake together all canals with blue water, and put $X := \partial W_{blue}$.

Let us accent some properties of the domain W_{blue} constructed in Example 2.15. Denote by W_{green} the union of the green lake with all of the canals starting therein and by W_{red} the respective union for the corresponding red lake, then W_{green} and W_{red} are simply connected bounded components of the set $\mathbb{C} \setminus \overline{W_{blue}}$. The construction of the

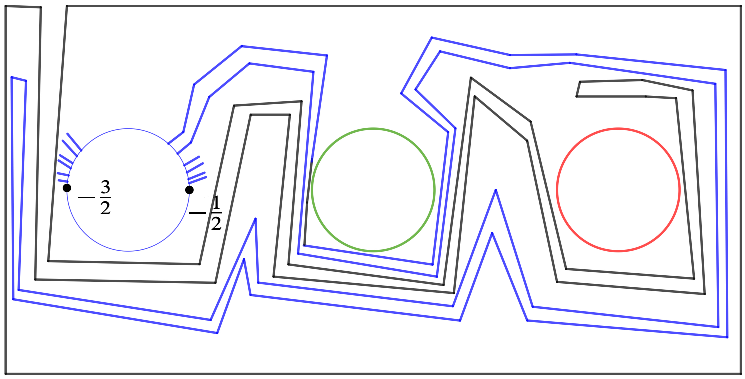

Figure 5. The first steps of the construction of the compact set $X = \partial W_{blue}$.

canals from the ocean implies that

$$\partial W_{blue} = \partial W_{green} = \partial W_{red} = \partial\Omega_\infty(\overline{W_{blue}}),$$

which yields that W_{blue} is a Carathéodory domain. Furthermore, ∂W_{blue} is a indecomposable continuum. If f is a conformal map from $\mathbb{D}$ onto W_{blue}, then f is continuous on $\overline{\mathbb{D}}$ except two points, says ζ_1 and ζ_2, where $C(f, \zeta_1) = C(f, \zeta_2) = X = \partial W_{green} = \partial W_{red}$, while $C(f, \zeta) \subset X$ for all $\zeta \in \mathbb{T}$.

By a suitable modification of the construction given in Example 2.15 it is possible to obtain a Carathéodory domain $G = f(\mathbb{D})$, for a conformal mapping f such that the (closed) set $T(f) = \{\zeta \in \mathbb{T} : C(f, \zeta) = \partial G\}$ is infinite. By a certain theorem by Rutt (see more details in [29]) in the case that the set $T(f)$ is not empty, the set ∂G is an indecomposable continuum, or the union of two indecomposable continua. Moreover, if ∂G is indecomposable, then $T(f) \neq \emptyset$. We do not know how big the set $T(f)$ may be for a Carathéodory domain in a general situation. We do not even know the answer to the following question.

Question II. Whether there exists a Carathéodory domain for which the set of prime ends of the first kind would be empty (so that the respective conformal map f cannot be continuously extended to any point of $\mathbb{T}$)?

The usual example of a domain of such kind (see, for example, [32, page 184]) is clearly not a Carathéodory domain.

Examples of this kind should not surprise the reader, since they are quite natural in a certain sense. To see this we need to use some results from plane topology.

Definition 2.16. One says that a set $E \subset \mathbb{C}$ possesses the *non-separation property* if for each closed subset $F \subset E$, such that $F \neq E$, the set $E^{\complement} \cup F$ is connected, that is the set $E \setminus F$ does not separate the plane. Otherwise, one says that E possesses the *separation property*.

Every Jordan curve possesses the non-separation property, but is it possible to assert the converse? This question was an open problem for some time at the beginning of the XX century. Its solution allows us to state the following result related with Carathéodory domain, which needs to be compared with [39, Proposition 10].

Theorem¶ 2.17. *Let K be a Carathéodory compact set. Then, one of the two following mutually exclusive conditions is fulfilled:*

(1) *K possesses the separation property;*

(2) *$K = \partial G$ for each component G of $\mathbb{C} \setminus K$.*

Moreover, if $\mathbb{C} \setminus K$ contains only two components, then K is a Jordan curve, while in the case that $\mathbb{C} \setminus K$ contains at least three components, the set K is an indecomposable continuum or the union of two indecomposable continua.

Proof. If $K^\circ \neq \emptyset$, then the condition (1) holds for such K. Therefore, let us consider the compact sets K possessing the non-separation property and having empty interior. If the set $\mathbb{C} \setminus K$ has only one component, then $K = \partial\Omega_\infty(K)$. Assume now that the set $\mathbb{C} \setminus K$ has a bounded component. Then, $\partial G \subset K$. If $K \neq \partial G$, then ∂G separates the plane, and hence $K = \partial G$.

To show that the conditions (1) and (2) are exclusive let us assume that the compact set K satisfy (2) and let F be some closed subset of K, different of K. Then, the set

$$K^{\complement} \cup F = \bigcup_G G \cup F,$$

where G runs over all components of the set $\mathbb{C} \setminus K$, is connected because $G \subset G \cup F \subset \overline{G}$ and both sets G and $\overline{G}$ are connected. So, K does not satisfy the condition (1).

Assume now that $K^{\complement} = G \cup \Omega_\infty(K)$, where G is simply connected domain. If the set $\partial G = K$ is not locally connected then there exists a sequence (F_n) of mutually disjoint closed sets with $F_n \subset K$, and a closed set $F \subset K$, $F \neq K$, such that $F_n \to F$ in the Hausdorff metric. Then,

$$\overline{K \setminus F} = K \quad \text{and} \quad K^{\complement} \cup F = (G \cup F) \cup \Omega_\infty(K)$$

with $(\overline{G \cup F}) \cap \Omega_\infty(K) = \emptyset$. Then, K possesses the separation property, which is impossible. Therefore, ∂G is locally connected, which yields, according to Corollary 2.13, that ∂K is a simple closed curve.

In the case that there exist three components of the set $\mathbb{C} \setminus K$, or more, than its common boundary is K and we need to refer the theorem stated in [78, page 590] in order to finish the proof. ∎

Let $\mathcal{E} \subset \mathbb{D}$ be an end-cut ending at some point $\zeta \in \mathbb{T}$, and let $f \in C(\mathbb{D})$. The cluster set $C_{\mathcal{E}}(f, \zeta)$ of f following $\mathcal{E}$ is defined as follows:

$$C_{\mathcal{E}}(f, \zeta) = \bigcap_{n=1}^{\infty} \overline{f\left(\left\{z \in \mathcal{E} : |z - \zeta| < \frac{1}{n}\right\}\right)}.$$

This set does not depend on the choice of the initial point of $\mathcal{E}$, so we can always assume that the initial point of $\mathcal{E}$ is the origin. It is easy to prove, and it is well-known (see, for instance, [32, Theorems 4.6 and 4.7]) that $C(f, \zeta) = C_{\mathcal{E}}(f, \zeta)$ for some $\mathcal{E}$. Moreover, by definition $\Pi(f, \zeta) = C_{[0,\zeta]}(f, \zeta)$.

The following result was communicated to us by Ch. Pommerenke.

Proposition¶ 2.18. *Let G be a Carathéodory domain, and let f be a conformal map from $\mathbb{D}$ onto G. Assume that there exist two points, say ζ_1 and ζ_2, in $\mathbb{T}$ such that $\zeta_1 \neq \zeta_2$ and for each $j = 1, 2$ there is an end-cut $\mathcal{E}_j$ in $\mathbb{D}$ ending at ζ_j and possessing the property*

$$C_{\mathcal{E}_1}(f, \zeta_1) \cup C_{\mathcal{E}_2}(f, \zeta_2) \subset E,$$

for some continuum $E \subset \partial G$. Then, for one of the open arcs Υ of $\mathbb{T} \setminus \{\zeta_1, \zeta_2\}$ one has

$$I(f, \zeta) \subset E \tag{2.4}$$

for each point $\zeta \in \Upsilon$.

Proof. We may assume that $\mathcal{E}_1 \cap \mathcal{E}_2 = \{0\}$. Take $F = \overline{f(\mathcal{E}_1)} \cup \overline{f(\mathcal{E}_2)} \cup E$. Then, F is a continuum that separates the plane. Let V be the bounded component of $F^{\complement}$ such that $f(\mathcal{E}_1) \cup f(\mathcal{E}_2) \subset \overline{V}$. Let $U \subset \mathbb{D}$ be the domain whose boundary is $\overline{\mathcal{E}_1} \cup \overline{\mathcal{E}_2} \cup \overline{\Upsilon}$, where Υ is one the arc of $\mathbb{T} \setminus \{\zeta_1, \zeta_2\}$ chosen in such a way that $f(U) \subset V$.

Let us now assume that (2.4) is false. Take a point $\zeta' \in \Upsilon$ and a sequence (z_n) such that $z_n \in U$ and $z_n \to \zeta'$ as $n \to \infty$ such that the sequence $(f(z_n))$ converges as $n \to \infty$ to some point $w \in (\overline{V} \setminus E) \cap \partial G$. So, there exists a closed disc $\overline{D(w, r)} \subset V$ such that $\overline{D(w, r)} \cap F = \emptyset$. Then, take a point $\alpha \notin \overline{G} \cap D(w, r)$. Since G is Carathéodory, there exists an infinite polygonal line $\mathcal{L} \subset \mathbb{C} \setminus \overline{G}$ that starts at α and goes to ∞. But therefore

$$\mathcal{L} \subset V \cup (\mathbb{C} \setminus \overline{V}), \quad \mathcal{L} \cap V \neq \emptyset, \quad \mathcal{L} \cap (\mathbb{C} \setminus \overline{G}) \neq \emptyset.$$

which gives a contradiction since $\mathcal{L}$ is connected. ∎

In the case that $I(f, \zeta_1) \cap I(f, \zeta_2) \neq \emptyset$ we can take as a candidate for E in the previous proposition the continuum $I(f, \zeta_1) \cup I(f, \zeta_2)$. For example, for the domains G_1 and G_2 in Figure 4 one can take as E the segments $[A, B]$ and $[C, D]$, respectively.

Under the assumptions of Proposition 1.15 it is possible to say more about the cluster set in the special point.

Proposition¶ 2.19. *Let G be a Carathéodory domain, and f be a conformal map from $\mathbb{D}$ onto G. Assume that B is a bounded component of $\mathbb{C} \setminus \overline{G}$ such that $\partial_a G \cap \partial B = \{w\}$. Let $\zeta \in \mathbb{T}$ be such that $f(\zeta) = w$. Then, $\partial B \subset I(f, \zeta)$.*

Proof. For simplicity let us assume that $\zeta = 1$, so that f has the radial limit w at 1. For $r \in (0, 1)$ let $\ell(r)$ stands for the length of the set $f(\{z \in \mathbb{D} : |z - 1| = r\})$. One of key points in the theory of conformal maps is the fact that

$$\int_0^1 \frac{\ell(r)^2}{r}\, dr < +\infty,$$

see [104, Proposition 2.2]. Then, there exists a sequences of cross-cuts $\mathcal{C}_n = f(\{z \in \mathbb{D} : |z - 1| = r_n\})$ such that $\ell(r_n) \to 0$ as $n \to \infty$. Each cross-cut $\mathcal{C}_n$ joins some point $\alpha_n \in \partial G$ with another point $\beta_n \in \partial G$, cuts the image $f([0, 1])$ in one point, and, finally, $\mathcal{C}_n$ tends to $\{w\}$. For each n take ε_n such that

$$\left(\overline{D(\alpha_n, \varepsilon_n)} \cup \overline{D(\beta_n, \varepsilon_n)}\right) \cap \partial B = \emptyset,$$

and $\varepsilon_n \to 0$ as $n \to \infty$. Going further we cover ∂B by a finite sequence of closed discs of radius ε_n in such a way that centers of these disks belong to G_∞. We can joint the centers of the constructed disks by polygonal lines in order to obtain a new polygonal line $\mathcal{L}_n \subset G_\infty$ such that $\mathcal{L}_n \cap \overline{D(\alpha_n, \varepsilon_n)} \neq \emptyset$ and $\mathcal{L}_n \cap \overline{D(\beta_n, \varepsilon_n)} \neq \emptyset$, and the compact set $\mathcal{L}_n \cup \overline{D(\alpha_n, \varepsilon_n)} \cup \overline{D(\beta_n, \varepsilon_n)} \cup \mathcal{C}_n$ separates the plane into two components. Denote by W_n the corresponding bounded component. This process can be done in such a way, that, moreover, $W_{n+1} \subset W_n$. Then, $f(\{z \in \mathbb{D} : |z - 1| < r_n\}) \subset W_n$. Taking into account the fact $\partial B \subset C(f)$, we conclude that

$$\partial B \subset \overline{f(\{z \in \mathbb{D} : |z - 1| < r_n\})}$$

for each r_n. So, $\partial B \subset I(f, 1)$. ∎

In general, in the above proposition the set $I(f, \zeta)$ is much bigger than ∂B.

The next example is new, but it is based on ideas of [26, Example 2]. This example shows that in the framework of hypotheses of Proposition 2.19 it can happen that $\partial_a G \cap \partial B = \{w\}$, but $\partial_a G \cap \partial_a B = \emptyset$, and B has different impressions of inaccessible points from B.

Example 2.20. Take $Q = \{z : 0 \leqslant \operatorname{Im} z < \pi,\ 0 < \operatorname{Re} z < 2\}$ and let $I_1, I_2, \ldots$ be a sequence of intervals

$$I_n = [ia_n, ib_n], \quad a_1 = 2, \quad a_n < b_n < a_{n+1} < \pi, \quad \lim_{n\to\infty} a_n = \pi.$$

Let $J_1, J_2, \ldots$ be a sequence of intervals

$$J_n = [ia'_n, ib'_n], \quad b'_1 = 1, \quad a'_n < b'_n, \quad b'_{n+1} < a'_n, \quad \lim_{n\to\infty} a'_n = 0.$$

Let, for each $n \geqslant 1$,

$$A_n = \{z : \operatorname{Im} z \in I_n,\ 0 \leqslant \operatorname{Re} z \leqslant 1\}, \quad z_n = \frac{1}{n+2} + i\frac{a_{n+1} + b_n}{2},$$
$$B_n = \{z : -1 < \operatorname{Re} z < 0,\ \operatorname{Im} z \in (ia'_n, ib'_n)\}, \quad z'_n = -\frac{n}{n+1} + i\frac{a'_n + b'_n}{2},$$
$$\widetilde{Q} = \left(\{z \in Q : \operatorname{Im} z \geqslant \operatorname{Re} z\} \Big\backslash \bigcup_{n=1}^{\infty} A_n\right) \cup \bigcup_{n=1}^{\infty} B_n,$$
$$F = \partial\widetilde{Q} \setminus ((0, 2+2i] \cup [2+2i, 2+\pi i] \cup (1+\pi i, 2+\pi i]).$$

Let now $L_1, L_2, \dots$ be a sequence of mutually disjoint closed intervals over the segment

$$\left\{z : 0 < \operatorname{Re} z < \frac{3}{2}, \operatorname{Im} z = \operatorname{Re} z\right\}$$

such that $L_n \to 0$.

Let $S_1 \subset \widetilde{Q}$ be narrow enough closed ribbon starting at L_1, entering in B_1 until the point $z'_1 \in S_1$, continuing thereafter and finishing at z_1, always without crossing the line $\{z : \operatorname{Im} z = a_2\}$. Assume that $S_1, S_2, \dots, S_n$ are already constructed. Then, $S_{n+1} \subset \widetilde{Q}$ is a narrow enough closed ribbon starting on L_{n+1} with the following properties:

(i) $z'_{n+1} \in S_{n+1}$;

(ii) S_{n+1} is always in the left-hand side of S_n. In particular, $S_{n+1} \cap S_j = \emptyset$ for each $j \leqslant n$;

(iii) S_{n+1} ends at the point z'_{n+1} without crossing the line $\{z : \operatorname{Im} z = a_{n+2}\}$.

(iv) $d_{\mathcal{H}}(\partial S_{n+1}, F) < \min\{1/n, d_{\mathcal{H}}(\partial S_n, F)\}$, where $d_{\mathcal{H}}$ is the Hausdorff distance.

This process can be continued indefinitely. Then, take

$$W_+ = \operatorname{Int}\left(\bigcup_{n=1}^{\infty} S_n \cup \{z \in Q : \operatorname{Re} z \geqslant \operatorname{Im} z\}\right).$$

Define, finally, $G = \exp(W_+ \cup W_- \cup (0,2))$, where W_- denotes the reflection of W_+ over the real axes. Now, the point $w = 1$ is an accessible point from G and $\mathbb{C} \setminus \overline{G} = B$ is a bounded component from which $[b, a)$ and $(c, 1]$ are inaccessible from B.

In Figure 6 the third step of the construction of W_+ was shown.

Figure 7 shows the domain G and the component B, this picture can help the reader to get a better understanding of the constructed domain.

Let now g map a given domain G conformally onto $\mathbb{D}$. The question whether g has a continuous extension to $\overline{G}$, or not, is also very interesting and important, however it usually not included in textbooks and courses on conformal maps.

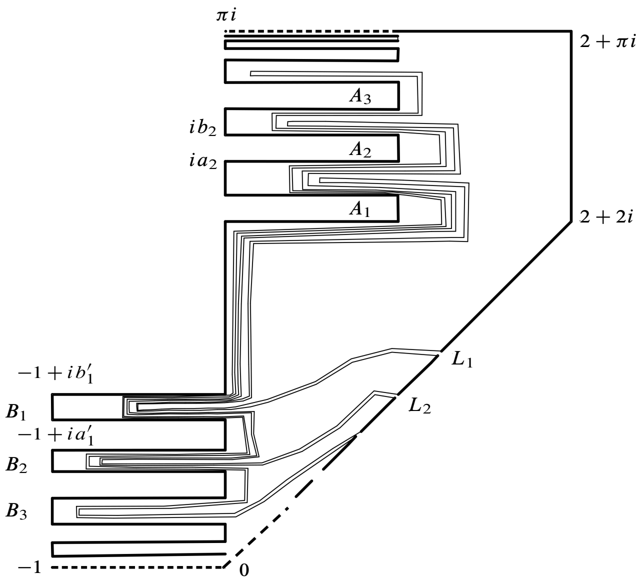

Figure 6. The third step of the construction of W_+.

Definition 2.21. Let G be a simply connected domain, and let f be a conformal map from $\mathbb{D}$ onto G. A point $w \in \partial G$ is said to be simple in the sense of Carathéodory if the set $\{\zeta \in \mathbb{T} : w \in C(f, \zeta)\}$ is a singleton.

The concept of a simple point in the sense of Carathéodory is independent of the choice of f. For the domain G_1, see Figure 4, all points in the arc $[A, B)$ are not simple in the sense of Carathéodory, while all other points in ∂G_1 are simple in this sense. To avoid confusion with other uses of the term "simple point" (see, for example, [115, Chapter 14]), we decide to use the term "simple in the sense of Carathéodory".

The next result was obtained in [44], it gives the criterion for continuity of g. However, this characterization is not completely topological. A proof can be found in [85].

Proposition 2.22. *Let G be a bounded simply connected domain and let g map G conformally onto $\mathbb{D}$. A continuous extension $\tilde{g}\colon \overline{G} \to \overline{\mathbb{D}}$ of g exists if and only if each point $w \in \partial G$ is a simple point in the sense of Carathéodory.*

In other words, the existence of a continuous extension of g is equivalent to the statement that distinct prime ends have disjoint impressions. Figure 1 can help to better understanding the previous result since there exists a continuous extension of g for G_1, but not for G_2.

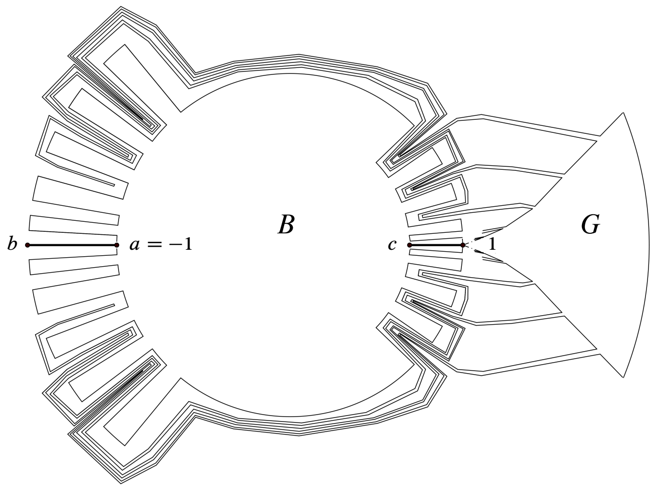

Figure 7. Inaccessible points from the bounded component B.

Proof. For the proof of necessity let us assume that some point $w \in \partial G$ is not simple. Then, $w \in C(f, \zeta_1) \cap C(f, \zeta_2)$, where $f = g^{-1}$ and ζ_1 and $\zeta_2 \neq \zeta_1$ are two points in $\mathbb{T}$. One can find two sequences, say (z_n) and (z'_n), such that $z_n \to \zeta_1$, $z'_n \to \zeta_2$, while $f(z_n) \to w$ and $f(z'_n) \to w$. In this case, the continuity of $\tilde{g}$ would imply that $\tilde{g}(w) = \lim z_n = \zeta_1$ and $\tilde{g}(w) = \lim z'_n = \zeta_2$ which is a contradiction.

The proof of sufficiency. Let us define $\tilde{g}(w) = \zeta$, where $\zeta \in \mathbb{T}$ is the unique point such that $w \in C(f, \zeta)$ in the case that $w \in \partial G$, while $\tilde{g}(w) = g(w)$ for $w \in G$. The continuity property of $\tilde{g}$ is not difficult, but some arguments from the theory of cluster sets are needed for the proof. ∎

Furthermore, in [44] Farrell proved the following result, which is related to the theorem about kernel convergence.

Theorem 2.23. *Let G be a Carathéodory domain such that each point in ∂G is simple in the sense of Carathéodory. Let $z_0 \in G$ and let (G_n) be a sequence of bounded simply connected domains, such that*

$$\overline{G} \subset G_{n+1} \subset \overline{G_n},$$

for $n \geqslant 1$, and $G_n \to G$ with respect to z_0. For $n \geqslant 1$, let g_n be the conformal map from G_n to $\mathbb{D}$ such that $g_n(z_0) = 0$ and $g'_n(z_0) > 0$. Denote by $\tilde{g}$ the extension of the conformal map from G onto $\mathbb{D}$ to $\overline{G}$ with $\tilde{g}(z_0) = 0$ and $\tilde{g}'(z_0) > 0$.

Then, $g_n \rightrightarrows \tilde{g}$ on $\overline{G}$.

Assume now that G is a Carathéodory domain, $z_0 \in G$ and let f be the conformal map from $\mathbb{D}$ onto G with the normalization $f(0) = z_0$ and $f'(0) > 0$. Furthermore, let $f^{-1}: G \to \mathbb{D}$ be the corresponding inverse map. The next result is a refinement of [26, Theorem 1].

Theorem¶ 2.24. *Let G be a Carathéodory domain and let (J_n) be a sequence of Jordan curves such that $D(J_n) \to G$ with respect to some point $z_0 \in G$ and $\overline{G} \subset D(J_n)$, $\overline{D(J_n)} \subset D(J_{n-1})$ for each $n > 1$. Let $f_n: \overline{\mathbb{D}} \to \overline{D(J_n)}$ be the extension of the respective conformal map with the normalization $f_n(0) = z_0$ and $f_n'(0) > 0$ inherited from f. Then, the following hold.*

(1) *If $\mathcal{E}$ is an end-cut in G, then f_n^{-1} converges uniformly on $\mathcal{E}$ to f^{-1}, in particular, $f_n^{-1}(z) \to f^{-1}(z)$ for each point $z \in \partial_a G$;*

(2) *If W is a bounded component of $\mathbb{C} \setminus \overline{G}$, then $|f_n^{-1}| \to 1$ uniformly on $\overline{W}$. However, in general it is not true that f_n^{-1} converges to some constant on $\overline{W}$.*

Since the proof of this theorem is essentially the same as the respective proof in [26], we present here only its sketch which highlights the keynote steps.

Sketch of the proof of Theorem 2.24. Without loss of generality we may also assume that $\mathcal{E}$ starts at the point z_0. Let now $b_0 \in \partial_a G$ be the end point of $\mathcal{E}$. Put $\varrho := f^{-1}(\mathcal{E})$ so that ϱ is an arc in $\mathbb{D} \cup \{\zeta_0\}$, where $\zeta_0 = f^{-1}(b_0)$, passing from 0 to ζ_0.

For each $m \in \mathbb{N}$ we consider a point $b_m \in J_m$ which is a nearest point to b_0. For each $m \geqslant 1$ we put $\mathcal{E}_m := \mathcal{E} \cup [b_0, b_m]$ and $\varrho_m := f_m^{-1}(\mathcal{E}_m)$. Let $\zeta_m = f_m^{-1}(b_m)$ and note, that each $\varrho_m = f_m^{-1}(\mathcal{E}) \cup f_m^{-1}([b_0, b_m])$ is the union of two consecutive arcs in $\mathbb{D} \cup \{\zeta_m\}$. It is clear, that the sequence (ϱ_m) accumulates to some subset Λ of $\overline{\mathbb{D}}$. It means that Λ is the set of all points $w \in \overline{\mathbb{D}}$ such that there exists a sequence (w_{m_j}) of points such that $w_{m_j} \in \varrho_{m_j}$ and $w_{m_j} \to w$ as $j \to \infty$.

The set Λ possesses some special properties. Namely, one has

(i) Λ is a continuum;

(ii) $\Lambda \subset \varrho \cup \mathbb{T}$;

(iii) $\varrho \subset \Lambda$;

(iv) The set $\Lambda \cap \mathbb{T}$ is connected.

Therefore, $\Lambda = \varrho \cup \gamma$, where γ is some closed subarc of $\mathbb{T}$. In order to prove the first assertion we need to show that $\Lambda = \varrho$ or, in other words, that $\gamma = \{\zeta_0\}$.

Let w_m' be a nearest point of the set ϱ_m to t_0 and let ϱ_m' be the subcontinuum $f_m^{-1}(\mathcal{E}_m')$, where $\mathcal{E}_m'$ is the segment $[f_m(w_m'), b_m]$ in the case when $f_m(w_m') \notin \mathcal{E}$, or the set $\mathcal{E}_m'' \cup [b_0, b_m]$ otherwise, where $\mathcal{E}_m''$ is the subarc of $\mathcal{E}$ that joints the points $f_m(w_m')$ and b_0.

We have that $f_m(w_m') \to b_0$ as $m \to \infty$ and therefore $\operatorname{diam}(f_m(\varrho_m')) \to 0$ as $m \to \infty$.

Notice that ϱ_m' is either an arc or the union of two consecutive arcs. Then, applying [103, Theorem 9.2] to ϱ_m', or to each of the arcs that form ϱ_m', we conclude, that $\operatorname{diam}(\varrho'_m) \to 0$ as $m \to \infty$, which means, that $\zeta_m \to \zeta_0$ as $m \to \infty$ and hence, $\gamma = \{\zeta_0\}$.

We are going to prove the assertion of the part (2). Assume that $|f_n^{-1}|$ does not converge uniformly to 1 on $\overline{W}$. Then, there exist a sequence (z_k) in W and a subsequence $(f_{n_k}^{-1})$ such that $|f_{n_k}^{-1}(z_k)| \leqslant r < 1$ for all k. Let $w_k := f_{n_k}^{-1}(z_k)$. Taking a subsequence of (w_k) if it is necessary we may assume that $w_k \to w_0$, $|w_0| \leqslant r < 1$. Since f_{n_k} converge uniformly on the compact set $\bigcup_{k=0}^{\infty}\{w_k\}$ to f we have

$$f(w_0) = \lim_{k\to\infty} f_{n_k}(w_k) \in \overline{W}.$$

But $f(w_0) \in G$ and $\overline{W} \cap G = \emptyset$, so that we arrive to a contradiction.

For the last assertion we must consider Example 2.15, where $W_{\text{green}} \cup W_{\text{red}} \subset D(J_n)$. Then, the sequence (f_n^{-1}) has two accumulation points, say ζ_1 and ζ_2 with the notation in the aforementioned example. To prove this some arguments are needed. However, we omit them, because we believe that this help will be enough for the reader. ■

Corollary¶ 2.25. *Let G be a Carathédory domain. Then, f and g can be extended to Borel measurable functions (denoted also by f and g) on $\mathbb{D} \cup F(f)$ and $G \cup \partial_a G$, respectively, and such that*

$$\begin{aligned} g(f(\xi)) &= \xi \quad \text{for all } \xi \in F(f), \\ f(g(\zeta)) &= \zeta \quad \text{for all } \zeta \in \partial_a G. \end{aligned}$$

The domain G_2 in Figure 4, which is not a Jordan domain, has the property $\partial_a G_2 = \partial G_2$. For such domains one has the following corollary.

Corollary¶ 2.26. *Let G be a Carathéodory domain such that $\partial_a G = \partial G$, and let f be some conformal map from $\mathbb{D}$ onto G. Then, f^{-1} can be extended to $\overline{G}$ and this extension belongs to the first Baire class on $\overline{G}$.*

Notice, that this corollary generalizes the Carathéodory extension theorem to the case that the domain under consideration is a Carathéodory domain with accessible boundary. It is clear, that this class of domain is substantially wider than the class of Jordan domain.