Chapter 3

Uniform and pointwise approximation on Carathéodory sets

3.1 Uniform approximation by polynomials

Problems on approximation of analytic functions by polynomials and rational functions were always of special importance during the development of contemporary analysis, but they have attracted special attention after the classical results about approximation in the complex domain obtained by Weierstrass and Runge at the end of 19th century. Let us recall, that Weierstrass proved that any continuous function defined on [0, 1] may be uniformly approximated on this segment by a sequence of polynomials. The *Runge's theorem* is as follows.

Theorem 3.1. *Let $K \subset \mathbb{C}$ be a compact set, and let $E \subset \mathbb{C}_\infty \setminus K$ be a set which contains, at least, one point of each component of $\mathbb{C}_\infty \setminus K$. If $f \in H(K)$, then for every $\varepsilon > 0$ there exists a rational function R with poles on E such that $\|f - R\|_K < \varepsilon$.*

This theorem was published in 1885, [116], the same year as the aforesaid result by Weierstrass. There are several proofs of Runge's theorem, see, for instance, [118, pages 171–177] for the proof which is close to the original one. See also [115, Chapter 13] for the proof using certain functional analysis methods, and [33, Chapter VIII] for a more direct and elementary proof.

The following properties follow directly from Runge's theorem. Let Ω be an open set in $\mathbb{C}$, let E be a set which contains one point of each component of $\mathbb{C}_\infty \setminus \Omega$. Then, for every function $f \in H(\Omega)$ one can find a sequence (R_n) of rational functions with poles lying only in E, such that $R_n \rightrightarrows f$ locally in Ω. In the special case when the set $\mathbb{C}_\infty \setminus \Omega$ is connected (note that this means that Ω is a simply connected set, but not necessarily a connected one), one can take $E = \{\infty\}$ and a sequence (K_n) of compact subsets of Ω such that $\bigcup_{n=1}^{\infty} K_n = \Omega$, and thus obtain a sequence (P_n) of polynomials such that $P_n \rightrightarrows f$ locally in Ω. Let us observe that the set $\mathbb{C}_\infty \setminus \Omega$ may have uncountably many components: for instance one can consider $\Omega = \mathbb{C} \setminus K$, where $K \subset [0, 1]$ is the linear $1/3$-Cantor set.

Note that the condition that the set $\mathbb{C}_\infty \setminus \Omega$ is connected cannot be relaxed in the latter statement. Namely, one has the following theorem.

Theorem 3.2. *Let $U \subset \mathbb{C}$ be an open set, and assume that for every $f \in H(U)$ there exists a sequence (P_n) of polynomials such that $P_n \rightrightarrows f$ locally in U. Then, the set $\mathbb{C}_\infty \setminus U$ is connected.*

Indeed, assume that the set $\mathbb{C}_\infty \setminus U$ is not connected, then $\mathbb{C}_\infty \setminus U = K \cup Y$, where K is a compact subset of $\mathbb{C}$, Y is closed set, $\infty \in Y$, and $K \cap Y = \emptyset$. By the *separation theorem*, see [136, page 108], there exists a Jordan curve $J \subset U$ such that $K \subset D(J)$. Let $a \in K$, then the function $h(z) = 1/(z-a)$ cannot be approximated by a sequence of polynomials uniformly on J. Indeed, let $C = \sup\{|z-a| : z \in J\}$ and $\rho = 1/(2C)$. If there exists $P \in \mathcal{P}$ such that $\|h - P\|_J < \rho$, then the inequality

$$|1 - p(z)(z-a)| < \rho|z-a| \leqslant \frac{1}{2}$$

holds for all $z \in J$. Therefore, by the maximum modulus principle, this inequality also holds for $z = a$, but this is a contradiction.

For further considerations we need to introduce several algebras of functions. Let $K \subset \mathbb{C}$ be a compact set. Denote by $P(K)$ the algebra of all functions which can be approximated uniformly on K by polynomials, so that $P(K)$ is the closure in $C(K)$ of the subspace $\mathcal{P}|_K$. Next, let $R(K)$ be the algebra consisting of all functions which can be approximated uniformly on K by rational functions with poles lying outside K. Furthermore, we put $A(K) = C(K) \cap H(K^\circ)$. It is clear that

$$P(K) \subset R(K) \subset A(K) \subset C(K). \tag{3.1}$$

All aforesaid algebras $A(\cdot)$, $R(\cdot)$ and $P(\cdot)$ may be defined in the same way for any closed subset of $\mathbb{C}_\infty$.

It can be readily verified that $P(\overline{\mathbb{D}}) = A(\overline{\mathbb{D}})$ and $R(\mathbb{T}) = C(\mathbb{T})$. Furthermore, the equality $P(K) = C(K)$ implies that the set $\mathbb{C} \setminus K$ is connected, while the Runge's theorem says that $P(K) = R(K)$ whenever the set $\mathbb{C} \setminus K$ is connected.

The question on for which compact sets K the approximation property $P(K) = A(K)$ is satisfied is quite natural. The investigation of this question was started in the 1920s by J. L. Walsh, who dealt with two important cases when K is the closure of a generic Jordan domain, and when K is a closed arc. In [129–131] Walsh proved several results, and his most general statement in this topic is as follows (for proofs and further details see [134, Chapter II]).

Theorem 3.3. *Let $Y \subset \mathbb{C}_\infty$ be a closed set such that ∂Y is a finite union of Jordan curves or closed arcs, no two of which have more than finitely many common points. Then, $A(Y) = R(Y)$. More precisely, let $E \subset \mathbb{C}_\infty \setminus Y$ be a set that contains at least one point of each component of $\mathbb{C}_\infty \setminus Y$. Then, for every $f \in A(Y)$ there exists a sequence (F_n) of rational functions with poles lying outside E such that $F_n \rightrightarrows f$ on Y.*

Let us comment how this result was proved in a particular case. If $Y = \overline{G}$ for some Jordan domain G, the proof runs as follows. Take a function $f \in A(Y)$ and a sequence (G_n) of Jordan domains such that $G_n \to G$. Let ψ_n be conformal map from G_n onto G as it was considered in Theorem 2.10. Then, each function $f \circ \psi_n$ is holomorphic in some neighborhood of Y (each function in its own one). Next, given an arbitrary $\varepsilon > 0$ Runge's theorem implies that there exists a polynomial P_n such that $|f(\psi_n(z)) - P_n(z)| < \varepsilon/2$ for all $z \in Y$. Finally, the fact that $\psi_n(z) \rightrightarrows z$ on Y and the uniform continuity of f on Y yield that $|f(\psi_n(z)) - f(z)| < \varepsilon/2$ for $z \in Y$.

The topological conditions imposed in Theorem 3.3 turned out to be not essential, since the following result was established in 1931 by F. Hartogs and A. Rosenthal, see [62]. If $K \subset \mathbb{C}$ is a compact set such that $\mathrm{Area}(K) = 0$, then $R(K) = C(K)$. What about compact sets X with empty interior for which $R(X) \neq C(X)$? Let us recall that the first example of such kind was constructed by A. Roth [110, page 97]. It was a compact set of the form $X = \overline{\mathbb{D}} \setminus \bigcup_{n\geqslant 1} D_n$, where each $D_n \Subset \mathbb{D}$ is some appropriately chosen open disk. The principal idea underlying Roth's example construction turned out to be crucial for a number of further constructions of examples of the failure of approximation. Let us note the construction of this kind given in [56, page 26]. In view of the shape of this compact set X, all such examples are called nowadays a "Swiss cheeses" or "Champagne bubbles".

Later on Walsh encouraged his student O. J. Farrell to study the problem of polynomial approximation to a function f holomorphic in a domain G but not necessarily continuous in $\overline{G}$ (but assuming only that f is bounded in G) and gave him some ideas how to proceed in this case. Farrell in [44] considered the problem on uniform approximation by polynomials of a conformal map from G onto the unit disk. As far as we know this is the second paper in the mathematical literature, where the notion of Carathéodory domain is important.

Theorem 3.4 (Farrell). *Let G be a bounded simply connected domain in $\mathbb{C}$, and let g map G conformally onto $\mathbb{D}$. Then, g has a continuous extension $\tilde{g}$ to $\overline{G}$ and $\tilde{g}$ may be approximated by polynomials uniformly on $\overline{G}$ if and only if G is a Carathéodory domain and all points in ∂G are simple in the sense of Carathéodory.*

Proof. Assume that the desired $\tilde{g}$ exists and that it can be approximated by polynomials uniformly on $\overline{G}$. Then, $\tilde{g}$ is continuous and Proposition 2.22 implies that all points in ∂G are simple. Moreover, $|\tilde{g}(w)| = 1$ for each $w \in \partial\Omega_\infty(\overline{G})$, then $|g(w)| < 1$ for $w \in \widehat{G} \cap \overline{G}$. So, $\partial G = \partial\Omega_\infty(\overline{G})$.

Conversely, fix a point $z_0 \in G$, take a sequence (G_n) of Jordan domains converging to G with respect to z_0 (see Theorem 2.11) and the corresponding sequence (g_n) of conformal maps from G_n onto $\mathbb{D}$. Since all point in ∂G are simple, then g has a continuous extension $\tilde{g}$ to $\overline{G}$, and in view of Theorem 2.23 for a given $\varepsilon > 0$ there exists such n that $|\tilde{g}(z) - g_n(z)| < \varepsilon$ for all $z \in \overline{G}$. Since $g_n \in H(G_n)$, it follows from Runge's approximation theorem that there exists $P_n \in \mathcal{P}$ such that $\|P_n - g_n\|_{\overline{G}} < \varepsilon$. Then, $\|\tilde{g} - P_n\|_{\overline{G}} < 2\varepsilon$ as desired. ■

Furthermore, [44, Theorem IV] may be stated as follows.

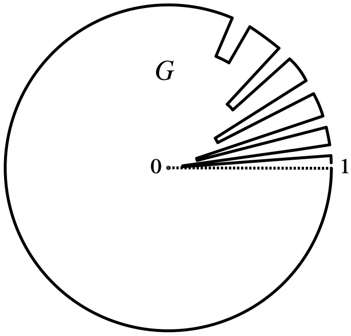

Figure 8. A counterexample to the opposite inclusion in (3.2).

Theorem 3.5. *Let G be a Carathéodory domain such that all points in ∂G are simple in the sense of Carathéodory, and let f be some conformal map from $\mathbb{D}$ onto G. Then,*

$$P(\overline{G}) \supset \{h \in A(\overline{G}) : h \text{ is constant on } I(\hat{f}(\zeta)) \text{ for each } \zeta \in \mathbb{T}\}. \tag{3.2}$$

Proof. Denote by B the set in the right-hand side of (3.2) and take $h \in B$. Put $F = h \circ f$. Then, F has a continuous extension $\widetilde{F}$ to $\overline{\mathbb{D}}$ because h is constant in each set $C(f, \zeta)$, $\zeta \in \mathbb{T}$. Then, $\widetilde{F} \in A(\overline{\mathbb{D}})$. Given $\varepsilon > 0$ let $P \in \mathcal{P}$ be such that $\|\widetilde{F} - P\|_{\overline{\mathbb{D}}} < \varepsilon$. Let $g = f^{-1}$, and let $\tilde{g}$ be the continuous extension of g to $\overline{G}$. Put $z = \tilde{g}(w)$ for $w \in \overline{G}$. Then,

$$|h(w) - P(\tilde{g}(w))| = |\widetilde{F}(z) - P(z)| < \varepsilon$$

for each $w \in \overline{G}$. Since $\tilde{g} \in P(\overline{G})$, then $P \circ \tilde{g} \in P(\overline{G})$. So, $h \in P(\overline{G})$. ∎

The opposite inclusion in (3.2) is not true in the general case. To construct a direct example, let us consider a sequence

$$1 > a_1 > b_1 > a_2 > b_2 > \cdots a_n > b_n > a_{n+1} > \cdots > 0$$

such that $a_n \to 0$, and define the domain G, see Figure 8, in such a way that

$$\overline{G} = \overline{\mathbb{D}} \Big\backslash \bigcup_{n=1}^{\infty} \Big\{ z = re^{it} : \frac{1}{n} < r \leqslant 1, t \in (b_n, a_n) \Big\}.$$

It is clear that the constructed domain G is a Carathéodory domain. The function $h(z) = \sqrt{1-z}$, defined on $\overline{G}$, belongs to $P(\overline{G})$, but it is not constant in $[0, 1]$ which is the impression of some prime end.

Remark[¶] 3.6. The set of the right-hand side of (3.2) seems to be very small in the case that f is not continuous on $\overline{\mathbb{D}}$. It looks quite plausible that this set is equal to the set of functions $F \circ f^{-1}$, where $F \in A(\overline{\mathbb{D}})$.

The following two theorems obtained by Lavrentiev [79] and Keldysh [72], respectively, turned out to be important milestones on the way of studying the problem of polynomial approximation on compact sets in the complex plane. In what follows they will be called *Lavrentiev's theorem* and *Keldysh's theorem*, respectively.

Theorem 3.7. *Let $K \subset \mathbb{C}$ be a compact set. Then, $P(K) = C(K)$ if and only if $K^\circ = \emptyset$ and $\mathbb{C} \setminus K$ is connected.*

Theorem 3.8. *Let $G \subset \mathbb{C}$ be a bounded domain. Then, $P(\overline{G}) = A(\overline{G})$ if and only if the set $\mathbb{C} \setminus \overline{G}$ is connected.*

Finally, the problem on characterization of such compact sets $K \subset \mathbb{C}$ for which it holds $P(K) = A(K)$ was completely solved by S. N. Mergelyan in 1952, see [90]. The following theorem summarize several Mergelyan's statements, it will be called *Mergelyan's theorem* in what follows.

Theorem 3.9. *Let $K \subset \mathbb{C}$ be a compact set.*

(1) *$P(K) = A(K)$ if and only if the set $\mathbb{C} \setminus K$ is connected.*

(2) *If $\mathbb{C} \setminus K$ has finitely many components, then $A(K) = R(K)$.*

(3) *Assume that there exists a decreasing sequence (δ_n) with $\delta_n \to 0$ such that for each point $b \in \partial K$ there exist an arc $\gamma_n \subset D(b, \delta_n) \cap K^{\complement}$ and a number $r_n > 0$ such that $\operatorname{diam}(\gamma_n) \geqslant r_n$. Let $f \in A(K)$ and let $\omega(f, \cdot)$ denotes its modulus of continuity. If*

$$\liminf_{n \to \infty} \omega(f, \delta_n) \left(\frac{\delta_n}{r_n} \right)^2 = 0, \tag{3.3}$$

then for every $\varepsilon > 0$ there exist $F \in \mathcal{R}$ with $\{F\}_\infty \subset K^{\complement}$ such that

$$\|f - F\|_K < \varepsilon.$$

Notice that the part (3) of Mergelyan's theorem yields that $R(K) = A(K)$ whenever all components of $\mathbb{C} \setminus K$ have diameter bigger than some given number $\delta > 0$.

Several proofs of Mergelyan's theorem may be found in the literature, see, for instance, [115, Chapter 20], [55, Chapter III], [32, Section 8.6], and [134, Appendix I]. Moreover, in [24] one can find the dual proof of this theorem, due to L. Carleson, see also [125, Chapter V].

Observe that using the ideas underlying the proof of Runge's theorem the statement of the part (3) of Theorem 3.9 may be improved in such a way that all poles of F can be chosen to belong to some prescribed set containing a point of each component of $\mathbb{C}_\infty \setminus K$.

Theorem 3.10. *Let K be a Carathèodory compact set, then $R(K) = A(K)$.*

Proof. For each $\delta > 0$ let $a \in \partial K = \partial\Omega_\infty(K)$. Then, take $a' \in \Omega_\infty(K)$ such that $|a - a'| < \delta/2$. Then, a' can be joined to ∞ by some infinite polygonal line L. The part of L that contains a' and ends in the first point, where L exists $\overline{D(a,\delta)}$ is an arc with diameter bigger than $\delta/2$. So, for each sequence (δ_n) in conditions of part (3) in Theorem 3.9 one can take $r_n = \delta/2$ and hence (3.3) holds for each function $f \in A(K)$. Then, $R(K) = A(K)$. ■

Corollary 3.11. *Let U be a Carathéodory open set, then $R(\partial U) = C(\partial U)$.*

The problem on characterization of those compact sets K for which it holds $R(K) = A(K)$ was solved in 1967 by A. G. Vitushkin in terms of the analytic capacity of the sets $D(a,r) \setminus K$ and $D(a,r) \setminus K^\circ$. We are not going to enter this topic, and we refer to [128] and [56, Chapter VIII] for the corresponding explanation. But one ought to pay attention to the following thing. For proving his result Vitushkin have proposed and elaborate the special approach to approximation, which is based on localization of singularities of the function being approximate, and further approximation of each localized functions. Using this approach one can obtain another proof of Theorem 3.10 without using Mergelyan's theorem. An example of the proof of such kind (in a different situation of approximation by polyanalytic rational functions) may be found in [28, Proposition 2.5]. In view of this it would be interesting to obtain the proof of Theorem 3.10 that avoids both the application of Mergelyan's theorem and Vitushkin's localization technique, at least in the case that $K = \overline{G}$ for a Carathéodory domain G.

3.2 Uniform harmonic approximation

An investigation of the problem on approximation of continuous functions by harmonic ones was started by Walsh in the 1920s. For an open set U let $\mathrm{Har}(U) = \mathrm{Har}(U,\mathbb{R})$ be the set of all real harmonic functions on U. Next, for a compact set $K \subset \mathbb{C}$ we denote by $\mathrm{Har}(K)$ the set of functions $u|_K$, where $u \in \mathrm{Har}(V)$ for some (depending on u) open set V containing K, and by $\overline{\mathrm{Har}(K)}$ the closure of $\mathrm{Har}(K)$ in $C(K)$. Then, $\overline{\mathrm{Har}(K)} \subset C(K) \cap \mathrm{Har}(K^\circ)$. By definition, a harmonic polynomial is $\mathrm{Re}\, P$, where $P \in \mathcal{P}$. For example the real polynomial $x^3 - 3xy^2 + x^2 + 2xy - y^2$ is harmonic, since it is a real part of $z^3 + (1-i)z^2$. Here, and in the sequel a real polynomial means a polynomial in two real variables x and y with real coefficients. A good reference for study of harmonic functions from the point of view of complex analysis is the book [107].

Let us also recall that a domain $G \subset \mathbb{C}$ is called n-connected, if the set $\mathbb{C}_\infty \setminus G$ has n components. A domain G is called finitely connected, if it is n-connected for some integer $n \geqslant 1$. Notice also that if G is a domain in $\mathbb{C}$, while K is some component of the set $\mathbb{C}_\infty \setminus G$ and K does not contain ∞, then K needs to be a compact subset of $\mathbb{C}$.

A bit of background about Dirichlet problem and harmonic measure

Let us recall some facts about harmonic functions, harmonic measure and the Dirichlet problem that we will use in what follows. Let U be a non-empty bounded open set of $\mathbb{C}$ and $f : \partial U \to \mathbb{R} \cup \{\pm\infty\}$ be an arbitrary function. Following the traditional terminology we will call such f a *boundary function*. Let us denote by $\overline{\mathcal{U}}_f$ the set of all functions h which are superharmonic or identically equal to $+\infty$ in each component of U with $\liminf_{y\to x} h(y) \geqslant f(x)$ for all $x \in \partial U$, and which are bounded from below on U. Furthermore, let $\overline{\mathcal{H}}_f$ be the function defined as follows: $\overline{\mathcal{H}}_f = \inf\{h : h \in \overline{\mathcal{U}}_f\}$. One says that $\overline{\mathcal{H}}_f$ is the upper solution of the generalized Dirichlet problems in U for the boundary function f. Next, similarly, one can define the set $\underline{\mathcal{U}}_f$ as the set of all functions h which are subharmonic or identically equals $-\infty$ in each component of U with $\limsup_{y\to x} h(y) \leqslant f(x)$ for all $x \in \partial U$, and bounded from above on U. Using this set we define the function $\underline{\mathcal{H}}_f := \sup\{h : h \in \underline{\mathcal{U}}_f\}$. Such function $\underline{\mathcal{H}}_f$ is called the lower solution of the generalized Dirichlet problem in U with the boundary function f. These definitions, as well as proofs of almost all results mention here in connection with Dirichlet problem may be found in [67, Chapter 8]. If $\overline{\mathcal{H}}_f = \underline{\mathcal{H}}_f$ and if both these functions are harmonic on U, then f is called a *resolutive boundary function*, while the function $\mathcal{H}_f = \overline{\mathcal{H}}_f = \underline{\mathcal{H}}_f$ is called the *solution of Dirichlet problem* with boundary function f (or, shortly, *Dirichlet solution* for f). The corresponding method to obtain a harmonic function from a boundary function f is called the *Perron–Wiener–Brelot* method.

Wiener's theorem says that any function $f \in C(\partial U)$ is a resolutive boundary function. Having this in mind we have the following statement, see [67, Lemma 8.12].

Lemma 3.12. *For $z \in U$ and $f \in C(\partial U)$ let $L_z(f) = \mathcal{H}_f(z)$. Then, L_z is a positive linear functional on the space $C(\partial U)$ and there exists a unique Borel probability measure μ_z on ∂U such that for all $z \in U$ and $f \in C(\partial U)$ it holds*

$$\mathcal{H}_f(z) = L_z(f) = \int f \, d\mu_z.$$

Moreover, we have (see [67, Theorem 8.14]).

Lemma 3.13. *If W is a component of U, then the class of Borel subsets of ∂U of μ_z-measure zero is independent of $z \in W$.*

Now, for $z \in U$ we define the set $\mathcal{F}_z$ of all sets having the form $(E \setminus N) \cup (N \setminus E)$ with $E \subset \partial U$ and $N \subset B$, where N and B are Borel sets such that $\mu_z(B) = 0$. Then, $\mathcal{F} := \bigcap_{z\in\partial U} \mathcal{F}_z$ is a σ-algebra containing all Borel subsets of ∂U, and the measure μ_z can be uniquely extended to $\mathcal{F}$.

Definition 3.14. The measure μ_z defined above is called the *harmonic measure* on ∂U relative to U and z, and it will be denoted in what follows by $\omega(z, \cdot, U)$.

Using Lemma 3.13 and the standard Radon–Nikodym theorem one can see that for every component W of U the measures $\omega(z_1, \cdot, U)$ and $\omega(z_2, \cdot, U)$ are mutually absolutely continuous for any points $z_1, z_2 \in W$. Moreover, the Radon–Nikodym derivative $h := d\omega(z_1, \cdot, U)/d\omega(z_2, \cdot, U)$ satisfies

$$\omega(z_1, \cdot, U) = h \cdot \omega(z_2, \cdot, U), \quad \text{and} \quad C^{-1} \leqslant |h(z)| \leqslant C \quad \text{for a.a. } z \in \partial W, \tag{3.4}$$

where $C > 0$ is some constant depending on z_1, z_2, W and U. Furthermore, $\omega(z, \cdot, U)$ has no atoms for each $z \in U$.

A keynote property of the harmonic measure is the following result.

Theorem 3.15. *Let U be a non-empty bounded open set. A boundary function f is resolutive if and only if it is $\omega(z, \cdot, U)$-integrable for some $z \in U$. If f is resolutive, then for all $z \in U$ it holds*

$$\mathcal{H}_f(z) = \int_{\partial U} f(\zeta)\, d\omega(z, \zeta, U).$$

To study the behavior of $\mathcal{H}_f(z)$ when $z \to \zeta \in \partial U$ we need the notion of regular point. Recall that a point $\zeta \in \partial U$ is said to be a *regular point*, if $\lim_{z\to\zeta} \mathcal{H}_f(z) = f(\zeta)$ for every function $f \in C(\partial U)$. A bounded set U is said to be *regular* (or *Dirichlet*) *open set*, if every point of ∂U is a regular one.

There are several sufficient conditions to conclude that a given point is regular, for example if there is a (half-opened) segment $[a, \zeta) \subset \mathbb{C} \setminus U$ with $\zeta \in \partial U$. However, the more useful condition is the following one given by A. Lebesgue.

Theorem 3.16. *Let $\zeta \in \partial U$ be such a boundary point that there exists a continuum $\mathcal{L}$ (consisting of more than one point) such that $\mathcal{L} \setminus \{\zeta\} \subset \mathbb{C} \setminus U$. Then, ζ is a regular point. In particular, if U is a simply connected set, then it is a Dirichlet open set.*

The proof of this theorem may be found in [33, Chapter X].

It follows from this theorem that any nonempty bounded open set $U \subset \mathbb{C}$ such that no component of ∂U reduces to a singleton is a Dirichlet set. For such open set U and for a function $f \in C(\partial U)$ let us define

$$\hat{f}(z) = \begin{cases} f(z) & \text{if } z \in \partial U, \\ \mathcal{H}_f(z) & \text{if } z \in U. \end{cases} \tag{3.5}$$

Then, $\hat{f} \in C(\overline{U}) \cap \operatorname{Har}(U)$. Moreover, for every $z \in U$ it holds

$$\hat{f}(z) = \int_{\partial U} f(\zeta)\, d\omega(z, \zeta, U). \tag{3.6}$$

In fact, we have the following corollary.

Corollary¶ 3.17. *Let U be a Carathéodory open set. Then, all points in ∂U are regular. So, U is a Dirichlet open set. Moreover, if B is a bounded connected component of $\mathbb{C} \setminus U$, then $\omega(z, \partial B, U) = 0$ for every $z \in U$.*

Uniform approximation by harmonic functions

Let us start with one suitable generalization of the fact, that the open connected set U is simply connected if and only if for every function $h \in H(U)$ there exists a sequence (P_n) of polynomials such that $P_n \rightrightarrows h$ locally in U. We have

Theorem 3.18. *Let G be a finitely connected domain, let E_j, $j = 1, \dots, N$, $N \geqslant 1$, are all bounded components of $\mathbb{C} \setminus G$, and let $a_j \in E_j$ for each $j = 1, \dots, N$. Then, any function $u \in \operatorname{Har}(G)$ can be uniquely expressed in G in the form*

$$u(z) = \operatorname{Re} h(z) + \sum_{j=1}^{N} c_j \log |z - a_j|, \quad z \in G, \tag{3.7}$$

where $h \in H(G)$ and $c_1, c_2, \dots, c_N$ are real numbers.

Furthermore, let $K \subset \mathbb{C}$ be a compact set, and let $G_1, G_2, \dots$ be all bounded components of the set $\mathbb{C} \setminus K$ (if exist). Let $a_j \in G_j$ for each j. Then, the set of functions of the form (3.7), where h runs over $R(K)$ and $c_j \in \mathbb{R}$, is dense in $\overline{\operatorname{Har}(K)}$. In particular, if $\mathbb{C} \setminus K$ is connected, then the harmonic polynomials are dense in $\overline{\operatorname{Har}(K)}$.

The first part of this theorem is a very classical result, it is known by the name of Logarithmic Conjugation theorem. However, it is not clear what is the most relevant reference to it prior to the paper [6], where one can find the history, the direct proof, and several consequences of this result. It seems that the first occurrence of the aforementioned result in the mathematical literature was in [132], but the assumption that the domain under consideration has analytic boundary was made therein.

The result of the second part of Theorem 3.18 is not a difficult fact, its detailed proof may be found in [18, Section 3.4]. Note, that this result can be proved using duality arguments as follows. Take a real valued measure μ on K which is orthogonal to the functions $\operatorname{Re} h$, $h \in R(K)$, and $\log |z - a_j|$ for all indices j. One can check that for the logarithmic potential of μ

$$\breve{\mu}(w) = \int \log |z - w| \, d\mu(z),$$

which is defined a.e. in $\mathbb{C}$, one has $\breve{\mu}(w) = 0$ for each $w \notin K$. This fact together with the formula

$$\int g \, d\mu = \frac{1}{2\pi} \int \Delta g \, \breve{\mu} \, dA,$$

which is valid for all compactly supported functions g of class C^2, implies that μ is orthogonal to $\overline{\mathrm{Har}(K)}$ (the symbol Δ stands, as usual, for the Laplace operator).

We are going now to proceed with the Walsh–Lebesgue theorem, which is one of the most famous and most important results about approximation of functions by harmonic polynomials. The name of Walsh–Lebesgue theorem is associated in the literature to several related results. In order to be more clear we present here three such results. The first one was proved in [132]. Later on L. Carleson in [24] made a new proof because he says that the original proof is not complete. Walsh repeatedly said in [132, 133] that his proofs are based on Lebesgue's important work [80]. This explains the reason why the name "Walsh–Lebesgue theorem" was subsequently adopted for the next Theorems 3.19, 3.21, 3.22, and 3.23.

Theorem 3.19 (Walsh–Lebesgue theorem; the first of such name). *Let $K \subset \mathbb{C}$ be a compact set with connected complement. Then, for every function $u \in C(\partial K, \mathbb{R})$ there exists a sequence (P_n) of harmonic polynomials such that $P_n \rightrightarrows u$ on ∂K.*

Scheme of the proof. Let (K_n) be a sequence of compact sets, each of which has a boundary consisting of a finite number of C^1-smooth Jordan curves, such that $K_{n+1} \subset K_n^\circ$ and

$$K = \bigcap_{n=1}^{\infty} K_n.$$

Each continuous function on ∂K can be approximated uniformly on ∂K by C^1-smooth functions. Then, one can assume that $u \in C^1(\mathbb{C})$. In each domain K_n° take u_n to be the solution of the Dirichlet problem with boundary data $u|_{\partial K_n}$. Each set K_n° is simply connected, then each function u_n is the real part of some holomorphic function f_n. Each of these functions f_n can be approximated by polynomials in view of Runge's theorem. The real part of these polynomials are harmonic polynomials, and they converge uniformly on ∂K to u_n. It remains to show that $u_n \rightrightarrows u$ on ∂K. This fact is a keynote point of the proof, and it is a consequence of the following lemma due to A. Lebesgue.

Lemma 3.20. *Let $K \subset \mathbb{C}$ be a compact set, and let (K_n) be such sequence of compact sets that ∂K_n consists of a finite number of smooth closed curves, $K_{n+1} \subset K_n^\circ$, and $\bigcap_{n=1}^{\infty} K_n = K$. Let $u \in C^1(\mathbb{C})$, and let u_n be the harmonic extension of $u|_{\partial K_n}$ to K_n°. If each $z \in \partial K$ satisfies the condition*

$$\int_S \frac{dr}{r} = +\infty, \tag{3.8}$$

where

$$S = \{r \in (0, +\infty) : \partial D(z, r) \cap K^{\complement} \neq \emptyset\},$$

then $u_n \rightrightarrows u$ on ∂K.

The detailed proof of this lemma may be found in [56, pages 35–36]. The condition (3.8) is called *Lebesgue's condition*. ■

Another proof of Theorem 3.19 was given in [24, pages 168–171]. This proof follows the pattern of the proof of the part (2) of Theorem 3.18.

Theorem 3.21 (Walsh–Lebesgue theorem; the second of such name). *Let $K \subset \mathbb{C}$ be a compact set such that the set $\mathcal{B}_K$ of all bounded components of $\mathbb{C} \setminus K$ is not empty. Let E be a set that contains one point for each $G \in \mathcal{B}_K$. Suppose that*

(a) *the set $\mathcal{B}_K$ is finite, and $u \in C(\partial K, \mathbb{R})$, or*

(b) *each component of K is finitely connected, and $u \in C(K, \mathbb{R}) \cap \mathrm{Har}(K^\circ)$.*

Then, u can be approximated uniformly on K by functions of the form (3.7) with such $h \in \mathcal{R}$ that all poles of h are inside E, the points $a_j \in E$ and $c_j \in \mathbb{R}$

Scheme of the proof. The proof of item (a) is given in [18, page 191] using the theory of representing measure for $R(K)$. For item (b) we follow the outline proposed by Walsh. Take a closed disc $\overline{D}$ with $K \subset \overline{D}$ and a continuous function u_0 defined on $\overline{D}$ that extends u. Then, there exists a real polynomial P that differs from u_0 by less than a given $\varepsilon > 0$. The next step is to construct a decreasing sequence (S_j) of closed sets, each of which is bounded by a finite number of non-intersecting Jordan polygonal lines (with wedges parallel to coordinate axis), such that $K = \bigcap_{j=1}^{\infty} S_j$. Let now h_j be the solution for the Dirichlet problem on $\mathrm{Int}(S_j)$ with the boundary function $P|_{\partial S_j}$. Then, $h_j \rightrightarrows P$ on ∂K. Then, take $k \in \mathbb{N}$ such that the difference between h_k and P is less than ε on ∂K. But h_k can be uniformly approximated on K by a function of such kind that were considered in Theorem 3.18 (for the points of E). It remains to modify this approximating function in such a way to settle its singularities to the given points in E. Then, the approximation is obtained on ∂K, but since $u \in C(K) \cap \mathrm{Har}(K^\circ)$, the approximation also holds on K. ■

Next result is stated in [133, page 518] and it is the oldest result were the notion of Carathéodory set plays a role. It can be proved using Theorem 3.19.

Theorem 3.22 (Walsh). *Let $G \subset \mathbb{C}$ be a bounded simply connected domain, and let K be a compact set in $\mathbb{C}$. Then, the following statements hold.*

(a) *Each function $u \in C(\overline{G}, \mathbb{R}) \cap \mathrm{Har}(\mathrm{Int}(\overline{G}))$ can be uniformly approximated on $\overline{G}$ by harmonic polynomials if and only if G is a Carathéodory domain.*

(b) *Each function $g \in C(K, \mathbb{R})$ can be uniformly approximated on K by harmonic polynomials if and only if K is a Carathéodory compact set and $K^\circ = \emptyset$.*

Proof. Let us prove the statement of part (a). For proving the statement of part (b) see the next theorem.

Assume that G is a Carathéodory domain, then $\operatorname{Int}(\overline{G}) = G$, and so, $u \in \operatorname{Har}(G)$. Put $K = \widehat{G}$. If the set $\mathbb{C} \setminus \overline{G}$ is connected then applying Theorem 3.19 we obtain a sequence (u_n) of harmonic polynomials such that $u_n \rightrightarrows u$ on $\partial K = \partial G = \partial \overline{G}$. Since u and all u_n are harmonic functions, the maximum modulus principle for subharmonic functions (see, for instance, [67, Theorem 7.10]) yields that this convergence is uniform on $\overline{G}$.

Suppose now that the set $\mathbb{C} \setminus \overline{G}$ is not connected. By Proposition 1.5, part (a), each bounded component G_1 of the set $\mathbb{C} \setminus \overline{G}$ is simply connected and $\partial G_1 \subset \partial G$. Then, we can solve, for each G_1, the Dirichlet problem with boundary values $u|_{\partial G_1}$. Therefore, one can define a function $\tilde{u}\colon K \to \mathbb{R}$ as $u(z)$ for $z \in \overline{G}$ and $\tilde{u}(z) = (u|_{\partial G_1})^\wedge(z)$ given by (3.5). The key point is that $K^\circ = G \cup \bigcup_j G_j$. So, $\tilde{u} \in C(K) \cap \operatorname{Har}(K^\circ)$. Since $\mathbb{C} \setminus K$ is connected and $\partial G = \partial K$, then there exists the sequence (u_n) of harmonic polynomials that converges uniformly on K to u, then in particular on $\overline{G}$.

Let now there exists $w \in \partial G$ such that $w \notin \partial G_\infty$. Take $r > 0$ such that $\overline{D(w, 2r)} \cap \partial G_\infty = \emptyset$. Let $\rho\colon \mathbb{C} \to \mathbb{R}$ be a continuous function such that $\rho \equiv 1$ on $\overline{D(w, r)}$ and $\operatorname{Supp}(\rho) \subset D(w, 2r)$. Consider as before the solution $\widehat{\rho}$ of the Dirichlet problem in G with boundary function $\rho|_{\partial G}$. If there exists such a sequence (u_n) that $u_n \rightrightarrows \rho$ on $\overline{G}$ then $u_n \rightrightarrows 0$ on ∂G_∞. Then, $u_n \rightrightarrows 0$ on $G_\infty^\complement$. In particular, $u_n(w) \to \rho(w) = 1$, which gives a contradiction. ∎

As a consequence, of the previous result we have the next theorem, which was not explicitly stated in [133]. Occasionally it is also referred as Walsh–Lebesgue theorem (see, for instance, [99, Section 1]) and nowadays it is this statement that is perceived by experts in the theory of approximation by analytical functions as the most complete and general form of the Walsh–Lebesgue theorem.

Theorem 3.23 (Walsh–Lebesgue theorem; the third of such name). *Let $K \subset \mathbb{C}$ be a compact set. Then, each function from the space $C(K) \cap \operatorname{Har}(K^\circ)$ can be approximated uniformly on K by harmonic polynomials if and only if K is a Carathéodory compact set.*

Proof. Let K be a Carathéodory compact set. The keynote ingredient here is Proposition 1.8, because one has $\operatorname{Int}(\widehat{K}) = \operatorname{Int}(K) \cup \bigcup_j G_j$ and $\partial G_j \subset \partial K = \partial \Omega_\infty(K)$. If $g \in C(K) \cap \operatorname{Har}(K^\circ)$, then we define the function $\hat{g}$ on $\widehat{K}$ in such a way that $\hat{g}(z) = g(z)$ if $z \in K$, while $\hat{g}(z)$ is the solution of the Dirichlet problem with boundary function $g|_{\partial G_j}$ for $z \in G_j$. Then, $\hat{g} \in A_h(\widehat{K})$ then the proof is finished as before, applying Theorem 3.19.

Going further assume that $\partial K \neq \partial \widehat{K}$. Then, take a disk $D(a, r) \subset \operatorname{Int}(\widehat{K})$ with $a \in \partial K$. Next, taking $b \in D(a, r) \setminus K$, let us consider the function $g \in C(K)$ such that $g(z) = \log|z - b|$ for each $z \in K$. Then, there exists a sequence (q_n) of harmonic polynomials such that $\|g - q_n\|_K \to 0$ as $n \to \infty$. Then, by the maximus modulus principle

$$\|q_n - q_m\|_{\widehat{K}} = \|q_n - q_m\|_{\partial K} \to 0$$

as $n, m \to \infty$. Then, (q_n) is a Cauchy sequence on $\widehat{K}$, then it converges uniformly on $\widehat{K}$ to the function $g_1 \in C(\widehat{K}) \cap \mathrm{Har}(\mathrm{Int}(\widehat{K}))$. But $g_1(z) = \log|z - b|$ if $z \in D(a, r)$ which is a contradiction. ■

Corollary 3.24. *Let K be a Carathéodory compact set. If $f \in C(K)$, then there exists a unique $u \in C(\widehat{K}) \cap \mathrm{Har}(\mathrm{Int}(\widehat{K}))$ such that $u(z) = f(z)$ for each $z \in K$.*

We must mention here that in [133] the condition in the part (b) of Theorem 3.21 stated by Walsh is different. He stated that "The compact K contains no region of infinite connectivity not included in a larger region of finite connectivity belonging to K. Then, in particular if K has no interior points, an arbitrary function $f(x, y)$ continuous on K can be so approximated". But the result with this formulation is not true, as one can see using the Deny's criterion for uniform approximation by functions harmonic in a neighborhood of K, see [36].

Ending our discussion on Walsh–Lebesgue theorem, let us mention the papers [95–97], where several interesting generalizations of this theorem were obtained in the situation when one deals with an approximation on boundaries ∂X of compact sets X in $\mathbb{C}$ with connected complement by functions of the form $P(\psi_1) + Q(\psi_2)$, where P and Q are polynomials in the complex variable, and ψ_1 and ψ_2 are two homeomorphisms of $\mathbb{C}$ to $\mathbb{C}$.

3.3 Pointwise polynomial approximation

Let us revert to the topic of approximation of functions by polynomials in the complex variable. We have seen in Theorem 3.2 that the locally uniform convergence of sequences of polynomials for each holomorphic function in a given open set implies certain topological restrictions on this set. But what can happen if we only suppose the pointwise convergence instead of the locally uniform one? Of course, the answer will depend on certain additional assumptions (such as, for instance, a boundedness of the corresponding sequence of approximating polynomials). In the most general case, when we did not demand anything else, the answer to this question was obtained by Montel: *For any open set $U \subset \mathbb{C}$ each function $f \in H(U)$ can be approximated by some sequence (P_n) of complex polynomials in such a way that $P_n(z) \to f(z)$ for every $z \in U$.* The proof of this fact may be obtained as follows. Let us take first some sequence (Y_n) of compact sets such that $\mathbb{C} \setminus Y_n$ is a connected set, $Y_n \subset U$, and $U = \liminf_{n\to\infty} Y_n$. A possible way to construct such sequence may be found in [85, Chaper IV, Section 2.3]. Next, Runge's theorem yields that for each n there exists a polynomial P_n such that $\|p_n - f\|_{Y_n} < 1/n$. Thus, the sequence (P_n) is as demanded. Notice that in such general setting we cannot conclude that (P_n) tends to f locally uniformly in U.

The most deep and important case of the aforesaid question arises when we assume that the function under approximation is bounded, and demand to approximate it by bounded sequence of polynomials. In this situation the picture changes completely. It became clear after works [45, 46] by O. J. Farrell in 1934–1935, where he proved the next Theorem 3.25. It is necessary to read simultaneously both papers to obtain the proof. However, these important papers are rarely mentioned in forthcoming works in the topic under consideration, so it causes errors in the attribution of who and what actually proved, see, for instance, [106, 112]. Farrell also mentioned that certain ideas of Carleman (see [23]) were of utility to prove both Theorems 3.25 and 4.1 below.

Theorem 3.25 (Farrell). *Let $G \neq \emptyset$ be a simply connected domain in $\mathbb{C}$. The following conditions are equivalent.*

(a) *For every function $f \in H^\infty(G)$ there exist a sequence of polynomials (P_n) such that $P_n(z) \to f(z)$ for each $z \in G$, and* $\limsup_{n\to\infty} \|P_n\|_G \leqslant \|f\|_G$.

(b) *G is a Carathéodory domain.*

Proof. Let G be a simply connected domain and put $T = \partial G_\infty$. Consider Q to be such component of $\mathbb{C} \setminus T$ that $G \subset Q$. Let f be some fixed conformal map from G onto $\mathbb{D}$. If the approximation properties stated in the part (a) holds, then there exists a sequence (P_n) of polynomials such that $P_n(z) \to f(z)$ and $|P_n(z)| \leqslant 2$ for each $z \in G$ and for each $n \in \mathbb{N}$ large enough. Then, $|P_m(z)| \leqslant 2$ for all $z \in \overline{G}$ and for some $m \in \mathbb{N}$, and $P_m \neq 0$, so that G needs to be bounded. Since $\partial Q \subset \overline{G}$, Montel's theorem (on the characterization of compact subsets of $H(G)$) shows that there exist a partial subsequence (P_{n_k}) and a holomorphic function $f_0\colon Q \to \mathbb{C}$ such that $P_{n_k}(z) \to f_0(z)$ for each $z \in Q$. Therefore, $f_0 = f$ in G, so that f_0 is non-constant. Moreover, since $Q \subset \widehat{G}$, then

$$|f_0(z)| = \limsup |P_{n_k}(z)| \leqslant \limsup \|P_{n_k}\|_Q \leqslant \limsup \|P_{n_k}\|_{\widehat{G}} \leqslant 1,$$

for each $z \in Q$. If we assume that G is not a Carathéodory domain, then $\partial G \setminus T \neq \emptyset$. Then, there exist $b \in \partial G \setminus T$ which is an accessible point from G by some end-cut $\mathcal{E}$ and there exists $\varepsilon > 0$ such that $D(b, \varepsilon) \subset Q$. Then,

$$|f_0(b)| = \lim_{\mathcal{E} \ni z \to b} |f_0(z)| = \lim_{z \to b} |f(z)| = 1,$$

but this is a contradiction since $|f_0|$ cannot achieve at the point b its maximum modulus over Q. Thus, the implication (a)$\Rightarrow$(b) is proved.

We are going now to prove the inverse implication. Take a function $f \in H^\infty(G)$. Fix $z_0 \in G$. Let us take, as usual, the sequence of Jordan curves (J_n) such that $D(J_n) \to G$ with respect to z_0 (in the sense of kernel convergence). Let $K = \widehat{G}$. Since $D(J_n)$ is a Jordan domain then $K \subset D(J_n)$ for all $n \geqslant 1$. Let us take the conformal maps φ_n from $D(J_n)$ onto $\mathbb{D}$ and the conformal map φ from $\mathbb{D}$ onto G such that $\varphi_n(z_0) = 0$ and $\varphi_n'(z_0) > 0$, while $\varphi(0) = z_0$ and $\varphi'(0) > 0$. Put $g_n = \varphi \circ \varphi_n$. Then, the function $f \circ g_n$ is holomorphic on $D(J_n)$, that is in an open neighborhood of K. Applying Runge's theorem, one can find a sequence of polynomials (P_n), such that

$$\|f \circ g_n - P_n\|_K < \frac{1}{n}. \tag{3.9}$$

From (3.9) it follows that $\|P_n\|_K \leqslant \frac{1}{n} + \|f\|_K = \frac{1}{n} + \|f\|_G$, which gives the conclusion of the theorem.

Notice that $g_n \rightrightarrows z$ in G. If $Y \subset G$ is a compact set then, for big enough n, one has

$$|f(g_n(z)) - f(z)| < \frac{1}{n}, \quad z \in Y. \tag{3.10}$$

From (3.9) and (3.10) we obtain that (P_n) converges uniformly on compact subsets of G to f. This implies the pointwise convergence in G. ∎

Paying more attention into the proofs given above and doing a bit more, the following result can be obtained.

Corollary¶ 3.26. *Let G be a simply connected domain in $\mathbb{C}$.*

(a) *Let f map G conformally onto $\mathbb{D}$. Assume that there exists a sequence of polynomials (P_n) such that*

$$\sup_{n\in\mathbb{N}} \|P_n\|_G \leqslant C \quad \text{and} \quad \lim_{n\to\infty} P_n(z) = f(z),\ z \in G, \tag{3.11}$$

for some constant C. Then, G is a Carathéodory domain.

(b) *Conversely, if G is a Carathéodory domain, then each function $h \in H^\infty(G)$ can be approximate by a sequence of polynomials (P_n) satisfying (3.11). In particular, if f is a conformal map from G onto $\mathbb{D}$, one can take the corresponding sequence in such a way that $C = 1$ in (3.11).*

Proof. Let us start with the part (a). We will use all notations introduced in the proof of the implication (a) $\Rightarrow$ (b) in Theorem 3.25. So, we take the partial sequence (P_{n_k}) and the function f_0 such that $P_{n_k} \rightrightarrows f_0$ on Q. Assume that $\partial G \setminus T \neq \emptyset$, then for each accessible point $w \in \partial G \setminus T$ we know that $|f_0(w)| = 1$. By continuity this is true for all $w \in \partial G \setminus T$. Then, take $b_1 \in \partial G \setminus T$ such that $f_0'(b_1) \neq 0$. Therefore, is it to possible to find $\varepsilon > 0$ and a small closed disk W such that

$$W \subset D(b_1, \varepsilon) \cap G \subset \overline{D(b_1, \varepsilon)} \cap G \subset Q$$

and f_0 is a holomorphic homeomorphism from W onto its image, so that $1 + \varepsilon < |f_0(z)|$ for all $z \in W$. Since the sequence (P_{n_k}) converges uniformly on W, then

$$1 + \frac{\varepsilon}{2} \leqslant |P_{n_k}(z)|, \quad z \in W,\ k \geqslant k_0.$$

Taking limits when $k \to \infty$ the previous estimate yields that there exist many points z, where $|f(z)| \geqslant 1 + \varepsilon/2$, which is a contradiction.

It remains to prove last assertion in the part (b). Let f be the conformal map from G onto $\mathbb{D}$. Since $f \circ g_n = \varphi_n$ is holomorphic in $D(J_n)$, then $\|\varphi_n\|_{\widehat{G}} \leqslant c_n < 1$. Then, take the corresponding polynomial P_n in such a way that

$$\|\varphi_n - P_n\|_{\widehat{G}} < 1 - c_n,$$

for every $N \geqslant 1$. Thus, $\|P_n\|_{\widehat{G}} \leqslant 1$. ■

This result for $C = 1$ is covered by the original proof in [45, 46], and it is [106, Theorem 2]. The author of the paper [106] and, highly likely, its referee were unaware that the respective result already has been proved 60 years prior to the publication of that paper.

Similar arguments can be used to prove the following result.

Proposition¶ 3.27. *Let G be a simply connected domain in $\mathbb{C}$. Assume that there is a subset $E \subset \partial G$ such that $\overline{E} = \partial G$ and for each point $a \in E$ the function $f(z) = \sqrt{z-a}$ can be boundedly approximated on G by a sequence of polynomials. Then, G is a Carathéodory domain.*

In [47] Farrell gave the estimate of the norm $\|f - p_n\|_G$ in terms of one special metrical concept. For a given domain G and a function $f \in H^\infty(G)$ let

$$D(f, \partial G) = \sup_{z \in \partial G} \operatorname{diam} C(f, z),$$

where $C(f, z)$ is the cluster set of f at the point z.

Theorem 3.28. *Let G be a Jordan domain and $f \in H^\infty(G)$. Then, there exists a sequence (P_n) of polynomials such that $P_n \rightrightarrows f$ in G and*

$$\limsup_{n \to \infty} \|f - P_n\|_G \leqslant D(f, \partial G). \tag{3.12}$$

Sketch of the proof. Fix $z_0 \in G$ and let (G_n) be the usual sequence of simply connected domains such that $G_n \to G$ with respect to z_0. Take the conformal map $\psi_n \colon G_n \to G$ such that $\psi(z_0) = z_0$ and $\psi'(z_0) > 0$. For the function f_n defined by the formula $f_n(z) = f(\psi_n(z))$ one can find an appropriate polynomial P_n in such a way that

$$|P_n(z) - f_n(z)| \leqslant \frac{1}{n}$$

for all $z \in \overline{G}$. Since $\psi_n \rightrightarrows z$ on $\overline{G}$ in view of Theorem 2.10, we have $P_n \rightrightarrows f$ locally in G.

The estimate (3.12) is obtained as a consequence of the following fact. If $w_0 \in \partial G$, if (z_n) is any sequence tending to w_0, and if (P_{k_n}) is a suitable subsequence of (P_n), then

$$\limsup_{n\to\infty} |f(z_n) - P_{k_n}(z_n)| \leqslant \operatorname{diam} C(f, w_0).$$

The omitted details may be found in [47]. ∎

Question III. Whether it is true, that if G is a Carathéodory domain such that $\mathbb{C} \setminus \overline{G}$ is connected and $f \in H^\infty(G)$, then there exists a sequence (P_n) of polynomials such that $P_n \rightrightarrows f$ in G and

$$\limsup_{n\to\infty} \|f - P_n\|_G \leqslant D(f, \partial G).$$

It seems that the answer is affirmative, but the proof given in the case of Jordan domains cannot be adapted directly.

Continuing the analysis of Farrell's results let us observe that the Carathéodory hull U^* of an open set U can be defined as follows:

$$U^* = \operatorname{Int}\big\{z_0 : |p(z_0)| \leqslant \sup_{z\in U} |p(z)|, \text{ for each } p \in \mathcal{P}\big\}.$$

As far as we know, the first occurrence of a related notion to Carthéodory hull (without the corresponding name) was in Theorem D in Farrell's work [46]. In this paper it was considered the component of the Carathéodory hull of a given domain that contains this domain itself. The concept of a Carathéodory hull of a set has appeared with this name in [120]. In [31] the set U^* was called the outer envelope of U. In [111, 112] this concept also appeared without name. Perhaps the name of "extended Carathéodory–Farrell hull" of U will be more honest and appropriated because if G is a Carathéodory domain, then G is only a component of a (sometimes) bigger open set G^*. However, in order to avoid a new name creation, the name of a Carathéodory hull is enough good and has been adopted to denote this set. Let us also note that the notation U^* for the Carathéodory hull of U coincides with the notation of [98], although in that paper it is not given a special name for this object.

The following properties are interesting and easy to prove (recall Proposition 1.5, see also [112]).

Lemma 3.29. *Let G be a bounded open set in $\mathbb{C}$. Then, the following hold.*

(i) $G^* = \mathbb{C} \setminus \overline{\Omega_\infty(\overline{G})}$, $\mathbb{C} \setminus \widehat{G} = \Omega_\infty(\overline{G})$, *and* $\partial G^* = \partial\Omega_\infty(\overline{G})$.

(ii) G^* *is a Carathéodory open set and* $(G^*)^* = G^*$.

(iii) *The set* G^* *is simply connected.*

The next lemma clarifies the usefulness of the concept given in Definition 1.4.

Lemma 3.30. *Let G be a bounded open set, let $f \in H(G)$, and let (P_n) be such sequence of polynomials that*

$$\sup_{n\in\mathbb{N}} \|P_n\|_G \leqslant C \quad \text{and} \quad P_n(z) \to f(z), \text{ for all } z \in G \tag{3.13}$$

for some constant C. Then, the following hold.

(a) $P_n \rightrightarrows f$ *locally in G as $n \to \infty$.*

(b) *There exists a function $f^* \in H(G^*)$ such that $f^*|_G = f$, that is f^* extends f to G^*.*

Proof. (a) Take a partial sequence (P_{n_k}) of the sequence (P_n). By Montel's theorem there exists a new partial sequence $(P_{n'_k})$ of this subsequence (P_{n_k}) such that $P_{n'_k} \rightrightarrows g$ locally in G for some function $g \in H(G)$. But $g(z) = f(z)$ in each component of G, then $g = f$ on G, and so, $P_{n'_k} \rightrightarrows f$ on G. Since it is true for all partial sequences of (P_n), the proof is completed.

(b) Let us observe that (3.13) together with the maximum modulus principle implies that $\|P_n\|_{\widehat{G}} = \|P_n\|_{\overline{G}} \leqslant C$ for all n. Then, there exists a partial sequence (P_{n_k}) such that $P_{n_k} \rightrightarrows f^*$ locally in G^* for some function f^* holomorphic on G^*. Since $G \subset G^*$, then f^* is an extension of f. Notice, that such extension in not unique in a general case. ∎

The final result by Farrell can be stated as follows.

Theorem 3.31. *Let $G \subset \mathbb{C}$ be a domain, and let $f \in H^\infty(G)$. The following conditions are equivalent.*

(a) *There exist a sequence of polynomials (P_n) such that (3.13) is satisfied.*

(b) *The function f is the restriction of some function belonging to $H^\infty(G^*)$.*

Near thirty years after publication of the above results, L. Rubel and A. Shields in [111, 112] obtained their generalization for a general bounded open sets. The following result is called nowadays *Farrell–Rubel–Shields theorem.*

Theorem 3.32. *Let $U \neq \emptyset$ be a bounded open subset of $\mathbb{C}$, and $f \in H^\infty(U)$. The following conditions are equivalent.*

(a) *There exists a sequence of (P_n), $P_n \in \mathcal{P}$, such that* $\sup_n \|P_n\|_U \leqslant \|f\|_U$ *and $P_n(z) \to f(z)$ for all $z \in U$.*

(b) *There exists such function $f^* \in H^\infty(U^*)$ that $f = f^*|_U$.*

The case that the set U is connected corresponds to the original Farrell's proof. There are two key points that distinguish the Rubel and Shields results from Farrell's ideas. The first one is the following thing. If U is an open set, then U^* is a Carathéodory open set, and hence each function $f \in H^\infty(U^*)$ can be bounded pointwise approximated by polynomials in U^*, but not only in U. The sequences of polynomials constructed in Farrell's proof cannot give directly the convergence in U^*. The second key point is related with the following observation. If U has infinitely many components G_j, then each $f|_{G_j}$ can be approximated by a sequence of polynomials $(P_{j,n})$. However, each of such sequence depends on j and it is not clear how to deal with all sequences. Rubel and Shields gave a clever idea how to avoid simultaneous work with several components of U.

Nowadays a proof of Theorem 3.32 using many important tools from the theory of uniform algebras consist in proving an abstract version of such theorem. From this abstract version the following result may be obtained which also gives Theorem 3.32 (the details of these proofs may be found in [56, pages 152–154]).

Theorem 3.33. *Let K be a finitely connected compact set in $\mathbb{C}$, and let $f \in H^\infty(K^\circ)$. Then, there is a sequence (f_n), $f_n \in R(K)$, such that $\sup_n \|f_n\|_K \leqslant \|f\|_\infty$ and $f_n(z) \to f(z)$ for all $z \in K^\circ$.*

Now, we will describe the pattern of the proof of Rubel–Shields theorem. We need to introduce yet one auxiliary construction.

Definition 3.34. Let U be a Carathéodory open set and let B be a component of U. The cluster $\mathcal{K}(B)$ is defined as the union of all components Q of U for which $Q \subset E_B$, where E_B is the component of $\overline{U}$ that contains B.

In order to illustrate this definition let us consider the outer snake (or cornucopia) Q_1 twisting around $\mathbb{D}$ with $Q_1 \subset D(0, 3/2)$ and another outer snake Q_2 with $Q_2 \subset D(3, 1)$; for example of the model for such Q_1 and Q_2 see G_1 on Figure 2. Take $U = \mathbb{D} \cup Q_1 \cup Q_2$. Then, $\mathcal{K}(\mathbb{D}) = \mathcal{K}(Q_1) = \mathbb{D} \cup Q_1$ and $\mathcal{K}(Q_2) = Q_2$.

The next result corresponds to Theorem 2.11 in the case of general Carathéodory open sets. It may be found in [112].

Theorem 3.35. *Suppose U be an open set.*

(a) *Let U be a Carathéodory open set. For each component B of U take a point $w_B \in B$. Then, there exists a sequence (U_n) of bounded simply connected open sets possessing the following properties:*

  (i) *$\overline{U} \subset U_n \subset \overline{U}_n \subset U_{n-1}$, $n \geqslant 2$;*

  (ii) *If B is any component of U and if B_n is the component of U_n containing B, then $\overline{B} \subset B_n \subset \overline{B}_n \subset B_{n-1}$, $n \geqslant 2$, and $B_n \to B$ with respect to w_B.*

(b) *If U is an open set such that there exists some sequence of open sets (U_n) satisfying the properties* (i) *and* (ii)*, then U is a Carathéodory open set.*

Notice, that in the frameworks of conditions of this theorem one has $\mathcal{K}(B) \subset B_n$ for every $n \in \mathbb{N}$.

The following lemma is one of key ingredients of the proof of Theorem 3.32.

Lemma 3.36. *Let E be a finite subset of U and let B be a component of U. Assume that $f = 1$ in all the other components of U and $\|f\|_U \leqslant 1$. Then, for each given $\varepsilon > 0$ there exists a polynomial P such that $|P(z)| \leqslant 1$ for each $z \in U$ and $|f(z) - P(z)| < \varepsilon$ for each $z \in E$.*

Sketched proof of Theorem 3.32. We need to prove that if U is a Carathéodory open set and $f \in H^\infty(U)$ with $\|f\|_U \leqslant 1$, then there exists a sequence of polynomials, uniformly bounded by 1 in U, and converging to f at each point of U. Let us assume that Lemma 3.36 is already proved.

Denote by $C_1, C_2, \dots$ some enumeration of all components of U, take a countable dense set $\{z_1, z_2, \dots\} \subset U$ and put $E_n = \{z_1, z_2, \dots, z_n\}$. Define the functions g_k, $k \in \mathbb{N}$, in such a way that $g_k(z) = f(z)$ for $z \in C_k$ and $g_k = 1$ in $U \setminus C_k$. Take (and fix) some $\varepsilon > 0$ and $n \in \mathbb{N}$. By Lemma 3.36 for every $k \in \mathbb{N}$ there exists a polynomial P_k such that $\|P_k\|_U \leqslant 1$ and $|g_k(z) - P_k(z)| < \varepsilon/n$ for each $z \in E_n$. Let now $f_n = g_1 g_2 \cdots g_n$ so that $f_n = f$ on $C_1 \cup C_2 \cup \cdots \cup C_n$, while $f_n = 1$ on each C_k with $k > n$. For the polynomial $\widetilde{P}_n = P_1 P_2 \cdots P_n$ we have $\|\widetilde{P}_n\| \leqslant 1$ in U and

$$f_n - \widetilde{P}_n = \sum_{j=1}^{n} P_1 \cdots P_{j-1} \cdot (g_j - P_j) \cdot g_{j+1} \cdots g_n,$$

which gives

$$|f_n(z) - \widetilde{P}_n(z)| < \varepsilon \quad \text{for each } z \in E_n.$$

Then, by Montel's theorem, each partial subsequence of $(\widetilde{P}_n)$ converges to a function h such that $f = h$ on E, so $\widetilde{P}_n \rightrightarrows f$ in U. ∎

It remains now to prove Lemma 3.36. To do this it is sufficient (in view of Runge's theorem) to verify the next statement.

Lemma 3.37. *Let E a finite subset of U and let B be a component of U. Assume that $f = 1$ in all other components of U and $\|f\|_U \leqslant 1$. Then, for any $\varepsilon > 0$ there exists a simply connected domain Q with $\overline{U} \subset Q$, and a holomorphic function g in Q with $\|g\|_Q \leqslant 1$ such that $|f - g| \leqslant \varepsilon$ on E.*

Let B be the component mentioned in Lemma 3.36. According to Theorem 3.35 one can take (U_n), (B_n), (φ_n) and φ, where φ_n is the conformal map from B_n onto $\mathbb{D}$ normalized at some point $w_B \in B$ as $\varphi_n(w_B) = 0$, and φ is the conformal map from B onto $\mathbb{D}$ normalized by the same way. Passing to an appropriate subsequence of (φ_n), we obtain that $\varphi_n \rightrightarrows \psi$ in $\mathcal{K}(B)$, where $\psi = \varphi$ in B. Let now $\{Q_j\}$ be the collection of all components that formed $\mathcal{K}(B)$. It holds that $\psi = \zeta_j$ with $|\zeta_j| = 1$ in Q_j for all indices j. Since E is finite, it meets only finite number of components of $\mathcal{K}(B)$, says for definiteness, $Q_1, \dots, Q_n$. Put $E' = \varphi(E \cap B) \subset \mathbb{D}$, so that E' is a finite set. Consider the function $F = f \circ \varphi^{-1}$ such that $\|F\|_{\mathbb{D}} \leqslant 1$. Using [112, Lemma 3.13] one can find a new function F_1 which is close to F on E', while it is close to 1 near the points $\zeta_1, \dots, \zeta_n$. Finally, for sufficiently large n the function g defined in such a way that $g = F_1 \circ \varphi_n$ in B_n and $g = 1$ in $U_n \setminus B_n$ is the desired approximant for f in Lemma 3.37. All omitted technical details may be found in [112, Lemmas 3.11, 3.12, and 3.13].

Example 3.38. Let G be the outer cornucopia, and $U = G^* = G \cup \mathbb{D}$. Then, there exists a sequence (P_n) of polynomials, uniformly bounded by 1 such that $P_n(z) \to 0$ if $z \in G$ and $P_n(z) \to 1$ if $z \in \mathbb{D}$.

The next statement is an application of Rubel–Shields theorem. But we encourage the interested reader to find a proof using only Farrell's ideas, as well as the another one basing only on Runge's theorem.

Corollary 3.39. *Let G be a Carathéodory domain and let $f \in H^\infty(G)$. Then, there exists a sequence of polynomials (P_n) such that $P_n \rightrightarrows f$ locally in G and for each bounded component B of $\mathbb{C} \setminus \overline{G}$ one has $P_n'(z) \to 0$ for each $z \in B$.*

We end this section mentioning several interesting and important concepts related with the topic on bounded pointwise approximation. The first one is the concept of a Farrell set, which was introduced by Rubel and studied, for example, in [126]. Later on, O'Farrell and Perez–Gonzalez defined Farrell pairs for general open sets and the notion of a Farrell–Rubel–Shields set. Notice that the family of Farrell–Rubel–Shields sets includes the family of Carathéodory domains. The paper [98] gives a comprehensive theorem on pointwise bounded-on-a-subset approximation for Farrell–Rubel–Shields sets.

3.4 Uniform algebras on Carathéodory sets

We start this section by mentioning some connections between the Walsh–Lebesgue theorem and the theory of uniform algebras. We recall some notions of that theory, whose exhaustive exposition may be found in [18, 56, 69, 125].

A uniform algebra $\mathcal{A}$ on a compact Hausdorff space X is a uniformly closed (with respect to the norm $\|f\| = \sup\{|f(x)| : x \in X\}$) subalgebra of $C(X)$ which contains constants and separates points of X. A set $E \subset X$ is called a boundary for $\mathcal{A}$ if for each $f \in \mathcal{A}$ there exists $y \in E$ such that $|f(y)| = \|f\|$. The minimum closed boundary of $\mathcal{A}$ (which always exists) is called the Shilov boundary of $\mathcal{A}$. A subset $F \subset X$ is called a peak set for $\mathcal{A}$ if there exists a function $f \in \mathcal{A}$ such that $\|f\| = 1$ and $F = f^{-1}(1)$. A point $x \in X$ is a peak point of $\mathcal{A}$ if $\{x\}$ is a peak set.

If $\mathcal{A}$ is a uniform algebra on a compact space X, the maximal ideal space of $\mathcal{A}$ can be identified with the space of non-zero complex-valued homomorphism of $\mathcal{A}$, which will be denoted by $M_{\mathcal{A}}$. If $\Psi \in M_{\mathcal{A}}$, then Ψ is continuous and $\|\Psi\| = 1 = \Psi(1)$. Moreover, there exists a probability measure μ on X such that $\Psi(f) = \int_X f\, d\mu$ for each $f \in \mathcal{A}$. This measure is call a representing measure for Ψ. The set of such measures is convex and weak-star compact but, in general, is not a singleton. The Choquet boundary of $\mathcal{A}$ is the set of all those $x \in X$ for which the evaluation functional $\tau_x(f) = f(x)$ has a unique representing measure, of course it is needed to be the unit point mass δ_x supported at the point x. Moreover, if X is a metrizable space, the Choquet boundary of $\mathcal{A}$ is also the set of all peak points of $\mathcal{A}$. It can be proved that it is a boundary for $\mathcal{A}$ and its closure coincides with the Shilov boundary.

Recall that $\mathcal{A}$ is called a Dirichlet algebra on X, if $\operatorname{Re} \mathcal{A}$ is dense in $C(X, \mathbb{R})$, while $\mathcal{A}$ is called a logmodular algebra on X, if

$$\left\{\log|f| : f \text{ is an invertible element of } \mathcal{A}\right\}$$

is dense in $C(X, \mathbb{R})$.

Let K be a compact subset of $\mathbb{C}$. We are going to discuss here several results related to [39]. For better understanding of the matter we emphasize the following facts.

(1) Let $g \in P(K)$. Then, there exists a sequence of polynomials that converges uniformly to g. By the maximum modules theorem this sequence also converges uniformly on $\widehat{K}$ to an extension $\hat{g} \in P(\widehat{K})$ of g which has the same norm. The isometry $g \mapsto \hat{g}$ allow us to identify $P(K)$ with $P(\widehat{K})$ or even with $P(\partial K)$. These identifications will be used in what follows without explicit reference.

(2) Returning to the algebras appearing in (3.1) let us note that the maximal ideal spaces for all of them are identified with K. Moreover, the Shilov boundaries for $P(K)$, $R(K)$, $A(K)$ and $C(K)$ are $\partial\widehat{K}$, ∂K, ∂K and K, respectively. For $P(K)$ and $C(K)$ the Choquet boundaries coincide with their Shilov boundaries, but for $R(K)$ and $A(K)$ the Choquet boundaries are more involved (see [56, page 205]).

For better understanding the next Proposition, we prove that the Choquet boundary of $P(K)$ is $\partial\widehat{K}$. First note that if x is a peak point of $P(K)$, then $x \in \partial\widehat{K}$. Let $x \in \partial\widehat{K}$ and let μ be a representing measure of τ_x. Since μ is real, then $\operatorname{Re} g(x) = \int \operatorname{Re} g\, d\mu$ for each $g \in P(K)$. Because $P(K)$ is a Dirichlet algebra, then $r(x) = \int_{\widehat{K}} r(y)\, d\mu(y)$ for each continuous function $r \in C_{\mathbb{R}}(\partial\widehat{K})$. It means that μ is also a representing measure of τ_x for the algebra $C_{\mathbb{R}}(\partial\widehat{K})$, so $\mu = \delta_x$.

If $\mathcal{A}$ is a Dirichlet algebra on X, then $\mathcal{A}$ is also a logmodular algebra on X, and X is the Shilov boundary of $\mathcal{A}$.

Theorem 3.23 tell us that $P(K)$ is a Dirichlet algebra on $\partial\widehat{K}$.

In view of the aforesaid, all ingredients are readily available to obtain the following statement which is worth comparing with [39, Theorem 4].

Proposition 3.40. *Let K be a compact set in $\mathbb{C}$, and let $\Gamma = \partial K$. The following conditions are equivalent.*

(a) *K is a Carathéodory compact set.*

(b) *The Choquet boundary of $P(K)$ is Γ.*

(c) *The Shilov boundary of $P(K)$ is Γ.*

(d) *$P(K)$ is a Dirichlet algebra on Γ.*

(e) *$P(K)$ is a logmodular algebra on Γ.*

(f) *Each point of Γ is a peak point for $P(K)$.*

Now, we are interested in the question about maximal subalgebras. We recall the concept of maximality in the theory of uniform algebras. Let K be a compact subset of $\mathbb{C}$. A closed subalgebra $\mathcal{A}$ of the algebra $C(K)$ is called *maximal* if for each closed subalgebra $\mathcal{B}$ of $C(K)$ such that $\mathcal{A} \subset \mathcal{B}$ it holds either $\mathcal{B} = \mathcal{A}$ or $\mathcal{B} = C(K)$. In [56, page 38] it is assumed that $\mathcal{A} \neq C(K)$, but seems more appropriate not to use this convention.

The question on maximality of the algebra $P(K)$, where K is a compact subset of $\mathbb{C}$, was initiated by J. Wermer, who proved that $A(\overline{\mathbb{D}})$ is maximal, considering as a uniform algebra on its Shilov boundary, $\mathbb{T}$, or with more generality for every closed subalgebra of $C(\mathbb{T})$ that contains an injective function. This result is known as Wermer's maximality theorem, see the first proof of it in [135]. Later on E. Bishop [15] (see Theorem 6 of the cited work) established the following result.

Theorem 3.41. *Let K be a compact subset of $\mathbb{C}$ such that both sets K° and $\mathbb{C} \setminus K$ are connected. Then, $P(\partial K)$ is maximal on $C(\partial K)$.*

Proof. We follow the proof which was done by Bishop that used ideas due to Hoffman. Other proof may be found in [125] (see Theorem 25.12 in this book). Let $\mathcal{B}$ be some closed subalgebra of $C(\partial K)$ such that $P(\partial K) \subset \mathcal{B}$ and put $G = K^\circ$. Then, we need to prove that $\mathcal{B} = C(\partial K)$ or $\mathcal{B} = P(\partial K)$. We know that every function from $P(\partial K)$ can be extended to some function belonging to $A(K)$. Then, for every point $a \in G$ the mapping $\varphi_a \colon P(\partial K) \to \mathbb{C}$ defined by $\varphi_a(h) = h(a)$, is a homomorphism of the algebra $P(\partial K)$. Now, we distinguish two cases.

Case 1. Assume that φ_a can be extended to the algebra $\mathcal{B}$ for any $a \in G$. Then, $|\varphi_a(h)| \leqslant \|h\|_{\partial K}$ for all $h \in \mathcal{B}$. Therefore, φ_a can be extended to a bounded linear functional (with norm equals 1) on the space $C(\partial K)$. It means that there exists a measure μ_a (with $\|\mu_a\| = 1$) on ∂K such that $\int h(z)\mu_a(z) = \varphi_a(h)$ for every $h \in \mathcal{B}$. Since

$\mu_a(\partial K) = \varphi_a(1) = 1$, and since $\|\mu_a\| = 1$, we have that μ_a is a positive measure (see [18, page 80]). Therefore, for each polynomial P we have

$$\operatorname{Re} P(a) = \operatorname{Re} \varphi_a(P) = \operatorname{Re} \int P d\mu_a = \int \operatorname{Re} P \, d\mu_a. \tag{3.14}$$

Let us denote by $\hat{f}$ the harmonic complex extension of f given by Corollary 3.24 of f to K. Take $h \in \mathcal{B}$ and consider $\hat{h} \in C(K) \cap \operatorname{Har}(K^\circ)$. By Theorem 3.23 and (3.14) one has $\hat{h}(a) = \int \hat{h} d\mu_a = \int h \, d\mu_a = \varphi_a(h)$ and, moreover, since φ_a is a multiplicative functional, $\widehat{zh}(a) = a\hat{h}(a)$. Thus, $\hat{h}$ and $z\hat{h}$ are harmonic in G. Hence,

$$0 = \partial\overline{\partial}(\widehat{zh}) = \partial\overline{\partial}(z\hat{h}) = \partial(z\overline{\partial}\,\hat{h}) = \overline{\partial}\,\hat{h} + z\partial\overline{\partial}\,\hat{h} = \overline{\partial}\,\hat{h}.$$

In G, which yields that $\hat{h}$ is holomorphic in G. Since $\mathbb{C} \setminus K$ is connected, we conclude from Mergelyan's theorem that $\hat{h} \in P(K)$.

Case 2. Assume that there exists a point $a \in G$ such that the homomorphism φ_a cannot be extended to $\mathcal{B}$. Consider in such a case the principal ideal in $\mathcal{B}$

$$\mathcal{J} = \{h \in \mathcal{B} : h(z) = h_1(z)(z-a), \ z \in \partial K, \ h_1 \in \mathcal{B}\}.$$

Assume that $\mathcal{J} \neq \mathcal{B}$, then there exists a maximal ideal $\mathcal{M}$ such that $\mathcal{J} \subset \mathcal{M}$. Then, there exists such homomorphism $\Phi\colon \mathcal{B} \to \mathbb{C}$ that $\ker \Phi = \mathcal{M}$. Then, $\Phi(\mathfrak{j}) = a$ (where, as before, $\mathfrak{j}(z) = z$) and therefore $\Phi(P) = P(a)$ for each $P \in P(\partial K)$. It means that Φ is an extension of φ_a which contradicts our assumption in Case 2. Thus, $\mathcal{J} = \mathcal{B}$ and $1 \in \mathcal{J}$. It means that $1/(\mathfrak{j} - a) \in \mathcal{B}$. In view of Mergelyan's theorem $C(\partial K)$ is the algebra generated by $\mathfrak{j}$ and $1/(\mathfrak{j} - a)$. Then, $\mathcal{B} = C(\partial K)$. ∎

In fact, the property that $P(K)$ is a maximal subalgebra of $C(K)$ imposes quite rigid topological restrictions on the compact set K. We prove now the converse statement for Wermer's maximality theorem, which was essentially obtained in [27].

Theorem¶ 3.42. *Let K be a compact subset of $\mathbb{C}$. If $P(K)$ is a maximal subalgebra of $C(K)$, then K is a Carathéodory compact set without interior. If, moreover, $K = \partial\Omega$, where Ω is a nonempty bounded open set in $\mathbb{C}$, then neither $\overline{\Omega}$ nor Ω does not separate the plane and both sets $\partial\Omega$ and Ω are connected.*

Proof. If $P(K) = C(K)$ then $K^\circ = \emptyset$ and $K = \widehat{K}$. Then, $\partial K = \partial\widehat{K}$.

Assume therefore that $P(K) \neq C(K)$. In such a case the set $\mathbb{C} \setminus \partial\widehat{K}$ has a bounded component. If $K^\circ = \emptyset$ this is a consequence of Lavrentiev's theorem. If $K^\circ \neq \emptyset$ we can choose a bounded component of K°. So, one has

$$P(\partial\widehat{K}) \neq C(\partial\widehat{K}). \tag{3.15}$$

By (3.15) there exists a measure μ on $\partial\widehat{K}$ such that $\mu \perp P(\partial\widehat{K})$ and $\mu \neq 0$ (the symbol $\perp$ expresses the fact of orthogonality of μ to the corresponding set of functions). Let us now assume that $\partial K \setminus \partial \widehat{K} \neq \emptyset$ or $K^\circ \neq \emptyset$ and let us take $a \in \partial K \setminus \partial \widehat{K}$ or $a \in K^\circ$. Then, there exists a function $f \in C(K)$ such that $f(a) = 1$ and $f|_{\partial \widehat{K}} = 0$.

Let now $\mathcal{B}$ be the closure of the set of functions having the form $\sum_{j=0}^{m} q_j f^j$, where $q_0, \dots, q_m$ are polynomials and $m \in \mathbb{N}$. Since $f \notin P(K)$, then $\mathcal{B} \neq P(K)$. Moreover, since $f|_{\partial \widehat{K}} = 0$, then

$$\int_K \left(\sum_{j=0}^{m} q_j f^j \right) d\mu = \int_{\partial \widehat{K}} q_0 \, d\mu + \sum_{j=1}^{m} \int_{\partial \widehat{K}} q_j f^j \, d\mu = 0.$$

Thus, $\mathcal{B} \neq C(K)$, and so, $P(K)$ is not maximal. Thus, $\partial K = \partial \widehat{K}$ and $K^\circ = \emptyset$.

Let now $K = \partial\Omega$, where $\Omega \neq \emptyset$ is a bounded open set.

Assume that $\overline{\Omega}$ separates the plane. Let G be a bounded component of the set $\mathbb{C} \setminus \overline{\Omega}$ and Ω_1 be some component of Ω. Take $z_1 \in \Omega_1$ and $z_2 \in G$. Consider the closed subalgebra $\mathcal{B}$ which is generated by $P(K)$ and by the function $g_1(z) = 1/(z - z_1)$, $z \in K$. Clearly $g_1 \notin P(K)$. Taking into account that $\partial G \subset K$, and $g_1|_{\overline{G}}$ is holomorphic in $\overline{G}$, an application of the Maximum modulus principle gives that the function $h(z) = 1/(z - z_2)$, $z \in K$, does not belong to $\mathcal{B}$. Therefore, $P(K)$ is not maximal, which gives a contradiction. Thus, $\overline{\Omega}$ does not separate the plane.

Let us assume that Ω separates the plane. In such a case $\mathbb{C} \setminus \Omega = F_\infty \cup F_1$, where F_∞ is a closed set such that $\overline{\Omega}_\infty \subset F_\infty$ and F_1 is a nonempty compact set such that $F_1 \cap F_\infty = \emptyset$. Since $F_1 \cap \partial\Omega$ is not empty, take a point $z \in F_1 \cap \partial\Omega$. Since $\partial\Omega$ is a Carathéodory compact set then $z \in \partial(\partial\Omega) = \partial\Omega = \partial(\widehat{\partial\Omega})$. Thus, there exists a sequence of points $\{z_n\}$ such that $z_n \notin \widehat{\partial\Omega}$ and $z_n \to z$ as $n \to \infty$. Since $z_n \in \Omega_\infty$, then $z \in \overline{\Omega}_\infty \cap F_1 = \emptyset$. Thus, a contradiction arises and therefore Ω does not separate the plane.

Going further let us assume that the set $\partial\Omega$ is not connected. Then, $\partial\Omega = F_1 \cup F_2$, where F_1 and F_2 are compacts sets and $F_1 \cap F_2 = \emptyset$. Then, $\Omega \cap \widehat{F_j} \neq \emptyset$ for $j = 1, 2$, because if $\Omega \cap \widehat{F_1} = \emptyset$, then $\mathbb{C} \setminus \Omega$ has a bounded component and Ω will separate the plane. Then, we consider the closed subalgebra

$$\mathcal{B} = \{ f \in C(\partial\Omega) : f|_{F_1} \in P(F_1) \}.$$

If we take the function $f(z) = 1/(z - a)$, where $a \in \Omega \cap \widehat{F_2}$, we can see that $\mathcal{B} \neq P(F_1 \cup F_2)$. Clearly, $\mathcal{B} \neq C(\partial\Omega)$, it may be readily verified by considering $g(z) = 1/(z - b)$, $b \in \Omega \cap \widehat{F_1}$. Thus, $P(\partial\Omega)$ would be not maximal. Therefore, the set $\partial\Omega$ is connected. The fact that the set Ω is connected may be proved by a similar way. ■

Corollary¶ 3.43. *If $\Omega \neq \emptyset$ is a bounded open set, then $P(\partial\Omega)$ is maximal subalgebra of $C(\partial\Omega)$ if and only if Ω is a Carathéodory domain which does not separate the plane.*

Notice that a slightly weaker version of Corollary 3.43 (in the case when Ω is a priory assumed to be a simply connected domain) was obtained in [39].

Remark 3.44. In the proof of Theorem 3.42 it was shown that if Ω is a Carathéodory domain, and if $\overline{\Omega}$ does not separate the plane, then Ω itself does not separate the plane either. In the general case the properties "Ω does not separate the plane" and "$\overline{\Omega}$ does not separate the plane" are independent because all four possible situations can occur. The same can be said concerning connectivity properties of $\partial\Omega$ and Ω.

3.5 Orthogonal measures on Carathéodory sets

Many results in approximation theory were obtained in the frameworks of so-called dual approach, which is based on studies of linear functionals orthogonal to certain spaces of functions. In the case of uniform approximation on compact sets in $\mathbb{C}$ any linear functional on the space $C(X)$ has the form $f \mapsto \int f \, d\mu$, where μ is some complex-valued Borel measure with support on X. So that it is interesting and important to study properties of measures on X which are orthogonal to spaces of polynomials or rational functions, or to some other spaces of functions. One important and deep theorem in this theory which we will need in what follows is the F. and M. Riesz theorem (for the proof see, for instance, [115, Chapter 17] or [77, Chapter II]). For the reader's convenience we state it in such a way which makes evident the starting point of the research made by E. Bishop in his three papers that we will discuss in this section.

Theorem 3.45 (F. and M. Riesz). *Let ν be a complex measure on $\mathbb{T}$ which is orthogonal to all polynomials, that is $\int_{\mathbb{T}} P(\zeta)\, d\nu(\zeta) = 0$ for every $P \in \mathcal{P}$. Then, the following hold.*

(a) *The measure ν is absolutely continuous with respect to the measure $m_{\mathbb{T}}$, that is there exists a Borel measurable function u such that*

$$\nu(E) = \int_E u(\zeta)\, dm_{\mathbb{T}}(\zeta) = \frac{1}{2\pi i}\int_E \overline{\zeta}\, u(\zeta)\, d\zeta$$

for every Borel set $E \subset \mathbb{T}$.

(b) *Let the function f be defined in $\mathbb{D}$ by the formula*

$$f(z) = \frac{1}{2\pi i}\int \frac{d\nu(\zeta)}{\zeta - z},$$

and let $f_r(\zeta) = f(r\zeta)$ for $r > 0$ and $\zeta \in \mathbb{T}$. Then, $f_r \to u$ as $r \to 1$ in $L^1(\mathbb{T})$.

(c) *For a.a. points ζ on $\mathbb{T}$, one has that $f(z) \to u(\zeta)$ when $z \in \mathbb{D}$ tends to ζ non-tangentially.*

The aim of Bishop's research was to obtain a generalization of the F. and M. Riesz theorem for measures living on boundaries of general compact sets. The first problem arising in this connection is that if ∂K is not a rectifiable set, then it is not clear what is the absolute continuity property (with respect to what measure?) that needs to be used. In [13–15] E. Bishop has provided a fruitful investigation of the structure of measures orthogonal to rational functions on Carathéodory compacts sets. He used many tools in conformal mappings, in the theory of Hardy spaces, in measure theory. One key point he introduced is the concept of an analytic differential $g(z)\,dz$ that represents some complex measure μ. Let us briefly recall this concept. An analytic differential in a domain $\Omega \subset \mathbb{C}$ is a differential form $g(z)\,dz$, where $g \in H(\Omega)$. One says that the analytic differential $g(z)\,dz$ represents the measure μ on $\partial\Omega$ if the sequence of measures $\{g(z)\,dz|_{\Gamma_j}\}$ converges in the weak-star topology of the space of measures on $\overline{\Omega}$ to μ, where $\{\Gamma_j\}$ is some sequence of rectifiable contours such that $D(\Gamma_j) \subset D(\Gamma_{j+1}) \subset \Omega$ and $D(\Gamma_j) \uparrow \Omega$ as $j \to \infty$. Observe, that the analytic differential $g(z)\,dz$ in Ω is defined even in the case when $\partial\Omega$ is not a rectifiable set. This concept is not used nowadays and it has been only occasionally used in the mathematical literature.

To present the Bishop's results we need to recall some definitions and fix some notation. We will use notation from Section 3.2 concerning harmonic measure. Let now G be a simply connected domain in $\mathbb{C}$ and let f be some conformal map from $\mathbb{D}$ onto G. Assume for a moment, that ∂G is locally connected. Then, by Theorem 2.5, f has a continuous extension to $\overline{\mathbb{D}}$ onto $\overline{G}$. Moreover,

$$\omega(w, E, G) = \omega(f^{-1}(w), f^{-1}(E), \mathbb{D}), \tag{3.16}$$

for every point $w \in G$ and every Borel set $E \subset \partial G$. The equality (3.16) is called the invariance principle of the harmonic measure under conformal mapping. It can be readily proved by comparing both harmonic functions by its values on ∂G. The right-hand side of (3.16) can be readily calculated since

$$\omega(a, F, \mathbb{D}) = \int_F \frac{1-|a|^2}{|e^{it}-a|^2}\,\frac{dt}{2\pi}, \qquad F \subset \mathbb{T},\ a \in \mathbb{D},$$

and, moreover, this quantity may be represented in geometric terms. In the case that ∂G is not locally connected we have the following result.

Theorem 3.46. *Let $G \subset \mathbb{C}$ be a simply connected domain, and let f be a conformal map from $\mathbb{D}$ onto G. Then, $\omega(z, \partial_a G, G) = 1$ for every $z \in G$. Moreover, if $E \subset \partial_a G$ is a Borel set, then (3.16) holds. In particular, if $f(0) = z_0 \in G$, then*

$$\omega(z_0, E, G) = \omega(0, f^{-1}(E), \mathbb{D}) = m_{\mathbb{T}}(f^{-1}(E)). \tag{3.17}$$

For a proof of this theorem see [104, Section 6.2] and [59, page 206].

In the case of Carathéodory open sets the following useful property of a harmonic measure is satisfied.

Proposition 3.47. *Let U be a Carathéodory open set, and let W_1 and W_2 be two different components of U. Then, the measures $\omega(a,\cdot,U)$ and $\omega(b,\cdot,U)$ are mutually singular for all points $a \in W_1$ and $b \in W_2$.*

Proof. We know that it is enough to prove the desired assertion for two fixed points a and b belonging to the different components of U. Take $a \in W_1$. We have that W_1 is a Carathéodory domain, the measure $\omega(a,\cdot,U)$ is concentrated in $\partial_a W_1$, and W_2 is a component of $\mathbb{C} \setminus \overline{W_1}$. So, we can apply Proposition 1.15 to obtain the result. ■

Notice that the result stated in Proposition 3.47 is clearly not true in the case, where the open set U is not assumed to be a Carathéodory open set. To better understand this curious behavior, the reader can remind the open set $U = \mathbb{D} \cup Q_1 \cup Q_2$ defined just after Definition 3.34. Another, slightly different, proof of Proposition 3.47 was given in [15, Lemma 10].

Let now G be a Carathéodory domain, let f be a conformal map from $\mathbb{D}$ onto G such that $f(0) = z_0 \in G$, and let $g = f^{-1}$ be the respective inverse mapping. In what follows we will (often implicitly) use all results about boundary behavior of f and g obtained in Chapter 2 (in particular, Theorem 2.24 and Corollary 2.25).

Take a function $h \in L^1(\mathbb{T})$ and consider the measure $h\,d\zeta$ on $\mathbb{T}$. Define the measure $f(h\,d\zeta)$ on ∂G by the formula

$$f(h\,d\zeta)(E) := \int_{g(E\cap\partial_a G)} h(\zeta)\,d\zeta = \int (\mathbf{1}_E \circ f)(\zeta)h(\zeta)\,d\zeta$$

for every Borel set $E \subset \partial G$ (where $\mathbf{1}_E$ stands of the characteristic function of E), or, equivalently,

$$\int \psi\, df(h\,d\zeta) = \int_{F(f)} \psi(f(\zeta))h(\zeta)\,d\zeta = \int_{\mathbb{T}} \psi(f(\zeta))h(\zeta)\,d\zeta \qquad (3.18)$$

for every function $\psi \in C(\bar{G})$. Note that (3.17) implies that

$$\int_{\partial G} \psi(z)\,d\omega(z_0,z,G) = \int_{\mathbb{T}} (\psi\circ f)(\zeta)\,dm_{\mathbb{T}}(\zeta) = \frac{1}{2\pi}\int_0^{2\pi} (\psi\circ f)(e^{i\vartheta})\,d\vartheta$$

in our situation.

We define the complex harmonic measure relative to G and z_0 as $\omega_c(z_0,\cdot,G) = f(d\zeta)$. Then,

$$\omega_c(z_0,\cdot,G) = 2\pi i g\,\omega(z_0,\cdot,G). \qquad (3.19)$$

Moreover, if $h \in L^1(\mathbb{T})$ then

$$f(h\,d\zeta) = (h\circ f^{-1})\,\omega_c(z_0,\cdot,G) = (h_0\circ f^{-1})\,\omega(z_0,\cdot,G), \qquad (3.20)$$

where

$$h_0(z) := 2\pi i z h(z).$$

In view of (3.19), the properties that some measure μ on ∂G is absolutely continuous with respect to $\omega_c(z_0, \cdot, G)$ and $\omega(z_0, \cdot, G)$ are equivalent. For simplicity in what follows we will denote the complex measure $\omega_c(z_0, \cdot, G)$ just by ω assuming that the point z_0 is clear from the context and fixed.

The following results are essentially (but only implicitly) stated in the [13–15]. A proof of Theorem 3.48 below based on studies of analytic differentials representing measures can be extracted from the aforementioned papers of Bishop. We consider that it is interesting and in certain sense important to present a direct proof of this theorem which is free from the concept of analytic differentials. It was done in [26], but here we made some modifications. In [15] it was not mentioned that $\mu_G = \mu_{|\partial G}$ (in the second part of Theorem 3.48). This fact was proved in [26]. For an open set U we denote by $\mathcal{C}(U)$ the collection of all components of U.

Theorem¶ 3.48. *Let G be a Carathéodory domain, while X be a Carathéodory compact set in $\mathbb{C}$.*

(1) *Let μ be a measure on ∂G such that $\mu \perp R(\overline{G})$. Then, there exists a function $h \in H^1$ such that*
$$\mu = (h \circ g)\,\omega. \tag{3.21}$$

(2) *Let $X^\circ \neq \emptyset$, and let μ be a measure on ∂X such that $\mu \perp R(X)$. Then,*
$$\mu = \sum_{G \in \mathcal{C}(X^\circ)} \mu_G, \tag{3.22}$$
where
$$\mu_G = \mu|_{\partial G}, \quad \mu_G \perp R(\overline{G}),$$
and the series in (3.22) converges in the space of measures on ∂X.

(3) *Let μ be a measure on ∂X such that $\mu \perp R(X)$. Then, $\mu = 0$ on $X \setminus \overline{X^\circ}$ and $\mu \perp R(\overline{X^\circ})$.*

We recall that the Cauchy transform of a measure μ is the function

$$\widehat{\mu}(z) = \frac{1}{2\pi i} \int \frac{d\mu(w)}{w - z}$$

which is well defined for A-a.a. $z \in \mathbb{C}$. It is well known, that $\widehat{\mu}$ is holomorphic outside of $\mathrm{Supp}(\mu)$ and $\overline{\partial}\widehat{\mu} = \frac{i}{2}\mu$ in the sense of distributions.

We also recall that for a given class $\mathcal{F}$ of continuous functions and for a given measure μ the expression $\mu \perp \mathcal{F}$ means that μ is orthogonal to $\mathcal{F}$, i.e., $\int f\, d\mu = 0$ for each $f \in \mathcal{F}$.

Sketched proof of Theorem 3.48. Let us denote by G_j, where $j \in J$ and $J \subset \mathbb{N}_0$ is some finite or countable set of indexes, each element of the set $\mathcal{C}(X^\circ)$. We know that every G_j, $j \in J$, is a Carathéodory domain. In the case of the part (1), one has $J = \{0\}$ and $G_0 = G$.

For each $j \in J$ let f_j be some conformal mapping from $\mathbb{D}$ onto G_j, such that $f_j'(0) > 0$, let $\psi_j = f_j^{-1}$ be the inverse mapping and let $h_j := (\widehat{\mu} \circ f_j)\, f_j'$.

The proof will consist of several steps.

Step 1. $h_j \in H^1$ *for every* $j \in J$.

Proof. Take and fix $j \in J$. In view of Proposition 1.18 there exists a connected Carathéodory compact set Y such that $X \subset Y$ and $X^\circ = Y^\circ$. Choose some sequence (Γ_m) of rectifiable contours such that $Y \subset D(\Gamma_m) \subset D(\Gamma_{m-1})$ and $\overline{D(\Gamma_m)}$ converges to Y as $m \to \infty$. Notice that for any point $z_j \in G_j$ the kernel of the sequence $(D(\Gamma_m))$ with respect to z_j is exactly G_j.

Let $z_0 = f_j(0)$. Let g_m be the conformal mapping from $D(\Gamma_m)$ onto $\mathbb{D}$ such that $g_m(z_0) = 0$, $g_m'(z_0) > 0$. By Carathéodory kernel theorem the sequence (g_m) converges to $\psi_j = f_j^{-1}$ locally uniformly in G_j. Take a point $w \in \mathbb{D}$ and set $z_m = g_m^{-1}(w)$. Then, the function

$$\begin{cases} a(z) = \frac{1}{g_m(z)-g_m(z_m)} - \frac{1}{g_m'(z_m)(z-z_m)} & \text{for } z \neq z_m, \\ a(z_m) = -\frac{g_m''(z_m)}{2g_m'(z_m)^2} \end{cases}$$

can be uniformly on X approximated by rational functions with poles lying outside X. Then, since $\mu \perp R(X)$, we have

$$\frac{1}{2\pi i}\int \frac{d\mu(z)}{g_m(z)-g_m(z_m)} = \frac{\widehat{\mu}(z_m)}{g_m'(z_m)}.$$

We define the measures ν_m supported on $\mathbb{D}$ by the formula $\nu_m(E) = \mu(g_m^{-1}(E \cap \mathbb{D}))$ for each Borel subsets E of $\mathbb{C}$. Taking into account the previous formula and the fact that $g_m(z_m) = w$, we have

$$\frac{1}{2\pi i}\int \frac{d\nu_m(\zeta)}{\zeta - w} = \widehat{\mu}(g_m^{-1}(w))(g_m^{-1})'(w). \tag{3.23}$$

Moreover, ν_m is orthogonal to polynomials and $\|\nu_m\| \leqslant \|\mu\|$.

Take now a weak-star limit point η of the sequence (ν_m). Then, $\operatorname{Supp}(\eta) \subset \mathbb{T}$ and η is orthogonal to polynomials. Thus, one can find a function $t_j \in H^1$ with the property $\eta = t_j\, d\zeta|_{\mathbb{T}}$. Passing to the limit in (3.23) we obtain

$$h_j(w) = \widehat{\mu}(f_j(w)) f_j'(w) = \widehat{\eta}(w) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{t_j(\zeta)\, d\zeta}{\zeta - w} = t_j(w)$$

for all $w \in \mathbb{D}$, so that $h_j \in H^1$. ∎

For each $j \in J$ we define the measures $\omega_j := f_j(d\zeta|_{\mathbb{T}})$ and $\mu_j := f_j(h_j d\zeta|_{\mathbb{T}}) = (h_j \circ \psi_j)\,\omega_j$.

Step 2. *One has*

(i) $\widehat{\mu}_j(z) = \widehat{\mu}(z)$ *for all* $z \in G_j$*;*

(ii) $\widehat{\mu}_j(z) = 0$ *for all* $z \notin \overline{G}_j$*, (that means that* $\mu_j \perp R(\overline{G}_j)$*).*

Proof. Take $z \notin \partial\overline{G}_j$. Then,

$$\widehat{\mu}_j(z) = \frac{1}{2\pi i}\int_{\partial G_j}\frac{h_j(\psi_j(\zeta))\,d\omega_j(\zeta)}{\zeta - z} = \frac{1}{2\pi i}\int_{\mathbb{T}}\frac{h_j(\zeta)\,d\zeta}{f_j(\zeta) - z} = 0,$$

because the function $w \mapsto h_j(w)/(f_j(w) - z)$ belongs to H^∞.

If $z \in G_j$ let us take $w_j = f_j^{-1}(z) \in \mathbb{D}$. Then, the function

$$\begin{cases} q(w) = \frac{w - w_j}{f_j(w) - f_j(w_j)}, & \text{for } w \neq w_j, \\ q(w_j) = \frac{1}{f_j'(w_j)} \end{cases}$$

belongs to H^∞. Therefore,

$$\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{h_j(\zeta)\,d\zeta}{f_j(\zeta) - f_j(w_j)} = \frac{1}{2\pi i}\int_{\mathbb{T}}\frac{h_j(\zeta)\,q(\zeta)\,d\zeta}{\zeta - w_j} = h_j(w_j)\,q(w_j).$$

It gives, that for $z \in G_j$ one has

$$\widehat{\mu}_j(z) = \frac{h_j(w_j)}{f_j'(w_j)} = \widehat{\mu}(f_j(w_j)) = \widehat{\mu}(z),$$

which ends the proof. ∎

We are ready now to prove the first assertion of the theorem. Recall, that $G = G_0$ and $X = \overline{G}$ in this case. It follows from Step 2, that $\widehat{\mu}(z) = \widehat{\mu}_0(z)$ for all $z \notin G$, consequently $\mu - \mu_0 \perp R(\partial G)$. Since ∂G is a Carathéodory compact, in view of Theorem 3.10 we have $R(\partial G) = C(\partial G)$ and hence $\mu = \mu_0$. For each finite subset $I \subset J$ put $W_I := \bigcup_{j \in I} G_j$. The following assertion is the direct consequence of [15, Lemma 7].

Step 3. *There exists a sequence* (r_k) *of functions from* $R(X)$ *such that* $\|r_k\|_X \leqslant 1$, $r_k \rightrightarrows 1$ *locally in* W_I *and* $r_k \rightrightarrows 0$ *locally in* $X^\circ \setminus W_I$*.*

Let denote by $\mu_I \perp R(X)$ a weak-star limit in the space of measures on X of the sequence of measures $(r_k\mu)$.

Step 4. *One has*

(i) $\widehat{\mu}_I(z) = \widehat{\mu}(z)$ *for all* $z \in W_I$*;*

(ii) $\widehat{\mu}_I(z) = 0$ *for all* $z \in X^\circ \setminus W_I$*.*

Proof. Denote also by $(r_k\mu)$ the partial sequence that converges in the weak-star topology to μ_I. If $z \notin \partial X$, then

$$\widehat{\mu}_I(z) = \lim_{k\to\infty}\left(\frac{1}{2\pi i}\int \frac{(r_k(\zeta)-r_k(z))\,d\mu(\zeta)}{\zeta - z} + \frac{r_k(z)}{2\pi i}\int \frac{d\mu(\zeta)}{\zeta - z}\right) = \widehat{\mu}(z)\lim_{k\to\infty} r_k(z),$$

gives the desired assertion. ∎

It follows from Steps 2 and 4, that

$$\widehat{\mu}_I(z) = \sum_{j\in I}\widehat{\mu}_j(z), \quad z \notin \partial X.$$

Since $R(\partial X) = C(\partial X)$, we conclude that

$$\mu_I = \sum_{j\in I}\mu_j. \tag{3.24}$$

Taking into account (3.24), Proposition 3.47 and the fact that $\mu_j \ll \omega_j$ we conclude, that $\mu_j \perp \mu_k$ for $j,k \in J$, $j \neq k$. Hence, we have

$$\sum_{j\in I}\|\mu_j\| = \left\|\sum_{j\in I}\mu_j\right\| = \|\mu_I\| \leqslant \|\mu\|,$$

which means that $\sum_{j\in J}\|\mu_j\| < \infty$. Let $\mu = \sum_{j\in J}\mu_j$. It is clear, that $\eta \perp R(X)$. For each $j \in J$ we have $\widehat{\eta}(z) = \widehat{\mu}_j(z)$ for all $z \in G_j$ and applying the result of Step 2 we conclude that $\widehat{\eta}(z) = \widehat{\mu}(z)$ on X. Then, $\eta = \mu$.

Take now $k \in J$. Since $\mu_j \perp \mu_k$ for $j \in J \setminus \{k\}$, then for every Borel set $E \subset \partial X$ we have

$$\mu_{|\partial G_k}(E) = \mu(E\cap\partial G_k) = \sum_{j\in J}\mu_j(E\cap\partial G_k) = \mu_k(E\cap\partial G_k) = \mu_k(E).$$

The remaining part (item (3)) follows from (3.22) if $X^\circ \neq \emptyset$, and from Theorem 3.10 otherwise.

Thus, the proof is finished. ∎

Moreover, it is possible to find out in [15] certain additional facts concerning the objects that were introduced in the proof of Theorem 3.48. We present only two of them. In fact, one has

$$\sum_{j\in J}\int_{f_j(\rho\mathbb{T})}|\widehat{\mu}(\zeta)|\,d\zeta \leqslant C\|\mu\|,$$

for each $\rho \in (0,1)$, and

$$\sum_{j\in J}\|h_j\|_1 \leqslant C\|\mu\|,$$

where $C > 0$ is some absolute constant.

Remark 3.49. The part (2) of Theorem 3.48 remind us the Decomposition theorem for orthogonal measures, see [56, Theorem 7.11, Chapter II], see also [60]. We have not made the connection between both results, probably it can give another proof of Bishop result. Also it is curious to observe that Bishop's papers were not mentioned in Gamelin's book.

We have seen in Bishop's Theorem 3.48 that if G is a Carathéodory domain and μ is a measure on ∂G such that $\mu \perp R(\overline{G})$, then $\mu \ll \omega(a, \cdot, G)$ for every point $a \in G$. It turns out, that some converse result is also true. More precisely we have the following result, which must be compared with [38, Theorem 1].

Proposition¶ 3.50. *Let Ω be a non-empty bounded domain in $\mathbb{C}$, and let $a \in \Omega$.*

(a) *If Ω is a Carathéodory domain, and the set $\mathbb{C} \setminus \overline{\Omega}$ is not connected, then there exist $\mu \in P(\overline{\Omega})^{\perp}$ such that μ is not absolutely continuous with respect to $\omega(a, \cdot, \Omega)$.*

(b) *Assume that every measure which is orthogonal to $P(\overline{\Omega})$ is absolutely continuous with respect to $\omega(a, \cdot, \Omega)$. Then, Ω is a Carathéodory domain and the set $\mathbb{C} \setminus \overline{\Omega}$ is connected.*

Proof. (a) Assume that $\mathbb{C} \setminus \overline{\Omega}$ has a bounded component Ω_1. Then, Ω_1 is a Carathéodory domain and its complement is connected. This fact together with the part (1) of Theorem 3.48 yields that every measure of the form $(h \circ \rho)\, \omega$, where $h \in H^1$, ρ is some conformal map from Ω_1 onto $\mathbb{D}$ and ω is the complex harmonic measure on $\partial\Omega_1$ with respect some point $b \in \Omega_1$, is supported in $\partial\Omega_1 \subset \partial\Omega$, it is orthogonal to $\mathcal{P}$ and it is not absolutely continuous with respect to $\omega(a, \cdot, \Omega)$, since $\omega(a, \partial\Omega_1, \Omega) = 0$.

(b) Let $\Omega_0 = \mathbb{C} \setminus \overline{\Omega}_\infty$, i.e., Ω_0 is the interior of the complement of the unbounded component of the set $\mathbb{C} \setminus \overline{\Omega}$. If Ω is not a Carathéodory domain then there exists $z_0 \in \Omega_0 \cap \partial\Omega$. Consider now the measure

$$\mu_0 := \omega(z_0, \cdot, \Omega_0) - \delta_{z_0}.$$

Then, for every $P \in \mathcal{P}$, one has

$$\int P(\zeta)\, d\mu_0(\zeta) = \int_{\partial\Omega_0} P(\zeta)\, d\omega(z_0, \zeta, \Omega_0) - P(z_0) = 0,$$

because P is a harmonic function on $\overline{\Omega}_0$. Then, $\mu \perp P(\overline{\Omega})$ and it is not absolutely continuous with respect to $\omega(z_0, \cdot, \Omega)$. When we know that Ω is a Carathéodory domain, we apply the result of part (a) in order to complete the proof. ∎

Remark 3.51. The class of Carathéodory domains Ω for which $\overline{\Omega}$ does not separate the plane is (in view of Proposition 3.50) the largest class of domains for which the well-known F. and M. Riesz theorem may be extended from the unit disk preserving its formulation.

At the end of this section we present one refinement of Rudin's converse of the maximum modulus principle, where the concept of a Carathéodory set and Theorem 3.48 plays a crucial role.

Let us briefly recall the story of the aforementioned result. Let Ω be a bounded domain in $\mathbb{C}$, and let $f \in C(\overline{\Omega}) \cap H(\Omega)$. The classical maximum modulus principle states that for any $z \in \Omega$ the inequality $|f(z)| \leqslant \|f\|_{\partial\Omega}$ is satisfied. Moreover, if this inequality turns into equality at least at one point $z \in \Omega$, then the function f is constant. The question on whether it is possible to invert this principle arises quite naturally. In other words this is the question on whether it follows from the condition $|f(z)| \leqslant \|f\|_{\partial\Omega}$ (or from its certain weaker versions; see below) that the function $f \in C(\overline{\Omega})$ is holomorphic in Ω. One of the best known results of this kind is the following theorem due to W. Rudin (see [115, Theorem 12.13]). As before, $\mathfrak{j}$ stands for the function $\mathfrak{j}(z) = z$.

Theorem 3.52. *Let $\mathcal{F}$ be a subspace of the space $C(\overline{\mathbb{D}})$. Assume that $\mathcal{F}$ satisfies the following three conditions:* (i) $1 \in \mathcal{F}$*;* (ii) *for every function $f \in \mathcal{F}$ it holds $\mathfrak{j} f \in \mathcal{F}$; and* (iii) *the inequality*

$$|f(z)| \leqslant \|f\|_{\mathbb{T}} \tag{3.25}$$

is satisfied for every $f \in \mathcal{F}$ and $z \in \mathbb{D}$. Then, each function of $\mathcal{F}$ is holomorphic in $\mathbb{D}$.

Let $\overline{\mathcal{F}}$ be the closure of $\mathcal{F}$ in $C(\overline{\mathbb{D}})$. Since the conditions (i) and (ii) of Theorem 3.52 imply that $\mathcal{P} \subset \mathcal{F}$, then $A(\overline{\mathbb{D}}) = P(\overline{\mathbb{D}}) \subset \overline{\mathcal{F}} \subset A(\overline{\mathbb{D}})$. So that, if a given closed subspace $\mathcal{X} \subset C(\overline{\mathbb{D}})$ satisfies all conditions of Theorem 3.52, then $\mathcal{X} = A(\overline{\mathbb{D}}) = P(\overline{\mathbb{D}})$.

Rudin's theorem was a starting point for a number of further studies in the line of inversion of the maximum modulus principle. These studies were mainly related with consideration of certain weaker versions of the inequality (3.25) instead of the original one. Let us mention in this occasion the work by J. Anderson, J. Cima, N. Levenberg, and T. Ransford [4]. In this paper the inequality $|f(z)| \leqslant C_z\|f\|_{\mathbb{T}}$, where C_z is some positive number (which may depend on the point $z \in \mathbb{D}$), is considered in place of the inequality (3.25), and meromorphic functions in $\mathbb{D}$ are included into consideration. The result in question is formulated as follows.

Theorem 3.53 (Anderson, Cima, Levenberg, Ransford). *Let U be an open subset of $\mathbb{D}$ and let $g \in C(U \cup \mathbb{T})$. Assume that for any point $z \in U$ there exists a constant C_z such that the inequality*

$$|f_1(z) + g(z) f_2(z)| \leqslant C_z \|f_1 + g f_2\|_{\mathbb{T}}$$

is satisfied for all functions $f_1, f_2 \in A(\overline{\mathbb{D}})$. Then, there exist two functions $u, v \in H^\infty$ such that $g = u/v$ in U and for a.a. points $\zeta \in \mathbb{T}$ the equality of angular boundary *values $g(\zeta) = u(\zeta)/v(\zeta)$ holds. In particular, the function g is holomorphic in U and extends meromorphically to $\mathbb{D}$.*

It is also interesting to extend Rudin's theorem to domains which are different from the unit disk. However, this question is unstudied as yet. In [39] A. Dovgoshei considered it for the first time for Carathéodory domains G which do not separate the plane. He proved the following statement.

Theorem 3.54 (Dovgoshei). *Let G be a Carathéodory domain with the boundary Γ, and let $\mathcal{A}$ be a closed subalgebra of the algebra $C(\overline{G})$ such that $1 \in \mathcal{A}$ and $\|f\|_{\overline{G}} = \|f\|_{\Gamma}$ for any function $f \in \mathcal{A}$. The following two conditions are equivalent:*

- (a) *if there exists a function $g \in \mathcal{A}$ such that g is injective on $\overline{G}$ and holomorphic in G, then $\mathcal{A} = P(\overline{G})$;*
- (b) *the set $\overline{G}$ does not separate the plane.*

Notice that in this theorem one considers subspaces of the space $C(\overline{G})$ possessing certain additional (with respect to Rudin's theorem) conditions. Thus, as distinct from Rudin's theorem, we are dealing in that case with a closed subalgebra $\mathcal{A} \subset C(\overline{G})$, but not with a subspace $\mathcal{F} \subset C(\overline{G})$. Moreover, in Theorem 3.54 the condition of closedness of $\mathcal{A}$ with respect to multiplication by $\jmath$ is replaced with the condition that $\mathcal{A}$ contains some univalent function. In fact, it was proved in [39] that for a Carathéodory domain G for which $\overline{G}$ does not separate the plane, the condition that a closed subalgebra $\mathcal{A} \subset C(\overline{G})$ contains some univalent function, yields that $\jmath \in \mathcal{A}$. This result may be obtained as the consequence of Theorem 1.7 (more precisely, as the consequence of the weaker version of this theorem obtained in [39]). Let us also notice that the result of Theorem 3.54 in the case when G is a Jordan domain was previously obtained by Rudin in [113]. It is worth to observe that the assumptions which are imposed to $\mathcal{A}$ in Theorem 3.54 can be weakened and formulated as in Rudin's theorem. Indeed, the following result holds, see [51, Theorem 1].

Theorem¶ 3.55. *Let G be a Carathéodory domain.*

- (a) *Let G be such that $\overline{G}$ does not separate the plane. If a subspace $\mathcal{F}$ of the space $C(\overline{G})$ satisfies the following three conditions:* (i) $1 \in \mathcal{F}$; (ii) *for every function $f \in \mathcal{F}$ it holds $\jmath f \in \mathcal{F}$; and* (iii) *the inequality $|f(z)| \leqslant \|f\|_{\partial G}$ is satisfied for all $f \in \mathcal{F}$ and $z \in G$; then each function in $\mathcal{F}$ is holomorphic in G.*
- (b) *A closed subspace $\mathcal{X} \subset C(\overline{\Omega})$ satisfying the conditions* (i)–(iii) *from the first part of the theorem (where $\mathcal{F}$ is replaced by $\mathcal{X}$) coincides with $P(\overline{G})$ if and only if $\overline{G}$ does not separate the plane.*

The proof of the direct statement in Theorem 3.55 is essentially based on the usage of Wermer's maximality theorem for Carathéodory domains that do not separate the plane. As it was shown previously in Theorem 3.42, the condition that $\overline{G}$ does not separate the plane cannot be dropped whenever we want to preserve the maximality theorem statement.

Since the notion of a Carathéodory domain has appeared in the same topics in complex analysis and theory of uniform algebras with theorems by Rudin, Wermer and Anderson–Cima–Levenberg–Ransford, it is quite natural to consider Carathéodory domains in the respective context. In fact, we have the following result, see [51, Theorem 2].

Theorem¶ 3.56. *Let G be a Carathéodory domain with the boundary Γ, and let U be an open subset of G. Let $g \in C(U \cup \Gamma)$. Assume that for any $z \in U$ there exists a constant C_z such that the inequality*

$$|f_1(z) + g(z)f_2(z)| \leqslant C_z \|f_1 + gf_2\|_\Gamma \tag{3.26}$$

is satisfied for any function $f_1, f_2 \in A(\overline{G})$. Then, there exist two functions $u, v \in H^\infty(G)$ such that the equality

$$g(z) = \frac{u(z)}{v(z)} \tag{3.27}$$

holds everywhere in U and a.e. on Γ in the sense of conformal mappings. The latter means that for a.a. points $\zeta \in \mathbb{T}$ the following equality of angular boundary values holds $g(f(\zeta)) = u(f(\zeta))/v(f(\zeta))$, where f is some conformal map from the disk $\mathbb{D}$ onto G. In particular, the function g is holomorphic in U and extends meromorphically to G.

In the case, when $M \subset G$ is some finite set and $U = G \setminus M$, Theorem 3.56 gives the description of meromorphic functions in G with poles in M. In particular, if the set M is empty, then the respective description of holomorphic functions in G originates from this theorem.

Observe that in the case when G is a Jordan domain with rectifiable boundary, the equality (3.27) may be realized directly as the equality of angular boundary values a.e. on ∂G.

Corollary 3.57. *Let G be a Carathéodory domain for which $\overline{G}$ does not separate the plane. Assume that a function $g \in C(\overline{G})$ is such that for any functions $f_1, f_2 \in A(\overline{G})$ and for any point $z \in G$ the inequality*

$$|f_1(z) + g(z)f_2(z)| \leqslant \|f_1 + gf_2\|_\Gamma \tag{3.28}$$

is satisfied. Then, the function g is holomorphic in G.

Notice, that the assertion of Rudin's theorem may be derived from this corollary, see [51, Section 3] for the details.

3.6 Approximation by polyanalytic functions

The topic on approximation of functions by polyanalytic polynomials and polyanalytic rational functions is the subject of active studying in contemporary complex analysis and approximation theory. The concept of a Carathéodory set appears in this topic very naturally. In this section let X be a compact set in the complex plane, and let $n \geqslant 1$ be an integer. We define

$$\mathcal{P}_n = \mathcal{P} + \bar{z}\,\mathcal{P} + \cdots + \bar{z}^{n-1}\,\mathcal{P},$$
$$\mathcal{R}_n = \mathcal{R} + \bar{z}\,\mathcal{R} + \cdots + \bar{z}^{n-1}\,\mathcal{R}.$$

The spaces $\mathcal{P}_n$ and $\mathcal{R}_n$ are modules of dimension n over $\mathcal{P}$ and $\mathcal{R}$, respectively, generated by the powers of the function $\bar{z}$. For a given integer $d \geqslant 1$ we will also consider modules $\mathcal{P}_{n,d}$ and $\mathcal{R}_{n,d}$ generated by powers of $\bar{z}^d$ instead of powers of $\bar{z}$. For instance,

$$\mathcal{P}_{2,d} = \mathcal{P} + \bar{z}^d\,\mathcal{P}, \quad \mathcal{R}_{2,d} = \mathcal{R} + \bar{z}^d\,\mathcal{R}, \ldots .$$

Let us recall, that a function f is said to be *polyanalytic of order n* (or, for the sake of brevity, *n-analytic*) in an open set $U \subset \mathbb{C}$, if it is of the form

$$f(z) = f_0(z) + \bar{z}\,f_1(z) + \cdots + \bar{z}^{n-1}\,f_{n-1}(z),$$

where $f_0, \ldots, f_{n-1} \in H(U)$. The functions $f_0, \ldots, f_{n-1}$ are usually called holomorphic components of f. As usual, n-analytic functions whose holomorphic components are polynomials and rational functions will be called polyanalytic polynomials and polyanalytic rational functions, respectively. In fact, a polyanalytic rational function is not, in the general case, a quotient of two polyanalytic polynomials. It can be readily verified that the set of all n-analytic function on an open set U coincides with the set of all functions $f \in C(U)$ each of which is satisfies in U (in the sense of distributions) the (elliptic) partial differential equation $\overline{\partial}^n f = 0$. One ought to notice right now, that elements of modules generated by $\bar{z}^d$ for every $d > 1$ no longer belong to the kernel of some elliptic differential operator with constant coefficients, but (under suitable additional assumptions) they belong to the kernel of the elliptic operator $f \mapsto \overline{\partial}(\bar{z}^{1-d}\,\overline{\partial} f)$.

Furthermore, for a closed set $E \subset \mathbb{C}$ we will denote by $\mathcal{R}_{n,d,E}$ the set of all functions $g \in \mathcal{R}_{n,d}$ such that all poles of all holomorphic components of g lies outside E. Finally, we put $\mathcal{R}_{n,E} = \mathcal{R}_{n,1,E}$ and define the space

$$A_n(X; \bar{z}^d) = C(X) \cap (H(X^\circ) + \bar{z}^d\,H(X^\circ) + \cdots + \bar{z}^{d(n-1)}H(X^\circ)),$$

and let $A_n(X) = A_n(X; \bar{z})$, so that $A_n(X)$ is the set of all functions which is continuous on X and n-analytic on its interior.

Let K be an arbitrary compact set in $\mathbb{C}$ containing X. It can be shown, that the uniform closures on X of the spaces $\mathcal{P}_{n,d}\,|_X$ and $\mathcal{R}_{n,d,K}\,|_X$ are contained in $A_n(X;\bar{z}^d)$.

Thus, the problem on to describe such compact sets X for which the set $\mathcal{P}_{n,d}\,|_X$ is dense in $A_n(X;\bar{z}^d)$ is of interest. We refer the reader to the recent survey paper [88], where the history of this problem and its state-of-the-art are established in details. Here, we only state two results, which highlight the role of Carathéodory sets in this topic. Before doing this let us present the following result which may be directly derived from the main results of [25] using the Runge's pole–shifting method.

Theorem 3.58. *Assume X to be such that the set $\mathbb{C} \setminus X$ is connected. Then, the following hold.*

(1) *For any integer $n \geqslant 1$ the space $\mathcal{P}_n\,|_X$ is dense in $A_n(X)$.*

(2) *For any integer $d \geqslant 2$ the space $\mathcal{P}_{2,d}\,|_X = (\mathcal{P} + \bar{z}^d\,\mathcal{P})|_X$ is dense in $A_2(X;\bar{z}^d)$.*

For formulation of next results we need the concept of a d-Nevanlinna domain. This is the special analytic characteristic of bounded simply connected domains in the complex plane which was originally introduced in the case $d = 1$ in [49] and [28], and later in [8] for $d > 1$. It will be clear from what follows, that this concept turned out to be crucial for the aforementioned problem.

Definition 3.59. Let $d \in \mathbb{N}$. A bounded simply connected domain $G \subset \mathbb{C}$ is called a d-Nevanlinna domain if there exists two functions $u, v \in H^\infty(G)$ such that the equality

$$\bar{z}^d = \frac{u(z)}{v(z)}$$

holds almost everywhere on ∂G in the sense of conformal mappings. The latter means, that the equality of boundary values $\overline{f(\zeta)}^d = (u \circ f)(\zeta)/(v \circ f)(\zeta)$ holds for almost all points $\zeta \in \mathbb{T}$, where f is some conformal mapping from $\mathbb{D}$ onto G.

The class of 1-Nevanlinna domains is just the class of Nevanlinna domains. Notice that properties of Nevanlinna domain and d-Nevanlinna domains has been studied in detail during the two last decades (see, for instance, [8–12, 50, 86, 87]).

Let us mentioned several simple examples. In fact, $\mathbb{D}$ is a d-Nevanlinna domain for all $d \geqslant 1$. At the same time, any domain bounded by an ellipse which is not a circle is not a d-Nevanlinna for any $d \geqslant 1$. Take any fixed $d > 1$. For a real $a > 1$ let g_a be the single valued branch of the function $\sqrt[d]{a - z}$ defined on $\mathbb{C} \setminus [a, +\infty)$ and such that $g_a(0) > 0$. Then, the domain $g_a(\mathbb{D})$ is a d-Nevanlinna, but not a Nevanlinna domain. At the first glance it seems that the concept of a Nevanlinna domain gives a slight refinement of the concept of a Schwarz function of an analytic arc (see, for instance, [35]), but it turns out that there exists Nevanlinna domains with not analytic, not smooth, not rectifiable boundaries and, moreover, Nevanlinna domains G such that the Hausdorff dimension of the set $\partial_a G$ could take any value in $[1, 2]$.

In the following statement we combine the results of [28, Theorem 2.2], [16, Theorem 4], and [26, Theorem 4]. Let $\mathcal{C}'_X$ be the set of all connected components of the set $\mathrm{Int}(\widehat{X})$ that are not contained in X, that is

$$\mathcal{C}'_X = \{\Omega \in \mathcal{C}(\mathrm{Int}(\widehat{X})) : \Omega \not\subset X\}.$$

Theorem¶ 3.60. *The following statements hold.*

(1) *Let X be a compact set in $\mathbb{C}$ such that the set $\mathcal{C}'_X$ is not empty. Then, the subspace $\mathcal{P}_n|_X$ is dense in $A_n(X)$ if and only if for every $\Omega \in \mathcal{C}'_X$ the space $\mathcal{R}_{n,\overline{\Omega}}|_{X\cap\overline{\Omega}}$ is dense in $A_n(X \cap \overline{\Omega})$.*

(2) *Let G be a bounded simply connected domain in $\mathbb{C}$. If G is a Nevanlinna domain, then $\mathcal{R}_{n,\overline{G}}|_{\partial G}$ is not dense in $C(\partial G)$ for any integer $n \geqslant 0$.*

(3) *Let G be a Carathéodory domain in $\mathbb{C}$. The subspace $\mathcal{R}_{n,\overline{G}}|_{\partial G}$ is dense in $C(\partial G)$ if and only if G is not a Nevanlinna domain.*

The same results hold in problem of approximating functions by elements of the space $\mathcal{P}_{2,d} = \mathcal{P} + \bar{z}^d\mathcal{P}$. In fact, we have (see [8, Theorems 1, 2, and Propositions 2, 3]).

Theorem¶ 3.61. *The following statements hold.*

(1) *Let X be a compact set in $\mathbb{C}$ such that the set $\mathcal{C}'_X$ is not empty. Then, the subspace $\mathcal{P}_{2,d}|_X$ is dense in $A_2(X; \bar{z}^d)$ if and only if for every $\Omega \in \mathcal{C}'_X$ the space $\mathcal{R}_{2,d,\overline{\Omega}}|_{X\cap\overline{\Omega}}$ is dense in $A_2(X \cap \overline{\Omega}; \bar{z}^d)$.*

(2) *Let G be a bounded simply connected domain in $\mathbb{C}$. If G is a d-Nevanlinna domain, then the space $\mathcal{R}_{2,d,\overline{G}}|_{\partial G}$ is not dense in $C(\partial G)$.*

(3) *Let G be a Carathéodory domain in $\mathbb{C}$. The subspace $\mathcal{R}_{2,d,\overline{G}}|_{\partial G}$ is dense in $C(\partial G)$ if and only if G is not a d-Nevanlinna domain.*

Notice that this result is established for modules of dimension 2 only. The general case remains open.

Remarks and hints concerning the proofs of Theorems 3.60 and 3.61. The first statements in Theorems 3.60 and 3.61 are proved using the following scheme consisting of two steps (see [16] and [8], respectively): at the first step it was proved that any measure on X which is orthogonal to $\mathcal{P}_n$ (respectively, to $\mathcal{P}_{2,d}$) is also orthogonal to $\mathcal{R}_{n,X}$ (respectively, to $\mathcal{R}_{2,d,X}$). The respective construction was essentially elaborated in [28] in the proof of Theorem 2.2 of this paper. At the second step, using the special refinement of the Vitushkin's localization technique, it was proved that the space $\mathcal{R}_{n,X}|_X$ is dense in $A_n(X)$ (respectively, the space $\mathcal{R}_{2,d,X}|_X$ is dense in $A_2(X; \bar{z}^d)$). The condition that $\mathcal{R}_{n,\overline{\Omega}}|_{X\cap\overline{\Omega}}$ is dense in $A_n(X \cap \overline{\Omega})$ (and the respective condition in the second case) allow us to construct the desired approximants.

In order to prove the second statements in both Theorems 3.60 and 3.61 it is sufficient to show, that if G is a d-Nevanlinna domain, then the function $(\bar{z}^d - \bar{a}^d)/(z - a)$, $a \in G$, cannot be approximated uniformly on ∂G by rational functions of the class $\mathcal{R}_{2,d,\bar{G}}$. The detailed exposition of this proof is in the proof of Theorem 4 in [26] and of the proof of Proposition 2 in [8].

Let us present the schematic exposition of the proof of the third statements of theorems under consideration, because in the respective constructions show the reasons why the Carathéodory domain and Nevanlinna domain concepts are important and crucial for the aforementioned topic.

Let f be a conformal mapping from $\mathbb{D}$ onto G. We recall, that Corollary 2.25 states that the functions f and f^{-1} can be extended to mutually inverse Borel measurable functions on $\mathbb{D} \cup F(f)$ and $G \cup \partial_a G$, respectively. Let $\omega = f(d\zeta)$ the complex harmonic measure with respect to $f(0)$, see (3.19). If the space $\mathcal{R}_{2,d,\bar{G}}$ is not dense in $C(\partial G)$, then there exists a non-zero measure μ on ∂G such that $\mu \perp \mathcal{R}_{1,\bar{G}}$ and $\bar{z}^d \mu \perp \mathcal{R}_{1,\bar{G}}$. In view of (3.21) there exists two functions $h_1, h_2 \in H^1$ such that $\mu = (h_1 \circ f^{-1})\, \omega$ and $\bar{z}^d \mu = (h_2 \circ f^{-1})\, \omega$. Therefore, for almost all $\zeta \in \mathbb{T}$ one has $\overline{f(\zeta)}^d h_1(\zeta) = h_2(\zeta)$. Going further, replacing the quotient h_2/h_1 by f_2/f_1 with $f_1, f_2 \in H^\infty$ and defining the functions u and v in G as follows: $u(z) = f_2(f^{-1}(z))$, $v(z) = f_1(f^{-1}(z))$ one obtains that $\bar{z}^d = u(z)/v(z)$ almost everywhere on ∂G in the sense of conformal mappings, as it is demanded. ∎

Finally, let X be a Carathéodory compact set. In such a case the set $\mathcal{C}'_X$ is exactly the set of all bounded connected components of the set $\mathbb{C} \setminus X$. Thus, the following statement is a direct corollary of Theorems 3.60 and 3.61:

Corollary¶ 3.62. *Let $X \subset \mathbb{C}$ be a Carathéodoty compact set.*

(1) *The space $\mathcal{P}_n|_X$ is dense in $A_n(X)$ if and only if each bounded connected component of the set $\mathbb{C} \setminus X$ is not a Nevanlinna domain.*

(2) *The space $\mathcal{P}_{2,d}|_X$ is dense in $A_2(X; \bar{z}^d)$ if and only if each bounded connected component of the set $\mathbb{C} \setminus X$ is not a d-Nevanlinna domain.*