Chapter 4

Approximation in L^p-norms on Carathéodory sets

In this chapter we consider the topic on approximation of functions on Carathéodory sets by rational functions or polynomials in L^p-norms for $0 < p < +\infty$.

For a bounded measurable set $E \subset \mathbb{C}$ let us denote by $L^p(E)$ the space of all measurable functions $f : E \to \mathbb{C}$ such that

$$\|f\|_{p,E} = \|f\|_{L^p(E)} = \left(\int_E |f(z)|^p \, dA(z) \right)^{\frac{1}{p}} < +\infty,$$

while by $A^p(E)$ we denote the space consisting of those functions in $L^p(E)$ that are holomorphic in the interior of E. In the case that E is a domain, the spaces $A^p(E)$ are usually called Bergman spaces. For $p \geqslant 1$ they are Banach spaces, but for $p \in (0, 1)$ the quantity $\|f\|_{p,E}$ is only a quasi-norm. The history and the state-of-the-art of the theory of Bergman spaces may be found in the books [41] and [66].

4.1 Approximation in Bergman spaces

Our first goal in this section is to prove and discuss the following result, which is due to O. J. Farrell [45, 46] and A. I. Markushevich [84], see also [85, Chapter v].

Theorem 4.1 (Farrell). *Let G be a Carathéodory domain and let $0 < p < +\infty$. For every function $f \in A^p(G)$ there exist a sequence (p_n) of polynomials such that*

$$\lim_{n\to\infty} \int_G |f(z) - p_n(z)|^p \, dA(z) = 0.$$

In order to find the original proof of this theorem given by Farrell, it is convenient to pass thought both his papers [45] and [46]. The case that $p = 2$ was considered independently by Markushevich, however, there are some evidences that he has proved the corresponding result in the general case too. Markushevich's proof given in his later book [85, Chapter v] uses some tools which are very useful in the case of Hilbert spaces.

Before proving Theorem 4.1 let us make some historical remarks concerning the matter. Let G be a bounded domain in the complex plane. As far as we know the first results on approximation of functions in the class $A^p(G)$, for a given domain $G \subset \mathbb{C}$ and a number $p \in (0, \infty)$, by polynomials were obtained in the beginning of the 1920s by T. Carleman [23]. He considered the case of Jordan domain starlike with respect to the origin. Since his result is completely covered by Theorem 4.1, we are not going to comment it or on the technique used in Carleman's proof. It is worth mentioning here, that in the case when $G = \mathbb{D}$, $p > 1$, and $f \in A^p(\mathbb{D})$, then one can take the sequence of Taylor polynomials of f (with the center at the origin) as the desired approximating sequence in Theorem 4.1. Notice also, that in the general case, if $f \in A^p(\mathbb{D})$ with $p \in (0, 1]$, the sequence of the Taylor polynomials of f does not converge to f. The details of these constructions may be found in [41, page 31]. Let us notice however, that the proof in the case $0 < p < \infty$ and $G = \mathbb{D}$ does show that f_ρ converges to f as $\rho \to 1$, where $f_\rho(z) = f(\rho z)$.

Proof of Theorem 4.1. Let the sequence (J_n) of Jordan curves such that $D(J_n)$ converges to G (in the sense of kernel convergence), the sequence (φ_n) of conformal maps from $D(J_n)$ onto $\mathbb{D}$, and the conformal mapping φ from $\mathbb{D}$ onto G be as in the proof of Theorem 3.25. Let $g_n, n \in \mathbb{N}$, be the function $g_n = \varphi \circ \varphi_n$ defined on $D(J_n)$. Then, $g_n(z) \to z$ and $g_n'(z) \to 1$ locally uniformly in G. Consider the function

$$f_n = (f \circ g_n)\,(g_n')^{2/p}$$

defined in $D(J_n)$, where the branch of ${g_n'}^{2/p}$ is taken in such a way that is positive at the point $z_0 = \varphi(0)$. Let

$$C_p = \max\{2^{p-1}, 1\},$$

so that $|a+b|^p \leqslant C_p(|a|^p + |b|^p)$ for every point $a, b \in \mathbb{C}$. Fix now $\varepsilon > 0$ and take $K \subset G$ to be the closure of some Jordan domain such that

$$C_p \int_{G \setminus K_1} |f(z)|^p \, dA(z) < \varepsilon/3, \tag{4.1}$$

where $K \subset K_1 \subset G$, and K_1 also is the closure of some Jordan domain. Choosing K_1 in such a way that $G \setminus g_n(K_1) \subset G \setminus K$ for all $n \geqslant n_0$ with some $n_0 \in \mathbb{N}$, one has

$$\begin{aligned}\int_G |f - f_n|^p \, dA &\leqslant \int_{K_1} |f - f_n|^p \, dA + \int_{G\setminus K_1} |f - f_n|^p \, dA \\ &\leqslant \int_{K_1} |f - f_n|^p \, dA + C_p \int_{G \setminus K_1} |f|^p \, dA + C_p \int_{G\setminus K_1} |f_n|^p \, dA.\end{aligned} \tag{4.2}$$

The last integral in (4.2) can be estimated using (4.1) and (2.1) as follows:

$$\begin{aligned}\int_{G\setminus K_1} |f_n|^p \, dA &= \int_{G\setminus K_1} |f(g_n(z))|^p \, |g_n'(z)|^2 \, dA(z) = \int_{g_n(G\setminus K_1)} |f|^p \, dA \\ &\leqslant \int_{G \setminus g_n(K_1)} |f|^p \, dA \leqslant \int_{G\setminus K} |f|^p \, dA \leqslant \frac{\varepsilon}{3C_p}.\end{aligned} \tag{4.3}$$

For $n \geqslant n_1$ with some $n_1 \in \mathbb{N}$. Since $f(z) - f(g_n(z))g_n'(z) \to 0$ uniformly on $z \in K_1$ we have

$$\int_{K_1} |f - f_n|^p \, dA(z) < \frac{\varepsilon}{3}, \tag{4.4}$$

for $n \geqslant n_2$ with some $n_2 \in \mathbb{N}$. Using (4.4), (4.3), and (4.2) we obtain

$$\int_G |f(z) - f_n(z)|^p \, dA(z) < \varepsilon, \quad n \geqslant n_3. \tag{4.5}$$

For $n \geqslant n_3 = \max\{n_0, n_1, n_2\}$.

Each function f_n is holomorphic in $D(J_n)$ which is an open simply connected neighborhood of $\widehat{G}$. Then, applying again the Runge's theorem, we conclude that there exists a polynomial P_n such that

$$\int_{\widehat{G}} |f_n(z) - P_n(z)|^p \, dA(z) < \varepsilon,$$

for $n \geqslant n_3$. Using this together with (4.5) we obtain

$$\int_G |f - P_n|^p \, dA \leqslant 2C_p\varepsilon,$$

for $n \geqslant n_3$, which completes the proof. ■

Let us observe that Theorem 3.25 can be regarded as the limiting case of Theorem 4.1 when $p = +\infty$.

It would be interesting to compare Theorems 3.25 and 4.1 with each other, as well as with Runge's theorem, at least in some simple cases. For instance, let us take $p = 1$ and put $G = \mathbb{D}$. In this case, the polynomials nz^n, $n \in \mathbb{N}$, converge to zero locally uniformly in $\mathbb{D}$, but they do not converge on $L^p(\mathbb{D})$ for any $p > 0$. On the other hand the polynomials z^n converge to zero locally uniformly in $\mathbb{D}$ and also converges in $L^p(\mathbb{D})$ for each $p > 0$. They do not, however, satisfy the estimate of Theorem 3.25. The polynomials z^n/n converge to zero locally uniformly in $\mathbb{D}$ and they satisfy the estimate of Theorem 3.25. Finally, the polynomials $p^n z^{nk^n}$ with $p > 0$ and $k \in \mathbb{N}$, converge in $L^p(\mathbb{D})$ when $p \leqslant k$ and diverge when $p > k$.

Since the convergence of some sequence of holomorphic functions defined on a given open set U in $L^p(U)$-norm (for some $p > 0$) implies the locally uniform convergence of this sequence in U, one can additionally conclude in Theorem 4.1 that the sequence (P_n) converges locally uniformly in G to f. The possibility of polynomial approximation in such theorem in the case when the set $\mathbb{C} \setminus \overline{G}$ has bounded components looks a bit surprising. For example, if we suppose G to be the left-hand-side domain in Figure 2 (the cornucopia), then the function $1/z$ can be approximated in $A^p(G)$ by a sequence of polynomials but, of course, it cannot be approximated by polynomials locally uniformly in G. Notice also, that in a given Carathéodory domain G there may exist a compact set $K \subset \overline{G}$ such that $\widehat{K}$ is not contained in $\overline{G}$ and the $L^p(G)$-convergence of some sequence of polynomials does not imply the uniform convergence of this sequence on $\widehat{K}$. It is worth comparing this observation with the next proposition.

Proposition¶ 4.2. *Let U be an open set in $\mathbb{C}$ and $K \subset U$ be a compact set. Let $p > 0$ and $f \in A^p(U)$. Assume that there exists a sequence (P_n) of polynomials such that*

$$\lim_{n\to\infty} \int_U |f - P_n|^p \, dA = 0. \tag{4.6}$$

Then, f has an analytic extension to $U \cup \widehat{K}$. Denoting again this extension by f one has

$$\lim_{n\to\infty} \int_{U\cup\widehat{K}} |f - P_n|^p \, dA = 0.$$

Proof. It is enough to consider the case $\widehat{K} \setminus U \neq \emptyset$. Take such $r > 0$ that $\operatorname{dist}(K, \partial U) > 2r$. We need the following statement asserting that functions in a Bergman space cannot grow too rapidly near the boundary (see [41, Theorem 1]).

Lemma 4.3. *Let $p > 0$. For each function $f \in A^p(U)$ and for each compact set $K \subset U$, we have*

$$\|f\|_K \leqslant \frac{\|f\|_{p,U}}{\pi^{1/p} \operatorname{dist}(K, \partial U)^{2/p}}. \tag{4.7}$$

In particular, if a sequence of functions (f_n), $f_n \in A^p(U)$, converges to f in $A^p(U)$, then $f_n \rightrightarrows f$ locally in U.

Consider the compact set

$$K_1 = K \cup \bigcup_{z\in K} \overline{D(z,r)} \subset U$$

and apply (4.7) to K_1. Then, one has

$$\|f - P_n\|_{K_1} = \sup_{z\in K_1} |f(z) - P_n(z)| \leqslant c_r \int_U |f - P_n|^p \, dA, \tag{4.8}$$

for some constant c_r depending on U, K, and p.

Then, (4.8) and (4.6) imply that (P_n) is a Cauchy sequence on K_1. By the maximum modulus principle the sequence (P_n) must converge uniformly to some holomorphic function g on $\operatorname{Int}(\widehat{K_1})$. Since $\widehat{K} \subset \operatorname{Int}(\widehat{K_1})$ then (P_n) converges uniformly on $\widehat{K}$, so it converges in the space $L^p(\widehat{K})$. By (4.8) it follows that $g(z) = f(z)$ on K, and the usage of the inequality

$$\int_{U\cup\widehat{K}} |f - P_n|^p \, dA \leqslant \int_U |f - P_n|^p \, dA + \int_{\widehat{K}} |g - P_n|^p \, dA$$

finishes the proof. ∎

We mention three simple examples showing the situation with polynomial approximation in L^p-norm in the case of non-Carathéodory domains.

Example 4.4. We mention three simple examples showing the situation with polynomial approximation in L^p-norm in the case of non-Carathéodory domains.

(i) Take $G_1 = \mathbb{D} \setminus (-1, 0]$ and $g(z) = 1/z$, $z \in G_1$. Then, $g \in A^p(G_1)$ for each p, $0 < p < 2$, but g cannot be approximated in the L^p-norm by polynomials for any $p \in (0, 2)$.

(ii) Let $g(z) = \log z$, $z \in G_1$ (where log stands for the principal branch of the log-function). Then, g cannot be approximated in the L^p-norm by polynomials for any $p > 0$.

(iii) Take $G_2 = \mathbb{D} \setminus (\overline{D(0, 1/2)} \cup [0, 1))$ and $g(z) = 1/z$, $z \in G_2$. Then, g cannot be approximated in the L^p-norm by polynomials for any $p > 0$.

The verification of all these statements may be done using (4.7).

The construction given in the first of the aforesaid examples may be refined and generalize by the following way.

Proposition¶ 4.5. *Let G be a Carathéodory domain and let $\mathcal{E}$ be some end-cut of G such that* Area$(\mathcal{E}) = 0$, *and let $G_{\mathcal{E}}$ be the corresponding slitted domain $G \setminus \mathcal{E}$. Then, the set of polynomials is not dense in $A^p(G_{\mathcal{E}})$ for any $p > 0$.*

Proof. Take a conformal map g from $G_{\mathcal{E}}$ to the unit disk. Then, $g \in A^p(G_{\mathcal{E}})$ for each $p > 0$. Assume that there exists a sequence of polynomials (P_n) that converges in $A^p(G_{\mathcal{E}})$ to the function g. Therefore, (P_n) is a Cauchy sequence in $A^p(G)$. So, it needs to converge uniformly on compact subsets of G to a function $\tilde{g} \in H(G)$ which coincides with g on $G \setminus \mathcal{E}$. But this is impossible since in each cut point $a \in \mathcal{E}$, the function g cannot be extended continuously to a neighborhood of a. ■

In view of this proposition one can ask whether the condition that a given domain G is a Carathéodory domain, is necessary in order to have polynomial approximation in $A^p(G)$. The answer to this question is negative, as it may be observed from several constructions of so-called moon-shaped domains.

Recall, that a domain $M \subset \mathbb{C}$ is called a *moon-shaped* domain, if it has the form $M = D(J_1) \setminus \overline{D(J_2)}$, where $J_{1,2}$ are two Jordan curves such that $J_1 \cap J_2 = \{\xi\}$ and $J_2 \subset D(J_1) \cup \{\xi\}$. In what follows it would be appropriate to say that the domain M is determined by the curves $J_1 = J_1(M)$ and $J_2 = J_2(M)$. It will be also useful to write $D_1 = D_1(M) = D(J_1)$ and $D_2 = D_2(M) = D(J_2)$ in the situation under consideration.

The simplest example of a moon-shaped domain is the domain

$$M_r := \mathbb{D} \setminus \overline{D(r, 1-r)}, \tag{4.9}$$

for $0 < r < 1$, see the left-hand side domain on Figure 9. In this situation $J_1 = \mathbb{T}$, while $J_2 = \{z : |z - r| = 1 - r\}$.

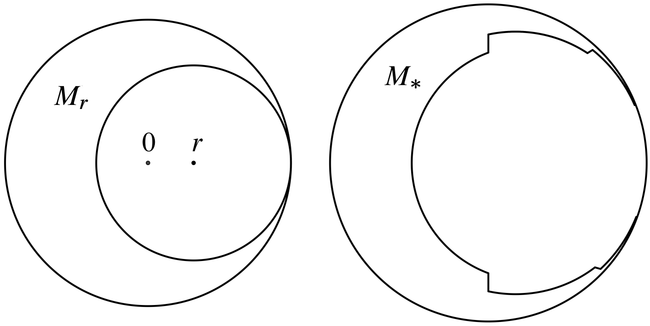

Figure 9. Two moon-shaped domains: M_r and M_*.

In the following two propositions we collect several results about moon-shaped domains, which are closely related with the topic on L^p-polynomial approximation being discussed.

Proposition¶ 4.6. *Let M be a moon-shaped domain, let $p \in (0, \infty)$ and $s = 2/(p + 2)$. The set of polynomials is dense in $A^p(M)$ if and only if there exists $b \in D_2(M)$ such that both functions $\varphi(z) = (z - b)^s$ and $\psi(z) = (z - b)^{-s}$ can be approximated by polynomials in $A^p(M)$.*

Sketch of the proof. The necessity of the stated condition is clear. For proving its sufficiency, let us note that the function φ maps conformally the given domain M onto some domain $W = \varphi(M)$. Since $s < 1$ then W is a Jordan domain. Put $w = \varphi(z)$, $z \in M$, then $z = \varphi^{-1}(w) = b + w^{1+p/2}$. Take $f \in A^p(M)$ and put $f_1(w) = f(\varphi^{-1}(w))$. Since $\varphi(z)^p(\varphi'(z))^2 = s^2$ for every $z \in M$, one has

$$\int_W |w f_1(w)|^p \, dA(w) = \int_M |\varphi(z)|^p |f(z)|^p |\varphi'(z)|^2 \, dA(z) = s^2 \int_M |f(z)|^p \, dA(z).$$

Then, $w f_1 \in A^p(W)$. Take any $\varepsilon > 0$. Since the set of polynomials is dense in $A^p(W)$ (because W is a Jordan domain), there exists a polynomial $P(w) = \sum_{k=0}^n a_k w^k$ such that

$$\int_W |w f_1(w) - P(w)|^p \, dA(w) = s^2 \int_M \left| f - a_0 \psi - \sum_{k=1}^n a_k \varphi^{k-1} \right|^p dA < \varepsilon.$$

Thus, the possibility of approximation of both function φ and ψ by polynomials in $A^p(M)$ implies the possibility of approximation in the desired sense of any function $f \in A^p(M)$. ∎

Note that in the case $p = 2$ it can be shown that is it sufficient to approximate just ψ in order to have the conclusion in Proposition 4.6. This will be used in proving part (3) of the next proposition. At the same time the mentioned arguments cannot be used in the case $p \neq 2$. In view of this reason it is not clear how to modify the construction of the domain M_* to obtain a corresponding example that covers the case $p \neq 2$ in part (3) of Proposition 4.7.

Proposition 4.7. *The following approximation properties hold.*

(1) *Let M be a moon-shaped domain, while ξ be the common point of $J_1(M)$ and $J_2(M)$. If there exist a rectifiable Jordan curve Γ in $M \cup \{\xi\}$ such that $\xi \in \Gamma$, and a number $\alpha > 0$ such that*

$$\int_\Gamma \operatorname{dist}^{-\alpha}(z, \partial M)\,|dz| < +\infty,$$

then the set of polynomials is not dense in $A^p(M)$ for every $p > 0$.

(2) *The set of polynomials is not dense in $A^p(M_r)$ for any $p > 0$ and for any $r \in (0, 1)$.*

(3) *There exists a moon-shaped domain M_* such that the set of polynomials is dense in $A^2(M_*)$.*

Sketch of the proof. Part (1) The proof of this statement can be obtained following the pattern of the verification of Example 4.4, which is based on usage of (4.7); the case $p = 2$ may be found in [91, page 116].

Part (2) Using the notation of the previous part, let us take

$$\Gamma(t) = \frac{r}{2} + \Big(1 - \frac{r}{2}\Big)e^{it}, \quad t \in [0, 2\pi],\ r \in (0, 1),$$

It can be verified that this curve satisfies the conditions of the previous statement for every $\alpha < \frac{1}{2}$.

Part (3) Denote by arg the branch of the argument function defined on $\mathbb{C} \setminus \{0\}$ such that $\arg z \in (-\pi, \pi]$ for $z \neq 0$. Let us construct a sequence (α_n) with $0 < \alpha_n < 1$, a sequence of polynomials (P_n), and three sequences of sets (D_n), Δ_n and Ω_n as follows. Let $\alpha_0 = 1/4$ and

$$D_1 = \{z \in \mathbb{D} : |z - \alpha_0| > 1 - \alpha_0,\ |\arg z| > \pi/2\}.$$

According to Runge's theorem there exists a polynomial P_1 such that

$$\|z^{-1/2} - P_1\|_{2,D_1} < 1/\sqrt{2}.$$

Next there exists a sufficiently small $\alpha_1 \in (0, 1)$ and the domain

$$\Delta_1 = \{z \in \mathbb{D} : |z - \alpha_1| > 1 - \alpha_1,\ |\arg z| < \pi/2\}$$

such that

$$\|z^{-1/2} - P_1\|_{2,\Delta_1} < 1/\sqrt{2}.$$

Finally, let $\Omega_1 := \{z \in \mathbb{D} : |z - \alpha_1| > 1 - \alpha_1,\ |\arg z| > \pi/4\}$. Assume, that all desired objects are already constructed for $n = 1, \dots, N-1$, $N \geqslant 2$. Put $D_N = D_{N-1} \cup \Omega_{N-1}$. There exists a polynomial P_N such that

$$\|z^{-1/2} - P_N\|_{2,D_N} < 2^{-N/2},$$

and a (sufficiently small) number $\alpha_N \in (0,1)$ such that

$$\|z^{-1/2} - P_N\|_{2,\Delta_N} < 2^{-N/2},$$

for the domain $\Delta_N = \{z \in \mathbb{D} : |z - \alpha_N| > 1 - \alpha_N,\ |\arg z| < \pi/2^N\}$. Defining

$$\Omega_N = \left\{z \in \mathbb{D} : |z - \alpha_N| > 1 - \alpha_N, |\arg z| > 1/2^{N+1}\right\}$$

we finish the construction. Now, we are able to define the domain M_* as $\bigcup_{n=}^{\infty} D_n$. Since $M_* \subset D_n \cup \Delta_n$ for every integer $n > 0$ one has

$$\|z^{-1/2} - P_n\|_{2,M_*} < 2^{(1-n)/2}.$$

Then, $\psi(z) = z^{-1/2}$ belongs to $A^2(M_*)$, so the proof of (3) is completed. ■

It is also worth mentioning here yet another example given in [71] (see also [91]). Taking $\alpha > 4$ and $\lambda \in (0,1)$ let

$$\Upsilon_{\alpha,\lambda} := \left\{z = x + iy \in \mathbb{C} : y^2 = (\lambda + x)(1-x)^{\alpha}\right\}.$$

The moon-shaped domain M determined by $J_1 = \mathbb{T}$ and $J_2 = \Upsilon_{\alpha,\lambda}$ is homeomorphic to the domain M_* defined in the proof of the part (3) of Proposition 4.7. But it was proved in [71] that the set of polynomials is not dense in $A^2(M)$. So, the question on L^p-approximation by polynomials depends on certain metric properties of the domain under consideration.

At the end of this section we present yet another two proofs of Theorem 4.1 in the Hilbert space setting, namely, in the case that $p = 2$. We do it in order to highlight certain special properties of Carathéodory domains and their conformal maps on which these proofs are based. The first proof was given by A. I. Markushevich [83], and it is based on the following lemma.

Lemma 4.8. *Let G be a Carathéodory domain and $z_0 \in G$. Take a sequence (G_n) of Jordan domains such that $G_n \to G$ with respect to z_0, and let g_n be the conformal map from G_n onto $\mathbb{D}$ normalized by the conditions $g_n(z_0) = 0$ and $g_n'(z_0) > 0$ for every $n \in \mathbb{N}$. Moreover, let g be the conformal map from G onto $\mathbb{D}$ with the same normalization, so that $g_n \rightrightarrows g$ locally in G. Then,*

$$\lim_{n\to\infty} g_n^k g_n' = g^k g' \quad \text{in } A^2(G), \text{ for } k \in \mathbb{N}_0. \tag{4.10}$$

Proof. Taking $k \geqslant 1$, one has

$$\begin{aligned}|g_n^k g_n' - g^k g'|^2 &\leqslant \left(|g_n^k(g_n' - g')| + |(g_n^k - g^k)g'|\right)^2\\ &\leqslant 2|g_n|^{2k}|g_n' - g'|^2 + 2|g'|^2|g_n^k - g^k|^2\\ &\leqslant 2|g_n' - g'|^2 + 2k^2|g'|^2|g_n - g|^2.\end{aligned}$$

We have used here that $|g_n(z)| < 1$ and $|g(z)| < 1$ for each $z \in G$. In fact, these inequalities also hold for $k=0$. Since $g_n'(z) \to g'(z)$ for each $z \in G$ and $\int_G |g_n'|^2\, dA = \int_G |g'|^2\, dA = \pi$ we have that $\int_G |g_n' - g'|^2\, dA \to 0$. In order to verify this one can use, for instance, the following fact on convergence which may be found in [115, page 76].

Lemma 4.9. *Let μ be a positive measure on some set E, let $p \in (0,\infty)$, and let $f \in L^p(E,\mu)$, $f_n \in L^p(E,\mu)$, $n \geqslant 1$, and, finally let $f_n(x) \to f(x)$ for μ-a.a. $x \in E$ and $\|f_n\|_p \to \|f\|_p$ as $n \to \infty$. Then, $\|f - f_n\|_p \to 0$ as $n \to \infty$.*

Thus, $\int_G |g_n - g|^2\, |g'|^2\, dA \to 0$, which is a consequence of Lebesgue dominated convergence theorem, since $|g'|^2 \in L^1(G, dA)$, and $|g_n - g| \leqslant 2$ and $|g_n(z) - g(z)| \to 0$ as $n \to \infty$ for each z. ∎

Sketched proof of Theorem 4.1 in the case $p = 2$. Take a function $h \in A^2(G)$. Using the conformal map $g\colon G \to \mathbb{D}$ let us "move" the function h to the unit disc. Namely we consider the function φ in $\mathbb{D}$ defined as follows:

$$\varphi(w) = (h \circ g^{-1})(w)(g^{-1})'(w), \quad |w| < 1.$$

Since

$$\int_{\mathbb{D}} |\varphi(w)|^2\, dA(w) = \int_G |h(z)|^2\, dA(z),$$

then $\varphi \in A^2(\mathbb{D})$. Take the Taylor expansion for φ at the origin $\varphi(w) = \sum_{k=0}^{\infty} a_k w^k$. Then, as it was mentioned above, the Taylor polynomials of φ converges in $A^2(\mathbb{D})$ to φ. Hence,

$$\int_{\mathbb{D}} \left|\varphi(w) - \sum_{k=0}^{N} a_k w^k\right|^2 dA(w) = \int_G \left|h(z) - \sum_{k=0}^{N} a_k g(z)^k g'(z)\right|^2 dA(z) \to 0.$$

Fixed N and using Lemma 4.8 each sum $\sum_{k=0}^{N} a_k g^k g'$ can be approximated in the space $A^2(G)$ by a function

$$h_n = \sum_{k=0}^{N} a_k g_n^k g_n'$$

for some value of n. Since the function g_n is defined on G_n and $\overline{G} \subset G_n$, one can use Runge's theorem to obtain a polynomial P_n which approximates the function h_n in $A^2(G)$. ∎

The next result is similar to Lemma 4.8. It was proved in [91]. We present here slightly different proof of this result working in the framework of more direct approach related with properties of Carathéodory domains.

Lemma 4.10. *Let G be a Carathéodory domain, let $z_0 \in G$, and let (G_n) be some sequence of Jordan domains such that $G_n \to G$ with respect to $z_0 \in G$. Let $\psi_n\colon G_n \to G$ be the conformal map normalized by conditions $\psi_n(z_0) = z_0$ and $\psi_n'(z_0) > 0$, $n \in \mathbb{N}$, and let h be some function of class $A^2(G)$. Then, $(h \circ \psi_n)\psi_n'$ converges to h in $A^2(G)$ as $n \to \infty$.*

Proof. Using the notations of Lemma 4.8 one can note that $\psi_n = g^{-1} \circ g_n$ for each $n \geqslant 1$. Then, $\psi_n(z) \to z$ for every $z \in G$, so that $h(\psi_n(z))\psi_n'(z) \to h(z)$ as $n \to \infty$ for every $z \in G$. Moreover, making the change of variables $w = \psi_n(z)$ we have

$$\int_G |h(\psi_n(z))\psi_n'(z)|^2\, dA(z) = \int_{\psi_n(G)} |h(w)|^2\, dA(w) \to \int_G |h(w)|^2\, dA(w)$$

as $n \to \infty$. At that point we can finish the proof applying Lemma 4.9, as it was done in the proof of Lemma 4.8. ∎

The second alternative proof of Theorem 4.1 will be presented in Section 4.2, where we will deal with certain aspects of the subject under consideration related with the Hilbert space structure of $A^2(G)$.

Approximation on Carathéodory compact sets

The next contribution to the theory of L^p-polynomial approximation on Carathéodory sets was made by S. O. Sinanjan in [123]. He proved the two following theorems, and the first one is a generalization of Theorem 4.1 for the case that $1 \leqslant p < \infty$.

Theorem 4.11 (Sinanjan). *If $K \subset \mathbb{C}$ is a Carathéodory compact set, then the set of polynomials is dense in $A^p(K)$ for every $1 \leqslant p < \infty$.*

Scheme of the proof. The proof follows more or less directly the pattern of the original proof of Mergelyan's theorem. Without loss of generality it may be assumed, that K is a continuum. Let $R > 0$ be such that $K \subset D(0, R/2)$. Take a function $f \in A^p(K)$ and define it also for all points $z \notin K$ by setting $f(z) = 0$. Take and fix an arbitrary $\delta > 0$. Set

$$f_\delta(z) = \int_{\mathbb{C}} f(\zeta)\, K_\delta(|\zeta - z|)\, dA(\zeta),$$

where

$$K_\delta(r) = \begin{cases} \frac{3}{\pi\delta^2}\left(1 - \frac{r}{\delta}\right), & \text{if } 0 \leqslant r \leqslant \delta \\ 0, & \text{if } r > \delta. \end{cases}$$

The function f_δ possesses the following important properties:

(1) $\|f_\delta\|_p \leqslant 3\|f\|_p$;

(2) $\|f_\delta - f\|_p \leqslant 3\omega_p(f,\delta)$, where $\omega_p(f,\delta)$ is the L^p-modulus of continuity of f;

(3) $\omega_p(f_\delta, r) \leqslant 3\omega_p(f,r)$;

(4) for any function $\psi \in C(\mathbb{C})$ with $\psi(z) = 0$ for $z \in \mathbb{C} \setminus \overline{D(0,2R/3)}$ it holds

$$\|\overline{\partial}\psi_\delta\|_p \leqslant 6\frac{\omega_p(\psi,\delta)}{\delta};$$

(5) $f_\delta = f$ in $U = \{z : \operatorname{dist}(z, K^{\complement}) > \delta\}$, while $f_\delta = 0$ in $U \cap K^{\complement}$.

In contrast to the original Mergelyan proof in the case under consideration one needs to consider yet another convolution of the approximable function with the kernel K_δ defined above.

$$f_\delta^*(z) = \int_{\mathbb{C}} f_\delta(\zeta)\, K_\delta(|\zeta - z|)\, dA(\zeta).$$

It follows from the aforesaid properties of the function f_δ that

$$\|f - f_\delta^*\|_p \leqslant \|f - f_\delta\|_p + \|f_\delta - f_\delta^*\|_p \leqslant 12\omega_p(f,\delta).$$

Thus, it is enough to find a polynomial Q such that $\|f_\delta^* - Q\|_p \leqslant A_1\omega_p(f,\delta)$ for some absolute constant $A_1 > 0$.

Take a conformal map from $\Omega_\infty(K)^* = \Omega_\infty(K) \cup \{\infty\}$ to $\mathbb{D}$. Then, the pre-images of the circles $|w| = 1 - 1/(n+1)$ under this transformation are denoted by $\varGamma_n$, they are analytic curves. Moreover, let $D_n = D(\varGamma_n)$. Now, the standard Cauchy–Green formula (see, for instance, [18, page 151]) gives

$$f_\delta^*(z) = \frac{1}{2\pi i}\int_{\varGamma_n} \frac{f_\delta^*(t)}{t - z}\, dt - \frac{1}{\pi}\int_{D_n} \frac{\overline{\partial} f_\delta^*(\zeta)}{\zeta - z}\, dA(\zeta), \quad z \in D_n. \tag{4.11}$$

Next one can choose a sufficiently large integer n in such a way that the following two conditions are fulfilled:

(a) $\operatorname{dist}(z, \Gamma_n) < \delta/2$ for each $z \in \partial K$;

(b) the following inequality holds

$$\int_K \left(\frac{1}{2\pi}\int_{D_n \setminus \widehat{K}} \left|\frac{\overline{\partial} f_\delta^*(\zeta)}{\zeta - z}\right| dA(\zeta)\right)^p dA(z) < \omega_p(f,\delta)^p. \tag{4.12}$$

Now, take $Y = \{z \in \widehat{K} : \operatorname{dist}(z, \partial K) < 2\delta\}$. At that point we need to recall the main [90, Lemma 2.2] (see also [115, Lemma 20.2]). The set Y can be covered by finitely many open discs $D(a_j, 2\delta)$, $1 \leqslant j \leqslant m$ with centers $a_j \in \Omega_\infty(K)$. Since

K is a Carathéodory compact set, then there exists a continuum (actually an arc) $L_j \subset D(a_j, 2\delta) \cap \Omega_\infty(K)$ such that diam L_j is comparable with δ. Using furthermore the conformal maps g_j, $j = 1, \ldots, m$, from $\mathbb{C}_\infty \setminus L_j$ to $\mathbb{D}$ such that $g_j(\infty) = 0$ and taking suitable linear combinations of g_j and g_j^2, one can find, for each point $\zeta \in Y$, a holomorphic (even rational) function R_ζ defined on the open set

$$\Omega := \mathbb{C}_\infty \Big\backslash \bigcup_j L_j \subset \mathbb{C}_\infty \backslash \widehat{K},$$

that R_ζ satisfies the following properties:

$$|R_\zeta(z)| \leqslant \frac{A_1}{\delta}, \quad \text{for } z \in \Omega, \tag{4.13}$$

$$\left| \frac{1}{\zeta - z} - R_\zeta(z) \right| \leqslant \frac{A_1 \delta^2}{|\zeta - z|^3}, \quad \text{for } z \in \Omega,\ |\zeta - z| \geqslant c_1 \delta, \tag{4.14}$$

where A_1 and c_1 are positive constants.

In view of (4.11) we define

$$Q_\delta(z) = -\int_Y \overline{\partial} f_\delta^*(\zeta)\, R_\zeta(z)\, dA(\zeta),$$
$$\varphi(z) = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{f_\delta^*(t)}{t - z}\, dt + Q_\delta(z),$$

where Q_δ is holomorphic in a neighborhood of $\widehat{K}$. Notice, that in order to prove the theorem it is sufficient to show that Q_δ is close to the function

$$\varphi_\delta(z) = -\frac{1}{\pi} \int_Y \frac{\overline{\partial} f_\delta^*(\zeta)}{\zeta - z}\, dA(\zeta),$$

namely, that

$$\|Q_\delta - \varphi_\delta\|_p \leqslant A_2 \omega_p(f, \delta) \tag{4.15}$$

for some constant $A_2 > 0$. Indeed, since φ is holomorphic in a neighborhood of $\widehat{K}$, Runge's theorem allow us to pick a polynomial P such that $\|\varphi - P\|_p \leqslant \omega_p(f, \delta)$. Therefore,

$$\|f - P\|_p \leqslant \|f - f_\delta^*\|_p + \|f_\delta^* - \varphi\|_p + \|\varphi - P\|_p \leqslant A_3 \omega_p(f, \delta),$$

for some positive constant A_3 because of (4.11), (4.12), (4.15), and the fact that $f_\delta^*(z) = 0$ for $z \notin Y$.

Thus, it remains to verify the estimate (4.15). In view of (4.13) and (4.14) we have

$$\|Q_\delta - \varphi_\delta\|_p \leqslant \|F_1\|_p + \|F_2\|_p + \|F_3\|_p \leqslant A_1(c_1 + 1)\|F_2\|_p + \|F_4\|_p, \tag{4.16}$$

where

$$F_1(z) = \int_{Y_1(z)} |\overline{\partial} f_\delta^*(\zeta)|\, |R_\zeta(z)|\, dA(\zeta),$$

$$F_2(z) = \int_{Y_1(z)} \left| \frac{\overline{\partial} f_\delta^*(\zeta)}{\zeta - z} \right| dA(\zeta),$$

$$F_3(z) = \int_{Y_2(z)} |\overline{\partial} f_\delta^*(\zeta)| \left| R_\zeta(z) - \frac{1}{\zeta - z} \right| dA(\zeta),$$

$$F_4(z) = \int_{Y_2(z)} \frac{|\overline{\partial} f_\delta^*(\zeta)|}{|\zeta - z|^3}\, dA(\zeta),$$

and $Y_1(z) = \{\zeta \in Y : |\zeta - z| < c_1\delta\}$, $Y_2(z) = \{\zeta \in Y : |\zeta - z| > c_1\delta\}$. The desired estimates of $F_2(z)$ and $F_4(z)$ was obtained in [123] as a result of using Hölder's inequality. Finally, the estimate (4.15) follows from (4.16), which finishes the proof. We skip here some details which can be found in [123]. ∎

Corollary 4.12. *Let U be a Carathéodory open set, and let $1 \leqslant p \leqslant +\infty$. Then, for each $f \in A^p(\overline{U})$ there exists a sequence of polynomials (P_n) such that $P_n \to f$ in $A^p(\overline{U})$ as $n \to \infty$.*

This result is a consequence of Sininjan's theorem and the fact that $K = \overline{U}$ is a Carathéodory compact set in the case under consideration. Even in the case that U is a domain this result cannot be obtained as a consequence of Theorem 4.1. The difference can happen if ∂U has positive area.

In [123] the following conjecture was made: *for $p \in (1, 2)$ and for every compact set K, the set of functions holomorphic in a neighborhood of K is dense in $A^p(K)$.* V. P. Havin in [63] solved this problem by proving the fact that $R^p(K) = A^p(K)$ for each compact set K and $p \in (1, 2)$. Here, $R^p(K)$ stands for L^p-closure of rational functions with poles lying outside K. The problem when $R^p(K) = A^p(K)$ for $p \in [2, +\infty)$ has a long history, and finally this problem was solved in terms of certain capacity conditions (or, in other words, in terms of $(1, q)$-stability), see, for instance, [65]. Since these results do not concern the class of Carathéodory sets, we will not continue this line of exposition.

Let us also say a few words about the case of harmonic polynomials. If E is a measurable set in $\mathbb{C}$, let us denote by $A^p_{har}(E)$ the set of all harmonic in E° functions of the class $L^p(E, \mathbb{R})$ The first result that the authors are aware of in connection with L^p-approximation by harmonic polynomials were obtained by A. L. Šaginjan in [117]. He proved that every bounded harmonic function on a given domain G belongs to $A^p_{har}(G)$ if G satisfies either of the following two conditions:

(i) G is a Carathéodory domain;

(ii) G is a moon-shaped domain and the real harmonic polynomials are dense in $A^p(G)$.

The next contribution was made by Sinanjan in [123, pages 99–101]. The respective result states as follows.

Theorem 4.13. *Let K be a Carathéodory compact set. Then, the set of real harmonic polynomials is dense in the space $A^p_{\mathrm{har}}(K)$ for every $p \geqslant 1$.*

Question IV. Whether the results stated in Theorems 4.11 and 4.13 hold true also for $p \in (0,1)$?

4.2 Some studies related with Hilbert space structure of $A^2(G)$

If $G \neq \emptyset$, then $A^2(G)$ is a separable Hilbert space with respect to the standard inner product $\langle f, g\rangle$ in $L^2(G)$, so that

$$\langle f, g\rangle = \int_G f\,\bar{g}\,dA.$$

We will use in this section all standard results from the Hilbert space theory without any special introduction and giving no references.

First of all let us give the second alternative proof of Theorem 4.1 in the case that $p = 2$ using some Hilbert space technique. In order to do that we need to show that the system of functions $\{1, z, z^2, \dots\}$ is complete in $A^2(G)$. The proof presented is due to A. L. Shaginyan, see [91].

Yet another proof of Theorem 4.1 for $p = 2$. Take a function $h \in A^2(G)$ and assume that $\langle h, z^m\rangle = 0$ for each $m \geqslant 0$. Then, it is enough to show that $h = 0$.

Given a big enough number $R > 0$, for w with $|w| > R$ we have

$$\int_G \frac{\overline{h(z)}}{z-w}\,dA(z) = -\sum_{n=0}^{\infty} \frac{1}{w^{n+1}} \int_G \overline{h(z)}\,z^n\,dA(z) = -\sum_{n=0}^{\infty} \frac{\langle z^n, h\rangle}{w^{n+1}} = 0.$$

Then,

$$\int_G \frac{\overline{h(z)}}{z-w}\,dA(z) = 0 \tag{4.17}$$

for every point $w \in G_\infty$, since G_∞ is a connected set.

Going further, take a sequence (Γ_n) of rectifiable Jordan curves such that the domains $G_n := D(\Gamma_n)$ converges to G with respect to some fixed point $z_0 \in G$ (in the sense of kernel convergence). Then, take as usual the sequence (ψ_n), $\psi_n\colon G_n \to G$, of conformal maps normalized by conditions $\psi_n(z_0) = z_0$ and $\psi_n'(z_0) > 0$. Multiplying (4.17) by $h(\psi_n(w))\psi_n'(w)$, integrating over Γ_n and applying Fubini's theorem and Cauchy integral formula we obtain

$$\int_G \overline{h(z)}\,h(\psi_n(z))\,\psi_n'(z)\,dA(z) = 0 \tag{4.18}$$

for all $n \geqslant 1$. By Lemma 4.10 we know that $(h \circ \psi_n)\psi_n' \to h$ as $n \to \infty$ in $A^2(G)$. Then, $\bar{h}(h \circ \psi_n)\psi_n' \to \bar{h}h$ as $n \to \infty$ in $A^1(G)$. By (4.18) one has $\int_G |h|^2\, dA = 0$, which yields $h = 0$. ∎

The next result is standard but it shows the special role that Carathéodory domains play in the theory.

Proposition 4.14. *Let G be a bounded domain in $\mathbb{C}$. Then, the following hold.*

(1) *In the space $A^2(G)$ there exists an orthonormal sequence of polynomials (P_n) such that* $\deg P_n = n$ *for all $n \geqslant 0$.*

(2) *This sequence (P_n) is uniquely determined whenever one demands that the coefficient of P_n at z^n is positive.*

(3) *If G is a Carathéodory domain, then this sequence (P_n) is an orthonormal basis.*

Proof. The construction of the desired system (P_n) is nothing else, then the standard Gram–Schmidt orthogonalisation process applied to the sequence of functions $\{1, z, z^2, \dots\}$ in the space $A^2(G)$. So, we need to verify the part (3) only.

Let G be a Carathéodory domain. We need to prove that the orthonormal system (P_n) such that $\deg P_n = n, n \geqslant 0$, is a basis in $A^2(G)$. Take a function $h \in A^2(G)$ and assume that $\langle h, P_n\rangle = 0$ for every $n \geqslant 0$. Since any polynomial Q of degree m may be represented as a linear combination of $P_0, P_1, \dots, P_m$, then $\langle h, Q\rangle = 0$, but since h can be approximated by a sequence of polynomials, then $\langle h, h\rangle = 0$, and hence $h = 0$. Thus, (P_n) is complete and hence it is an orthonormal basis for $A^2(G)$. ∎

Going further let us observe that the space $A^2(G)$ has a reproducing kernel for every nonempty domain G. Recall, that the reproducing kernel for $A^2(G)$, which is usually called the *Bergman kernel* for G, is a function $K: \mathbb{C}^2 \to \mathbb{C}$ such that $K(\cdot, w) \in A^2(G)$ for every $w \in G$ and $h(w) = \langle h, K(\cdot, w)\rangle$ for every $h \in A^2(G)$ and $w \in G$. It is well-known, that if (v_n) is some orthonormal basis in $A^2(G)$, then $K(z, w) = \sum_{n=0}^{\infty} \overline{v_n(w)} v_n(z)$.

Let now G be a Carathéodory domain. According to Proposition 4.14 there exists the orthonormal basis (P_n) in $A^2(G)$ consisting of polynomials (with $\deg P_n = n$). In this case, we have

$$K(z, w) = \sum_{n=0}^{\infty} \overline{P_n(w)} P_n(z). \tag{4.19}$$

Using this representation of reproducing kernel we are able to obtain the explicit expression for the conformal radius of G and for the corresponding conformal map from G onto $D(0, R)$.

Take a point $a \in G$, let R be the conformal radius of G with respect to a, and let g_0 be the conformal map from G onto $D(0, R)$ with the standard normalization

$g_0(a) = 0$ and $g_0'(a) = 1$. It follows from Proposition 2.1 that

$$\inf\left\{\int_G |h'(z)|^2\, dA(z) : h \in A^2(G),\ h(a) = 1\right\} = \int_G |g_0'(z)|^2\, dA(z) = \pi R^2. \quad (4.20)$$

Let us define the functions

$$K_m(z, w) = \sum_{n=0}^{m} \overline{P_n(w)} P_n(z), \quad m \in \mathbb{N}_0. \quad (4.21)$$

Then, using (4.20) for some appropriate h and making a bit of computations, we have

$$\int_G \left|\frac{K_m(z,a)}{K_m(a,a)}\right|^2 dA(z) = \frac{1}{\sum_{n=0}^{m} |P_n(a)|^2} \geqslant \pi R^2.$$

It gives

$$M_a := \sum_{n=0}^{\infty} |P_n(a)|^2 \leqslant \frac{1}{\pi R^2}.$$

Now, since $g_0' \in A^2(G)$ we have, in particular, $g_0'(z) = \sum_{n=0}^{\infty} c_n P_n(z)$ for all $z \in G$, where $c_n = \langle g_0', P_n \rangle$. Therefore,

$$\pi R^2 \leqslant \int_G \left|\frac{K(z,a)}{M_a}\right|^2 dA(z) = \frac{1}{M_a} \leqslant \sum_{n=0}^{\infty} |c_n|^2 = \int_G |g_0'|^2\, dA(z) = \pi R^2.$$

Therefore, $M_a = 1/(\pi R^2)$ and $K(z, a) = M_a g_0'(z)$ for all $z \in G$. So that we have proved the following result.

Proposition 4.15. *Let G be a Carathéodory domain, let $a \in G$, let R be the conformal radius of G with respect to a, and let g_0 be the conformal map from G onto $D(0, R)$ with the standard normalization $g_0(a) = 0$ and $g_0'(a) = 1$. Then,*

$$R = \frac{1}{\sqrt{\pi M_a}}, \quad \text{where } M_a = \sum_{k=0}^{\infty} |P_k(a)|^2;$$

$$K(z, a) = M_a g_0'(z) \quad \text{for all } z \in G;$$

$$g(z) = \frac{1}{M_a} \sum_{k=0}^{\infty} \overline{P_k(a)} \int_a^z P_k(\zeta)\, d\zeta \quad \text{for all } z \in G.$$

The representation of $K(z, w)$ in terms of conformal mapping (and vice-versa) given in this proposition may be adapted in a clear way for conformal mappings normalized by other ways. Thus, if g is the conformal map from G onto $\mathbb{D}$ such that $g(a) = 0$ and $g'(a) > 0$, then

$$g'(z) = \sqrt{\frac{\pi}{K(a,a)}} K(z, a), \quad K(z, a) = \frac{g'(a)}{\pi} g'(z). \quad (4.22)$$

In the case of a general conformal mapping g from G onto $\mathbb{D}$ (without special normalization) one has

$$K(z,a) = \frac{g'(z)\overline{g'(a)}}{\pi(1-\overline{g(a)}g(z))^2}. \tag{4.23}$$

In the simplest case that $G = D(0,R)$ for some $R > 0$ the corresponding function $K(\cdot,\cdot)$ and basis (P_n) in $A^2(G)$ may be easily computed:

$$P_n(z) = \frac{\sqrt{n+1}}{\sqrt{\pi}R^{n+1}}z^n, \quad n \geqslant 0 \quad \text{and} \quad K(z,w) = \frac{R^2}{\pi(R^2 - z\bar{w})^2}.$$

Remark 4.16. Going further we need to make the following observation.

(1) Let G_1 and G_2 be two simply connected domains, and let $\psi\colon G_2 \to G_1$ be a conformal map. Then, the map $f \mapsto (f \circ \psi)\,(\psi')^{2/p}$ provides an isometry of $A^p(G_1)$ onto $A^p(G_2)$ for each p, $0 < p < +\infty$.

(2) For instance, if $\{v_n : n \in \mathbb{N}\}$ is some orthonormal system in $A^2(G_1)$, then the system $\{(v_n \circ \psi)\,\psi' : n \in \mathbb{N}\}$ is an orthonormal system in $A^2(G_2)$.

(3) Let G be a simply connected domain, and let g be a conformal map from G onto $\mathbb{D}$ such that $g(a)=0$ and $g'(a)>0$ for some $a \in G$. Since $(\sqrt{(n+1)/\pi}z^n)$ is the orthonormal basis in $A^2(\mathbb{D})$, then the system of functions

$$\omega_n(z) = \sqrt{\frac{n+1}{\pi}}g(z)^n\, g'(z), \quad n \in \mathbb{N}_0$$

forms an orthonormal system in $A^2(G)$.

Example 4.17. In order to obtain yet another example of the orthonormal basis (P_n) in $A^2(G)$ for certain special domain G, let us consider the Cassini's oval $\{z : |z-1| \cdot |z+1| < \alpha\}$ with $\alpha \in (0,1]$. Let $G = O_\alpha$ be the component of this Cassini's oval lying in the right half-plane. The function $g\colon O_\alpha \to \mathbb{D}$ defined by $g(z) = (z^2-1)/\alpha$ gives the conformal map such that $g(1) = 0$ and $g'(1) > 0$. Then, according to the statement of the part (3) of Remark 4.16, an orthogonal basis in $A^2(O_\alpha)$ is formed from the polynomials

$$P_n(z) = \frac{2\sqrt{n+1}}{\alpha^{n+1}\sqrt{\pi}}\, z(z^2-1)^n, \quad n \geqslant 0.$$

The Bergman kernel for O_α may be also expresses explicitly:

$$K(z,w) = \frac{4\alpha^2\bar{w}z}{\pi(\alpha^2 - (\bar{w}^2-1)(z^2-1))^2}.$$

Let us now briefly describe the concept of a *Bieberbach polynomials* and their relations with Carathéodory domains. Let G be a domain in $\mathbb{C}$ and let $a \in G$. For each $n \geqslant 2$, let

$$\mathcal{P}_n(a) = \{P \in \mathcal{P} : \deg P = n,\ P(a) = 0,\ P'(a) = 1\}.$$

Definition 4.18. A polynomial $\pi_n \in \mathcal{P}_n(a)$ solving the following extremal problem

$$\int_G |\pi_n'(z)|^2 \, dA(z) = \inf\left\{\int_G |P'(z)|^2 \, dA(z) : P \in \mathcal{P}_n(a)\right\} \tag{4.24}$$

is called the nth Bieberbach polynomial (with respect to G and a).

The solution π_n of the extremal problem (4.24) always exists, because it is the primitive of a polynomial which is the orthogonal projection of 0 onto $\mathcal{P}_n'(a) = \{P' : P \in \mathcal{P}_n(a)\}$ in $A^2(G)$.

It turned out that in the case of Carathéodory domains the Bieberbach polynomials possess certain interesting and important properties, as it is shown in the following statement. For a given domain $G \subset \mathbb{C}$ let us recall that (P_n) is the orthonormal basis in $A^2(G)$ consisting of polynomials with deg $P_n = n$ and that the function $K_n(z, w)$, $n \in \mathbb{N}_0$ is defined by (4.21).

Proposition 4.19. *Let G be a Carathéodory domain, $a \in G$ and let g_0 map G conformally onto $D(0, R)$, where R is the conformal radius of G with respect to a (so that $g_0(a) = 0$ and $g_0'(a) = 1$). Then,*

$$\pi_n(z) = \pi_n(z; G, a) = \sum_{j=0}^{n-1} \frac{\overline{P_j(a)}}{K_{n-1}(a, a)} \int_a^z P_j(\zeta) \, d\zeta.$$

Moreover, $\pi_n' \to g_0'$ in $A^2(G)$, and hence $\pi_n \rightrightarrows g_0$ locally in G.

The proof of this proposition may be found in several sources, for example in [61, Chapter iii, Section 1].

If G is a bounded domain in $\mathbb{C}$ such that the space $A^2(G)$ admits an orthonormal basis consisting of polynomials, one can prove the existence of Bieberbach polynomials for such a domain. So, Carathéodory domain is one of the most suitable class of domains when the aforementioned condition is always fulfilled.

Let us make one more remark about the conditions of Proposition 4.19. Let G be a Carathéodory domain and let $\mathcal{E}$ be some end-cut of G such that Area$(\mathcal{E}) = 0$. Take $G_1 = G \setminus \mathcal{E}$. Then, the conditions determining the Bieberbach polynomials for G and for G_1 are the same (the corresponding extremal problem "does not see" $\mathcal{E}$), but it is clear that conformal maps from G onto $\mathbb{D}$ and from G_1 onto $\mathbb{D}$ differ significantly. So, certain condition that prevent "cuts" in domains under consideration is needed if we want to have results similar to Proposition 4.19, where the condition that G is a Carathéodory domain guaranties that G has no "cuts".

Let us give two examples showing how the Bieberbach polynomials look like.

(1) Let $G = \mathbb{D}$ and $a \in \mathbb{D}$, and let $b_n = \sum_{k=0}^{n-1}(k+1)|a|^{2k}$ for $n \in \mathbb{N}$, then

$$\pi_n(z; \mathbb{D}, a) = \frac{z-a}{b_n} + \frac{1}{b_n}\sum_{k=1}^{n-1} \bar{a}^k \left(z^{k+1} - a^{k+1}\right).$$

(2) Furthermore, the Bieberbach polynomials may be explicitly computed if the domain under consideration is $D(\Gamma_{\alpha,\beta})$, where $\Gamma_{\alpha,\beta}$ is the ellipse with semi-axes α and β for some $\alpha > \beta > 0$ having foci at the point ± 1 (so that $\alpha^2 - \beta^2 = 1$).
Let T_n and U_n, $n \geqslant 1$ stand for the Tchebyshev polynomials of the first and second kind, respectively. We recall, that $T_n(z) = \cos(n \arccos z)$ if $|\mathrm{Re} z| < 1$, and $U_n(z) = (n+1)^{-1} T'_{n+1}(z) = (1-z^2)^{-1/2} \sin((n+1) \arccos z)$.
It holds that the Bergman kernel for the domain $D(\Gamma_{\alpha,\beta})$ is

$$K(z,a) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{T'_{n+1}(z)\overline{U_n(a)}}{\rho^{n+1} - \rho^{-(n+1)}}, \qquad \rho = (\alpha+\beta)^2,$$

while the respective Bierberbach polynomials have the form

$$\pi_n(z; D(\Gamma_{\alpha,\beta}), a) = \frac{1}{K_{n-1}(a,a)} \sum_{j=0}^{n-1} \frac{(T_{j+1}(z) - T_{j+1}(a))\overline{U_j(a)}}{\rho^{j+1} - \rho^{-(j+1)}}.$$

Moreover, if g maps $D(\Gamma_{\alpha,\beta})$ conformally onto $\mathbb{D}$ with $g(0) = 0$ and $g'(0) > 0$, then

$$g(\cos w) = \frac{\pi}{2\sqrt{d}} \sum_{n=0}^{\infty} \frac{(-1)^n \cos((2n+1)w)}{\rho^{2n+1} - \rho^{-(2n+1)}},$$

where w belongs to the rectangle $\{w : 0 < \mathrm{Re}\, w < \pi,\ |\mathrm{Im} w| < c\}$ such that $\cosh c = \alpha$, while

$$d = \sum_{n=0}^{\infty} \frac{2n+1}{\rho^{2n+1} - \rho^{-(2n+1)}}.$$

The proof of these statements uses the fact that the system $(c_n U_n)$, where

$$c_n = \frac{4\pi}{n+1}\left(\rho^{n+1} - \rho^{-(n+1)}\right)$$

forms a basis in the space $A^2(D(\Gamma_{\alpha,\beta}))$, see [93, page 258].

According to Farrell's theorem (see Theorem 3.4) in order to have uniform convergence of the sequence (π_n) on $\overline{G}$, it is necessary that G is a Carathéodory domain and all prime ends of G are simple. A natural question arises now: *Are these conditions sufficient to have uniform approximation of the corresponding conformal mapping by Bieberbach polynomials*? The answer to this question is negative. In [71] a starlike Jordan domain G was constructed whose boundary is analytic except at one point such that the corresponding sequence of Bierberbach polynomials diverges on some dense subset of ∂G. We refer the reader who is interested in more information about Bierberbach polynomials, to the book [127].

One moment problem in $A^2(G)$ and in $A^1(G)$

Observe that Proposition 4.14 yields immediately the following proposition.

Corollary 4.20. *Let G be a Carathéodory domain. Let $h \in A^2(G)$ be such that*

$$\int_G h(z)\,\bar{z}^m\,dA(z) = 0, \quad \text{for each } m = 0, 1, 2, \dots . \tag{4.25}$$

Then, $h = 0$ in G.

Indeed, let (P_n) be the orthogonal basis in $A^2(G)$ given by Proposition 4.14. Thus, (4.25) implies the property $\langle h, P_m\rangle = 0$ for each $m \in \mathbb{N}_0$. Then, $h = 0$ in $A^2(G)$.

It is natural to consider the following question.

Question V. Let G be a Carathéodory domain and let $h \in A^1(G)$. Is it true that the condition (4.25) implies that $h = 0$ in G?

We are able to give a partial answer to this question by proving the following statement. The proof presented below is based on some results about pointwise approximation and it is quite short and simple. A different proof without using these tools may be found in [124, page 261].

Theorem 4.21. *Let G be a Carathéodory domain and let φ be a conformal map from G onto $\mathbb{D}$ such that $\|\varphi'\|_G \leqslant C$ for some constant $C > 0$. If the function $h \in A^1(G)$ is such that (4.25) is fulfilled, then $h = 0$ in G.*

In order to prove this theorem we need the following lemma.

Lemma 4.22. *Let G be a simply connected domain and assume that there exists $a \in G$ such that the Bergman kernel $K(\cdot, a)$ is bounded. Then, for all $b \in G$ the function $K(\cdot, b)$ is also bounded. Moreover, if $h \in A^1(G)$ then*

$$h(a) = \int_G h(z)\overline{K(z,a)}\,dA(z), \quad a \in G.$$

Proof. Let g_a be the conformal map from G onto $D(0, R_a)$ with $g_a(a)=0$, $g_a'(a)=1$ (so that R_a is the conformal radius of G with respect to a). Taking into account (4.22) and the hypothesis that $K(\cdot, a)$ is bounded, we obtain $|g_a'(z)| = \pi R_a^2 |K(z,a)| \leqslant C$ for each $z \in G$. Here, and in the sequel in this proof $C, C', \dots$ stand for some positive constants which may differ in different formulae. Take an arbitrary $b \in G$ and consider the analogous conformal map g_b constructed with respect to b. Then, $g_b \circ g_a^{-1}$ maps $D(0, R_a)$ onto $D(0, R_b)$, therefore this function is the restriction of a Möbius transformation, and hence,

$$|(g_b \circ g_a^{-1})'(w)| \leqslant C$$

for each $w \in D(0, R_a)$. Then, for every $w \in D(0, R_a)$ and $z = g_a^{-1}(w)$ we have (in view of (4.22)) that

$$|K(z,b)| = \frac{1}{\pi R_b^2}|g_b'(z)| \leqslant C|g_a'(z)| \leqslant C'.$$

So that $K(\cdot, b)$ is bounded.

Let now $h \in A^1(G)$. Put $g := g_a$ and $R := R_a$ and define $G^{(r)} := \{z \in G : |g(z)| < r\}$ for $0 < r < R$. Now, using (4.22) once again we obtain

$$\begin{aligned}
&\pi R^2 \int_G K(z,a)\, h(z)\, dA(z) \\
&= \lim_{r\to R} \int_{G^{(r)}} \overline{g'(z)}\, h(z)\, dA(z) \\
&= \lim_{r\to R} \int_{D(0,r)} \overline{g'(g^{-1}(w))}\, h(g^{-1}(w))|(g^{-1})'(w)|^2\, dA(w) \\
&= \lim_{r\to R} \int_{D(0,r)} \frac{h(g^{-1}(w))}{g'(g^{-1}(w))}\, dA(w) = \pi R^2 h(a),
\end{aligned}$$

where we have used (2.1) and, further, the mean area value property in $D(0, r)$. ■

Proof¶ *of Theorem* 4.21. Put $z_1 = \varphi^{-1}(0)$. In view of (4.22) we have

$$|K(z, z_1)| \leqslant \frac{1}{\pi}|\varphi'(z)\varphi'(z_1)| \leqslant C_1,$$

where $C_1 = C^2/\pi$. By virtue of Lemma 4.22 just proved, $K(\cdot, w)$ is bounded for all $w \in G$. Fix $a \in G$ and take the conformal map g from G onto $D(0, R)$ such that $g(a) = 0$, $g'(a) = 1$. By Lemmas 4.22 and (4.22), one has

$$h(a) = \int_G h(z)\, \overline{K(z,a)}\, dA(z) = \frac{1}{\pi R^2} \int_G h(z)\overline{g'(z)}\, dA(z).$$

Since g' is bounded (because $K(z, a)$ is also bounded), Theorem 3.25 tells us that there exists a sequence of polynomials (P_n) such that $P_n(z) \to g'(z)$ and $|p_n(z)| \leqslant C'$ for each $z \in G$ and for some positive constant C'. Then,

$$h(a) = \int_G h(z)\, \overline{g'(z)}\, dA(z) = \lim_{n\to\infty} \int_G h(z)\, \overline{p_n(z)}\, dA(z) = 0.$$

Because this holds for each $a \in G$ we have $h = 0$. ■

One estimate for polynomials forming orthogonal basis in $A^2(G)$

The main aim of the subsection is to prove the following result.

Theorem 4.23. *Let G be a Carathéodory domain, and let (P_n) be the orthogonal basis consisting of polynomials such that each P_n has degree n and its coefficient at z^n is positive. Then, for each $h \in H(\mathbb{C})$ there exists a sequence (a_n) such that*

$$h(z) = \sum_{n=0}^{\infty} a_n P_n(z),$$

where the series converges locally uniformly in G. Moreover, for every $\rho > 1$ there exists $C > 0$ such that

$$|P_n(z)| \leqslant C\rho^n, \quad n = 0, 1, 2, \dots, \; z \in \overline{G}.$$

Before proving this theorem we need to recall one construction related with the certain lemma due to Bernstein. Let K be a continuum and let $\Omega'_\infty(K) = \Omega_\infty(K) \cup \{\infty\}$. Then, there exists a unique conformal map Φ from $\Omega'_\infty(K)$ onto $\mathbb{C}_\infty \setminus \overline{\mathbb{D}}$ such that $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$. For a given number $\rho > 1$ let us define the set

$$L_\rho = \{z \in \Omega_\infty(G) : |\Phi(z)| = \rho\}.$$

Lemma 4.24 (Bernstein). *Let K be a continuum and let $F\colon G_\infty(K) \to \mathbb{C}$ be holomorphic function having a pole of order $n \geqslant 1$ at infinity. Assume that*

$$\lim_{\rho \to 1+} \sup_{z \in L_\rho} |F(z)| = M < +\infty.$$

Then, for every $\rho > 1$ it holds that $|F(z)| \leqslant M\rho^n$ for each $z \in L_\rho$.

Proof. Put $f(w) = F(\Phi^{-1}(w))/(\Phi^{-1}(w))^n$ for $|w| > 1$. Then, f is bounded in $\mathbb{C}_\infty \setminus \mathbb{D}$ and $\limsup_{|w| \to 1+} |f(w)| = \lim_{\rho \to 1+} \sup_{z \in L_\rho} |F(z)| = M$. Finally, the maximum modulus theorem implies that $|F(\Phi^{-1}(w))| \leqslant M|\Phi^{-1}(w)|^n$ for each $|w| = \rho$ as desired. ■

Proof of Theorem 4.23. Let $z^n = \sum_{k=0}^{n} b_k^n P_k$ for each n, and let (a'_n) be the Taylor coefficients of h. Then, $a_k = \sum_{n \geqslant k} a'_n b_k^n$. Let us prove the growth estimate for P_n. Let g be some fixed conformal map from G onto $\mathbb{D}$. Put $G^{(r)} := g^{-1}(D(0,r))$ for $0 < r < 1$. For $r \in (0,1)$ take some conformal map Φ_r from $G_\infty^{(r)} \cup \{\infty\}$ onto $\mathbb{C}_\infty \setminus \overline{\mathbb{D}}$, and take some conformal map Φ from $G_\infty \cup \{\infty\}$ onto $\mathbb{C}_\infty \setminus \overline{\mathbb{D}}$. Take an arbitrary $\rho > 1$ and define $L_{r,\rho} := \{z : |\Phi_r(z)| = \rho\}$ and $L_\rho = \{z : |\Phi(z)| = \rho\}$. Furthermore, let $G_{r,\rho} := D(L_{r,\rho})$ and $G_\rho := D(L_\rho)$. Then, $\overline{G} \subset G_{r,\rho}$ for some r sufficiently close to 1. Note that from (4.19) we have $|P_n(z)|^2 \leqslant K(z,z)$ and (4.23) implies that

$$\lim_{\rho \to 1} \sup_{z \in L_{r,\rho}} |P_n(z)| \leqslant \sup_{z \in \overline{G^{(r)}}} \frac{|g'(z)|}{\sqrt{\pi}(1 - |g(z)|^2)} = C_r < +\infty.$$

Therefore, one can apply Lemma 4.24 to $K = \overline{G^{(r)}}$ in order to obtain that $|P_n(z)| \leqslant C_r\rho^n$ for each $z \in G_{r,\rho}$ and for each $n \geqslant 0$. Since $\overline{G} \subset G_{r,\rho}$, the proof is finished. ■

4.3 Topics on weighted Bergman spaces

Let U be an open set, and let $\mathbf{w}\colon U \to [0, \infty)$ be a measurable function (a weight). For p, $1 \leqslant p < +\infty$, the weighted space $A^p(U, \mathbf{w})$ is defined as follows:

$$A^p(U,\mathbf{w}) = \left\{ f \in H(U) : \|f\|_{p,\mathbf{w}} = \left(\int_U |f(z)|^p \, \mathbf{w}(z) \, dA(z) \right)^{1/p} < +\infty \right\}.$$

In order that the space $A^p(U, \mathbf{w})$ to be complete with respect to the norm $\|\cdot\|_{p,\mathbf{w}}$ one needs to assume that $\mathbf{w}$ satisfies the condition that for each compact set $K \subset U$ there exists a constant $c_K > 0$ such that

$$c_K |f(a)|^p \leqslant \int_U |f(z)|^p \, \mathbf{w}(z) \, dA(z) \quad \text{for each } a \in K \text{ and } f \in A^p(U,\mathbf{w}). \tag{4.26}$$

This inequality may be regarded as an analogue of the estimate (4.7). It yields, in particular, that the convergence in $A^p(U, \mathbf{w})$ implies the locally uniform convergence in U. This fact shows that the space $A^p(U, \mathbf{w})$ is a Banach space for all p under consideration, while for $p = 2$ it is also a Hilbert space with respect to the inner product $\langle f, g\rangle = \int_U f \, \bar{g} \, \mathbf{w} \, dA$.

One important family of weights is the family $\{\mathbf{w} = |h| : h \in A^1(U) \text{ and } h \not\equiv 0\}$. In this case, (4.26) holds, and its proof is analogous to the proof of (4.7).

In the case, where $p = 2$ it is convenient to consider the weighted Bergman spaces $A^2(U, \mathbf{w})$ with respect to the weight $\mathbf{w} = |h|^2$, where $h \in A^2(U)$ and $h \not\equiv 0$. Such weights are called an analytic weights.

Denote by $P^p(U, \mathbf{w})$ the closure of the set $\mathcal{P}|_U$ in $A^p(U, \mathbf{w})$. The following question arises in a natural way: to describe U, p and $\mathbf{w}$ such that

$$P^p(U, \mathbf{w}) = A^p(U, \mathbf{w}). \tag{4.27}$$

Note that for $\mathbf{w} = 1$, Theorem 4.1 gives a sufficient condition for (4.27), which is that U needs to be a Carathéodory domain. But the problem just stated is very far from completely solved. In this section we are going to present some results concerning the matter which have certain connections with Carathéodory sets.

The equality (4.27) implies some restrictions on $\mathbf{w}$ and to U.

Proposition 4.25. *The following statements hold.*

(a) *Let* $\mathbf{w} = |h|^2$ *with* $h \in A^2(U)$. *If* h *has zeros in* U, *then* $P^2(U, \mathbf{w}) \neq A^2(U, \mathbf{w})$.

(b) *If* U *is not simply connected, then* (4.27) *does not hold for any* $p \geqslant 1$ *and for any weight* $\mathbf{w}$.

From now on we will assume that the open set U is simply connected, while $\mathbf{w}\colon U \to (0, +\infty)$. Let now G be a Carathéodory domain and $\mathcal{E}$ be some end-cut in G with $\operatorname{Area}(\mathcal{E}) = 0$. Then, in view of Proposition 4.5, we have $P^2(\Omega) \neq A^2(\Omega)$ for

$\Omega = G \setminus \mathcal{E}$. Thus, the hypothesis that G is a Carathéodory domain plays some role in the theory. However, for a $\mathbf{w} \neq 1$ the situation is more complicated, as it may be seen from the following example.

Example 4.26. Take $t \in (0, 1)$ and consider the function

$$h_t(z) = \exp\left(\frac{z+1}{z-1}\right)^t \quad \text{for } z \in \mathbb{D}.$$

If $0 < t < 1$ then $P^p(\mathbb{D}, |h_t|) = A^p(\mathbb{D}, |h_t|)$ and $P^p(\mathbb{D}, |h_1|) \neq A^p(\mathbb{D}, |h_1|)$ for each p with $1 \leqslant p < +\infty$.

The proof of the fact that $P^2(\mathbb{D}, |h_1|) \neq A^2(\mathbb{D}, |h_1|)$ is given in [73], where it was proved that the function

$$f(z) = \exp((1+z)/(2(1-z)))$$

does not belong to $P^2(\mathbb{D}, |h_1|)$. This proof may be also extended to all values of p under consideration. The first assertion follows from one result of Hedberg, that we will see in Example 4.37.

Let us discuss the case $p = 2$ and $\mathbf{w} = |h|^2$, $h \in A^2(U)$, in more detail. In this case, the map $f \in A^2(U, |h|^2) \mapsto fh \in A^2(U)$ is an isometry between the respective Hilbert spaces. This fact allow us to use the general Hilbert space tools for study the approximation problem under consideration. In particular, the equality (4.27) can be verified using the construction of orthogonal basis, or Bessel's inequality, or Parseval's formulae in $A^2(G)$ or in $A^2(U, \mathbf{w})$. The following lemma shows how conformal maps may be used in the theory.

Lemma 4.27 (Keldysh). *Let G be a simply connected domain, let f maps $\mathbb{D}$ conformally onto G, and let $\mathbf{w}$ be defined on G. Put $g = f^{-1}$. If $P^2(\mathbb{D}, \mathbf{w} \circ f) = A^2(\mathbb{D}, \mathbf{w} \circ f)$ and if $g^m g' \in P^2(G, \mathbf{w})$ for each $m \in \mathbb{N}_0$, then $P^2(G, \mathbf{w}) = A^2(G, \mathbf{w})$.*

Outline of the proof. Let $F \in A^2(G, \mathbf{w})$. Using (2.1) we have

$$\int_G |F(z)|^2\, \mathbf{w}(z)\, dA(z) = \int_{\mathbb{D}} |F(f(v))|^2\, \mathbf{w}(f(v))|f'(v)|^2\, dA(v) < +\infty,$$

so $(F \circ f)f' \in A^2(\mathbb{D}, \mathbf{w} \circ f)$. Then, given $\varepsilon > 0$ one can find a polynomial Q such that

$$\int_{\mathbb{D}} |F(f(v))f'(v) - Q(v)|^2\, \mathbf{w}(f(v))\, dA(v) =$$
$$= \int_G |F(z) - Q(g(z))g'(z)|^2\, \mathbf{w}(z)\, dA(z) < \varepsilon.$$

Since $(Q \circ g)g'$ is a sum of functions of the type $g^m g' \in P^2(G, \mathbf{w})$, $m \in \mathbb{N}_0$, we conclude that $F \in P^2(G, \mathbf{w})$. ■

Going further and working with space $A^2(U, |h|^2)$, let us consider the sequence (ψ_k), $\psi_k(z) = h(z)z^k$, $k \in \mathbb{N}_0$. Then, the Gram–Schmidt procedure in $A^2(U)$ applied to this sequence gives a new sequence (φ_k), $\varphi_k(z) = h(z)q_k(z)$, $q_k \in \mathcal{P}$, $\deg q_k = k$, and we can assume that the coefficient at z^k of q_k is positive. In [124, Section 3.1.8] one can find all details of this procedure. But for general simply connected open set U the sequence (φ_k) is not a basis for $A^2(U)$. Let us formulate the criterion in order that the equality (4.27) holds for $U = \mathbb{D}$ and $p = 2$.

Theorem 4.28. *Let $h \in A^2(\mathbb{D})$ and let h have no zeros in $\mathbb{D}$. The equality*

$$P^2(\mathbb{D}, |h|^2) = A^2(\mathbb{D}, |h|^2)$$

holds if and only if one of the following conditions is fulfilled:

(1) *There exists a sequence (u_n), $u_n \in \mathcal{P}$, such that* $\lim_{n\to\infty} \int_{\mathbb{D}} |1 - hu_n|^2 \, dA = 0$.

(2) *The sequence (φ_n), $\varphi_n = hq_n$, defined above is a basis for $A^2(\mathbb{D})$.*

(3) *It holds that $\sum_{k=0}^{\infty} |h(0)|^2 |q_k(0)|^2 = 1/\pi$.*

A few remarks about the proof of Theorem 4.28. Assume that the weight function h is such that $P^2(\mathbb{D}, |h|^2) = A^2(\mathbb{D}, |h|^2)$. Since $1/h \in A^2(\mathbb{D}, |h|^2)$ then there is a sequence of polynomials (u_n) such that

$$\int_{\mathbb{D}} \left| \frac{1}{h} - u_n \right|^2 |h|^2 \, dA = \int_{\mathbb{D}} |1 - hu_n|^2 \, dA \to 0.$$

Since hq_n, $n \in \mathbb{N}_0$, form an orthonormal system in $A^2(\mathbb{D})$, then q_n form an orthonormal system in $A^2(\mathbb{D}, |h|^2)$. Moreover, the construction of q_n implies that the closed linear span of (q_n) in $A^2(\mathbb{D}, |h|^2)$ coincides with the closure of $\mathcal{P}$ in this space. Therefore, (q_n) is a basis for $A^2(\mathbb{D}, |h|^2)$ and hence (φ_n) is a basis for $A^2(\mathbb{D})$. If (hq_k) is a basis for $A^2(\mathbb{D})$, then the Parseval's equality for $f = 1$ gives $\pi = \sum_{n=1}^{\infty} |\langle 1, hq_k \rangle|^2$. The fact that $\langle 1, hq_k \rangle = \pi h(0) q_k(0)$ implies (3).

Let us check the sufficiency of the conditions stated. Assume that (φ_n), $\varphi_n = hq_n$, is a basis for $A^2(\mathbb{D})$. Take $g \in A^2(\mathbb{D}, |h|^2)$ and observe that gh is the sum of its Fourier series $\sum c_n hq_n$ in $A^2(\mathbb{D})$. Since the partial sums of this series converges to gh in $A^2(\mathbb{D})$, then the partial sums $\sum_{n=1}^{m} c_n q_n$ converges to g in $A^2(\mathbb{D}, |h|^2)$. Thus, $g \in P^2(\mathbb{D}, |h|^2)$.

Assume now that (1) is satisfied. If $\lim_{n\to\infty} \int_{\mathbb{D}} |1 - h(z)u_n(z)|^2 \, dA(z) = 0$, then

$$\lim_{n\to\infty} \int_{\mathbb{D}} |z^m - h(z)z^m u_n(z)|^2 \, dA(z) = 0.$$

So, each polynomial is the limit in $A^2(\mathbb{D})$ of functions hp, where $p \in \mathcal{P}$. Since $A^2(\mathbb{D}) = P^2(\mathbb{D})$ then the set of hp, $p \in \mathcal{P}$, are dense in $A^2(\mathbb{D})$. It means that (hq_n) is a basis for $A^2(\mathbb{D})$.

If (3) holds, then the conclusion follows from the fact that if Bessel's inequality becomes the equality for a certain orthonormal system, then this system is a basis. ■

For a general simply connected domain we have the following sufficient condition.

Theorem 4.29. *Let G be a simply connected domain, and let h and g be as before. Assume that $g^m g' \in P^2(G)$ for each $m \in \mathbb{N}_0$. Then, $P^2(G,|h|^2) = A^2(G,|h|^2)$ if and only if there exists a sequence (u_n), $u_n \in \mathcal{P}$, such that*

$$\lim_{n\to\infty} \int_G |1 - hu_n|^2 \, dA = 0. \tag{4.28}$$

In view of (2.1) we have $\int_G |g^m(z)g'(z)|^2 \, dA(z) = \int_{\mathbb{D}} |z^m| \, dA(z) < +\infty$, so that $g^m g' \in A^2(G)$. Therefore, $g^m g' \in P^2(G)$ by Theorem 4.1. Then, Theorem 4.29 implies the following consequence.

Theorem 4.30. *Let G be a Carathéodory domain and let h be as before. Then, the equality $P^2(G,|h|^2) = A^2(G,|h|^2)$ holds if and only if there exists a sequence of polynomials (u_n) such that (4.28) is satisfied.*

Sketch of the proof. The proof of Theorem 4.29 is based on the following observation which is also a consequence of (2.1).

Let G_1 and G_2 be two simply connected domains, let f be some conformal map from G_1 onto G_2, and let $g = f^{-1}$. Take $h \in A^2(G_2)$. Then, the spaces $A^2(G_2,|h|^2)$ and $A^2(G_1,|h \circ f|^2)$ are isometric by means of the map $F \to (F \circ f)f'$. Its inverse is $R_1 \to (R_1 \circ g)g'$.

The fact that $g^m g' \in P^2(G)$ means that given $\varepsilon > 0$ there exists $q \in \mathcal{P}$ such that $\|g^m g' - q\|_{2,G} < \varepsilon$. In view of (2.1) one has

$$\int_G |g^m g' - q|^2 \, dA = \int_{\mathbb{D}} |z^m - q(f(z))f'(z)|^2 \, dA(z).$$

That means that the closed subspace generated in $L^2(\mathbb{D})$ by $\{(q \circ f)f : q \in \mathcal{P}\}$ is the same as the one generated by $\mathcal{P}$. The fact that $P^2(G,|h|^2) = A^2(G,|h|^2)$ is equivalent (in view of the isometry described above) to the fact that $P_1^2(\mathbb{D},|h \circ f|^2) = A^2(\mathbb{D},|h \circ f|^2)$, where $P_1^2(\mathbb{D},|h \circ f|^2)$ is the closure of the set $\{(q \circ f)f' : q \in \mathcal{P}\}$ which is dense. Then, one can find a basis in $P_1^2(\mathbb{D},|h \circ f|^2)$ and after that the proof may be finished using ideas from the proof of Theorem 4.28. ■

Let us now present two examples of situation when the condition (4.28) is fulfilled for a general Carathéodory domain G.

Example 4.31. Let G be a Carathéodory domain, and let $\alpha_1, \alpha_2, \ldots$ be real numbers such that $\alpha_k > -1$, $k \in \mathbb{N}$, among which there is only a finite number of negative ones, and $\sum_{k=1}^{\infty} \alpha_k < +\infty$. Take a sequence of points (z_k), $z_k \notin G$, in such a way that α_k needs to be integer if $z_k \in \partial_a G$. Finally, fix a point $a \in G$. Then, the function

$$\Phi(z) = \sum_{k=1}^{\infty} \alpha_k \log\left(1 - \frac{z-a}{z_k - a}\right), \quad z \in G,$$

where the branch of logarithm is defined on G and equal to zero at a, is well defined, and the weight

$$h(z) = C \prod_{k=1}^{\infty} \left(1 - \frac{z-a}{z_k - a}\right)^{\alpha_k}$$

is such that (4.28) is satisfied, and, therefore, the equality $A^2(G, |h|^2) = P^2(G, |h|^2)$ holds.

The proof of the fact that this function h satisfies (4.28) is rather involved. All details may be found in [124, Section 3.2.3]. The starting step of this proof is to show that $h \in A^2(G)$ which is not difficult. Later on it is needed to consider three consecutive cases. The first one is related with the simplest possible function $\Phi(z) = \alpha \log(1 - (z-a)/(z_1 - a))$ constructed by one point z_1. This case is analyzed with the help of the special analogue of Mergleyan's key lemma. The second case is related with the finite set of points $\{z_1, \dots, z_n\}$. The important ingredient of the proof in the general case is the fact that the product of two functions h_1 and h_2 satisfying (4.28) is again the function satisfying this property.

Example¶ 4.32. Let G be a Carathéodory domain. If $h \in H(\overline{G^*})$, and $h(z) \neq 0$ for each $z \in G$, then the property (4.28) holds for h, and hence $A^2(G, |h|^2) = P^2(G, |h|^2)$.

It turns out that for general weights $\mathbf{w}$, the assumption that G is a Carathéodory domain is not necessary in order to have (4.27) because the following result holds.

Proposition 4.33. *Let $G \neq \emptyset$ be a simply connected domain in $\mathbb{C}$. Then, there exists a weight $\mathbf{w}$ such that $P^2(G, \mathbf{w}) = A^2(G, \mathbf{w})$.*

Outline of the proof. Let g be (as before) a conformal mapping from G onto $\mathbb{D}$, and take an increasing sequence (ρ_n) of positive real numbers. Define $G_n = \{z \in G : |g(z)| < \rho_n\}$. For each k the function $g^k g'$ is holomorphic in $\overline{G_n}$, so there exist polynomials $Q_{n,k}$ such that

$$\int_{G_n} |g^k g' - Q_{n,k}|^2 \, dA < \frac{1}{2^n}.$$

Also it is possible to find $\alpha_k > 0$ such that

$$\alpha_k \int_{G \setminus G_n} |g^k g' - Q_{n,k}|^2 \, dA < \frac{1}{2^n}.$$

Then, the function h may be defined on $G_{n+1} \setminus G_n$ as $h(z) = \min\{1, \alpha_1, \alpha_2, \ldots, \alpha_n\}$, the details of the proof may be found in [73]. ■

Now, we will mention some general results for $1 \leqslant q < +\infty$ that were proved in [64]. Let G be a Carathéodory domain, let $\mathbf{w}\colon G \to (0, +\infty)$ be continuous functions such that $\mathbf{w} \in L^1(G, dA)$. Define

$$M(\mathbf{w}, z, r) = \frac{1}{r^2} \int_{|w| \leqslant r} \mathbf{w}(z + \zeta)\, dA(\zeta), \quad z \in G,\ r > 0,$$

where $\mathbf{w}(z + \zeta) = 0$ if $z + \zeta \notin G$.

Theorem 4.34. *Let G, $\mathbf{w}$, M be as before, and let $g \in A^q(G, \mathbf{w})$.*

(1) *Let $q > 1$. If $\sup_{r>0} \int_G |g(z)|^q\, M(\mathbf{w}, z, r)\, dA(z) < +\infty$, then $g \in P^q(G, \mathbf{w})$.*

(2) *Let $q = 1$. If $\int_G |g(z)| \sup_{r>0} M(\mathbf{w}, z, r)\, dA(z) < +\infty$, then $g \in P^1(G, \mathbf{w})$.*

One of the crucial ingredient of the proof of this theorem is the fundamental Mergelyan's lemma. The proof is obtained as an appropriate combination of this lemma, duality arguments and standard L^p-estimates.

The next result gives yet another generalization of Theorem 4.1.

Corollary 4.35. *Let G and $\mathbf{w}$ be as before, let $1 \leqslant p < +\infty$, and assume that $\mathbf{w} \in L^s(G)$ for some s, $1 < s \leqslant +\infty$. If $v \in A^p(G, \mathbf{w}) \cap L^{pt}(G)$, where $1/s + 1/t = 1$, then $v \in P^p(G, \mathbf{w})$.*

Notice that Theorem 4.34 give a sufficient approximation condition for individual functions. For the special classes of weights it is also possible to find sufficient approximation conditions for classes of functions.

Theorem 4.36. *Let G be a Carathéodory domain, and let $\mathbf{w} = |h|$, where $h \in A^1(G)$ and $|h(z)| > 0$ for all $z \in G$.*

(1) *If there exist $\varepsilon > 0$ such that*

$$\sup_{r>0} \int_G |h(z)|^{-\varepsilon}\, M(|h|, z, r)\, dA(z) < +\infty,$$

then $P^q(G, |h|) = A^q(G, |h|)$ for all $q \in (1, +\infty)$.

(2) *If there exist $\varepsilon > 0$ such that*

$$\int_G |h(z)|^{-\varepsilon} \sup_{r>0} M(|h|, z, r)\, dA(z) < +\infty,$$

then $P^1(G, |h|) = A^1(G, |h|)$.

Let us also present some class of weights $\mathbf{w} = |h|$ constructed in [64] such that $A^q(G, |h|) = P^q(G, |h|)$ for all $q \in [1, \infty)$ for a given general Carathéodory domain G.

Example 4.37. Let G be a Carathéodory domain, $q \in [1, \infty)$, and let $v \in H(G)$ be such that $\operatorname{Re} v > 0$ in G. Put $h(z) = e^{-v(z)}$ and assume that there exist two positive constants, says c_1 and c_2, such that $|\operatorname{Im} v| \leqslant c_1 \operatorname{Re} v + c_2$ in G. Then, $P^q(G, |h|) = A^q(G, |h|)$.

In order to prove that $P^q(G, |h|) = A^q(G, |h|)$ we need firstly to show that $v \in P^q(G, |h|)$. Notice, that the function $v_1 = (v-1)/(v+1)$ is bounded by 1 in G. Then, $v_1^m \in P^q(G, |h|)$ for all $m \in \mathbb{N}$. It gives that

$$v = \frac{1+v_1}{1-v_1} = 1 + 2\sum_{n=1}^{\infty} v_1^n \in P^q(G, |h|).$$

By induction one can prove that $v^n \in P^q(G, |h|)$. Now, for each $z \in G$, we have

$$\sum_{n=0}^{N} \frac{t^n v(z)^n}{n!} \to e^{tv(z)} = \frac{1}{|h(z)|^t}.$$

This convergence holds in $L^q(G, |h|)$ for each t such that $0 \leqslant t \leqslant t_h = (qc_1 + q)^{-1}$, it can be proved using Lebesgue's dominated convergence theorem. Thus, $|h|^{-t_h} \in P^q(G, |h|)$. Some more argument is needed to conclude the desired approximation result from Theorem 4.36 with $\varepsilon = 1/(c_1 + 1)$.

4.4 Topics on Hardy spaces

We start by recalling some basic facts about Hardy spaces in general domains in the complex plane. An appropriate reference for the next statements is [54]. During this section p will denote a number belonging to $(0, \infty)$ (we will mention below only the special restrictions on p, if needed). Let $G \subset \mathbb{C}$ be a bounded domain. *By definition* $H^p(G)$ is the space consisting of all functions $f \in H(G)$ for which there exists a positive harmonic function u in G such that $|f(z)|^p \leqslant u(z)$ for each $z \in G$. Such function u is called a harmonic majorant of $|f|^p$ in G. If $f \in H^p(G)$, then there exists a unique least harmonic majorant u_f such that $|f|^p \leqslant u_f$ in G. Then, we put $\|f\|_{H^p(G)} = (u_f(z_0))^{1/p}$, where $z_0 \in G$ is some fixed point. In the case that $p \geqslant 1$ this quantity is a norm in $H^p(G)$ and the resulting topology is independent on the choice of z_0.

Lemma 4.38. *Let K be a compact subset of G. Then, there exist a constant $C = C(K, p)$ such that for every $f \in H^p(G)$ and $z \in K$ one has $\|f\|_K \leqslant C\|f\|_{H^p(G)}$.*

Using this lemma it may be readily obtained that $H^p(G)$ is a Banach space for $p \geqslant 1$. Also it is well-known that $H^p(G)$ is conformally invariant, that is if φ is a conformal map from some domain G_1 onto another domain G_2, then $H^p(G_1) = \{g \circ \varphi : g \in H^p(G_2)\}$ and both these spaces are isometric.

Lemma 4.39. *Let $f \in H(G)$. Then, $f \in H^p(G)$ if and only if for each C^1-exhaustion (Ω_n) of G there exists a constant C such that for some point $a \in G$ one has*

$$\int_{\partial\Omega_n} |f|^p(\zeta)\, d\omega(a,\zeta,\Omega_n) \leqslant C.$$

In the case when $G = \mathbb{D}$ the spaces $H^p(\mathbb{D})$ are the classical Hardy spaces in the unit disk, and they are denoted usually by H^p.

Lemma 4.40. *If $f \in H^p$ then there exists a sequence of polynomials (P_n) which converges to f in norm in H^p.*

The principal part of the proof of this lemma is to prove the fact that $f_r \to f$ in H^p as $r \to 1$, where $f_r(z) = f(rz)$, see, for instance, [77, page 71]. After that it remains to observe that the Taylor series of f_r converges uniformly on $\overline{\mathbb{D}}$, and hence the Taylor polynomials of f give the desired approximation.

Let us also recall, that the Hardy spaces $H^p(\mathbb{T})$ on the unit circle are the spaces consisting of all functions $h \in L^p(\mathbb{T})$ such that $\int_{\mathbb{T}} h(\zeta)\overline{\zeta}^n\, dm_{\mathbb{T}}(\zeta) = 0$ for every integer $n < 0$. According to Fatou's theorem, every function $f \in H^p$ has a.e. on $\mathbb{T}$ angular boundary values, which determine a function in the class $H^p(\mathbb{T})$. The mapping which maps a function $f \in H^p$ to its boundary function is an isometric isomorphism between the spaces H^p and $H^p(\mathbb{T})$. When $p = \infty$, this mapping is also a weak-star homeomorphism. In what follows, functions in H^p and their boundary functions will be denoted by the same symbols.

Weak-star generators in H^∞

The space H^∞ is isometric to $H^\infty(\mathbb{T})$, while this space is a subset of $L^\infty(\mathbb{T})$ which is isometric to the dual space of $L^1(\mathbb{T})$. Then, we can consider the weak-star topology in $L^\infty(\mathbb{T})$, with a basis of neighborhoods of zero formed by sets

$$\left\{ f \in L^\infty(\mathbb{T}) : \left| \int_0^{2\pi} f(e^{it}) g_j(e^{it})\, dt \right| < r_j,\ j = 1,\dots,n \right\}$$

for all possible choice of numbers $r_1,\dots,r_n > 0$ and functions $g_1, g_2, \dots, g_n \in L^1(\mathbb{T})$. The weak-star topology in H^∞ is that induced by the isometry. For some mote detailed explanation of weak topologies see, for instance, [114, Chapter 3]. Notice also, that the aforesaid week topologies are not metrizable in the general case. Then, the sequences are not enough to manage with this topology. For sequences in H^∞ the convergence in the weak-star topology can be easily characterized.

Lemma 4.41. *The following statements hold.*

(a) *Let $f_n \in H^\infty$. Then, the sequence (f_n) converges in the weak-star topology to f if and only if this sequence is uniformly bounded in $\mathbb{D}$ and $f_n(z) \to f(z)$ for each $z \in \mathbb{D}$.*

(b) *Let $f_\alpha \in H^\infty$ be a net. Then, if (f_α) converges in the weak-star topology to some f, then $\sup_{\alpha, z\in\mathbb{D}} |f_\alpha(z)|$ is finite and $f_\alpha(z) \to f(z)$ for each point $z \in \mathbb{D}$.*

This is a well-known result. The proof of the part (a) may be found in [120]. The proof of the part (b) is similar.

The next definition was given by D. Sarason in [119].

Definition 4.42. Let $\varphi \in H^\infty$. Then, the following hold.

(a) φ is a weak-star generator if the set $\{P \circ \varphi : P \in \mathcal{P}\}$ is weak-star dense in H^∞.

(b) φ is a (weak-star) sequential generator if every function in H^∞ is the weak-star limit of a sequence of polynomials in φ.

It is clear that a sequential generator is a weak-star generator, but the converse is very far to being true, as it will be shown later. The main reason to introduce the concept of a weak-star generator was because of its relations with the theory of invariant subspaces for certain multiplication operators. Let us recall, that for a given function $\varphi \in H^\infty$ the operator $S_\varphi\colon L^2(\mathbb{T}) \to L^2(\mathbb{T})$ acts as follows: $S_\varphi\colon h \mapsto \varphi h$, while the Toeplitz operator $T_\varphi\colon H^2 \to H^2$ is defined as follows: $T_\varphi\colon h \mapsto P_+(\varphi h)$, where P_+ stands for the orthogonal projection from $L^2(\mathbb{T})$ to H^2. In the special case when $\varphi = \jmath$ the operator S_z is called the *bilateral shift*, while the operator T_z is called the *unilateral shift*. The study of shift-invariant subspaces in H^2 was initiated by Beurling, Helson–Lowdenslager and Halmos. The following simple fact whose proof may be found in [69, page 106] shows the specific role of the unilateral shift in the topic under consideration.

Lemma 4.43. *Let E a closed subspaces of H^2. Then, E is T_z-invariant if and only if it is T_φ-invariant for all $\varphi \in H^\infty$.*

The descriptions of shift-invariant (closed) subspaces of H^2 and $L^2(\mathbb{T})$ are well-known; they are given by the following nowadays become classical results whose proofs may be found in [69, Chapter 7].

Theorem 4.44 (Beurling). *Let E be a non-empty closed subspace of H^2. Then, E is T_z-invariant if and only if $E = K_\Theta = \Theta H^2$, where Θ is an inner function.*

We recall, that a function $\Theta \in H^\infty$ is said to be *inner*, if $|\Theta(\zeta)| = 1$ for a.a. $\zeta \in \mathbb{T}$.

Theorem 4.45. *Let W be a closed S_z-invariant subspace of $L^2(\mathbb{T})$.*

(a) *If $zW = W$, then $W = \{f \in L^2(\mathbb{T}) : f|_B = 0\}$, where $B \subset \mathbb{T}$ is some Borel set.*

(b) *If $zW \neq W$, then $W = FH^2$, where F is a measurable function on $\mathbb{T}$ of modulus 1.*

For non closed subspaces the situation is fairly different. Let us mention in this connection the next result obtained in [119].

Proposition 4.46. *Let $\varphi \in H^\infty$. Then, the following are equivalent.*

(a) *φ is a weak-star generator of H^∞.*

(b) *The operator S_φ has the same invariant subspaces as S_z.*

(c) *The operator T_φ has the same invariant subspaces as T_z.*

Two simple necessary conditions for a function φ to be a weak-star generator of H^∞ were obtained in [119].

Proposition 4.47. *Let $\varphi \in H^\infty$ be a weak-star generator of H^∞. Then, the following hold.*

(i) *φ in univalent on $\mathbb{D}$.*

(ii) *There exists a set $I \subset \mathbb{T}$ such that $m_{\mathbb{T}}(I) = 0$ and $\varphi|_{\mathbb{T}\setminus I}$ is injective.*

If some function φ satisfies the second condition of this proposition we will call it *univalent almost everywhere on* $\mathbb{T}$.

Sketch of the proof of Proposition 4.47. Let φ be a weak-star generator of H^∞ and assume that $\varphi(a) = \varphi(b)$ for some $a, b \in \mathbb{D}$ with $a \neq b$. Then, there exist a family $\{P_\alpha\}$ of polynomials such that the net $P_\alpha(\varphi)$ converging to z in the weak-star topology. Fix $a \in \mathbb{D}$. Then, the point evaluation functional $f \mapsto f(a)$, defined for each $f \in H^\infty$, is weak-star continuous because $f(a)$ is obtained via the standard Poisson formula and the Poisson kernel belongs to $L^1(\mathbb{T})$. This continuity implies $a = \lim P_\alpha(\varphi(a)) = \lim P_\alpha(\varphi(b)) = b$, which gives a contradiction.

Because the evaluation at an arbitrary point e^{it} is not defined for $f \in H^\infty$ in the general case, the proof of the second condition needs to be different from the previous one. Let E be the closed span of the elements $\{1, \varphi, \varphi^2, \dots\}$ in $L^2(\mathbb{T})$. So that $S_\varphi E \subset E$. If φ is a weak-star generator then, by Proposition 4.46, the space E is also S_z-invariant, and hence $z \in E$. Then, there exists a sequence (P_n) of polynomials that converges to z in $L^2(\mathbb{T})$. So, there is a partial subsequence (that will be denoted by the same symbol) such that

$$P_n(\varphi(e^{it})) \to e^{it} \quad \text{for a.e } e^{it} \in \mathbb{T}. \tag{4.29}$$

If we assume that for each measurable set $M \subset \mathbb{T}$ of positive measure there exist two points $e^{it} \in M$ and $e^{is} \in M$ with $e^{it} \neq e^{is}$ and $\varphi(e^{it}) = \varphi(e^{is})$ we arrive to a contradiction with (4.29). So, the second property also holds. ■

If φ a weak-star generator of H^∞, then $G = \varphi(\mathbb{D})$ is a simply connected domain and ∂G cannot have a lot of cut points. For example, the set of all cut points of ∂G should have harmonic measure zero with respect to $\varphi(0)$. It implies that each conformal map from $\mathbb{D}$ onto $\mathbb{D} \setminus [0, 1)$ is not a weak-star generator. The statement of the part (ii) of Proposition 4.47 was improved in [108] in the way shown in the next theorem.

Theorem 4.48. *The following statements hold.*

(a) *If φ is a weak star generator of H^∞ then the boundary function defined on $F(\varphi)$ is one to one.*

(b) *There exists a bounded univalent function φ in $\mathbb{D}$ such that $F(\varphi) = \mathbb{T}$ and φ is injective on $\overline{\mathbb{D}}$, but it is not a weak-star generator of H^∞.*

The proof of the part (a) is a bit technically involved since it uses classical results about conformal maps together with certain tools from ordinal number theory. To verify the statement (b) it is enough to take the conformal map from $\mathbb{D}$ onto the domain G_2 in Figure 1, but some work is needed in order to show that it is not a weak-star generator.

Following Sarason let us pay attention to the sequential generators of H^∞ because it admits certain characterizations in topological terms.

Proposition 4.49. *Let φ be a conformal map from $\mathbb{D}$ onto a simply connected domain $G \subset \mathbb{C}$. Then, φ is a sequential generator of H^∞ if and only if G has the following property: for every $h \in H^\infty(G)$, there is a sequence of polynomials which is uniformly bounded on G and converges to f at every point of G.*

Proof. Assume that φ is a sequential generator and let $h \in H^\infty(G)$. Then, $h \circ \varphi \in H^\infty$ and let (P_n) be such sequence of polynomials that $P_n(\varphi(z)) \to h(\varphi(z))$ for every $z \in \mathbb{D}$, and $p_n \circ \varphi$ is uniformly bounded on $\mathbb{D}$. Then, $P_n(w) \to h(w)$ for every $w \in G$ and P_n is uniformly bounded in G. For the converse it enough to consider $g \circ \varphi^{-1}$ for $g \in H^\infty$. ∎

Corollary 4.50. *A conformal map onto a Jordan domain is always a weak-star generator. A conformal map onto a moon-shaped domain is never a weak-star generator.*

Now, combining Proposition 4.49 with Theorem 3.31 we arrive at the following result (see [120, Proposition 2]).

Proposition 4.51. *Let G be a bounded simply connected domain and let φ a conformal map from $\mathbb{D}$ onto G. Then, φ is a sequential generator of H^∞ if and only if G is a component of the set G^* (the latter property exactly means that G is a Carathéodory domain).*

We will denote by $\widetilde{G}$ the component of G^* that contains G. With the same notations as in Proposition 4.49 the following result holds.

Proposition 4.52. *Let $h \in H^\infty$. Then, h is the weak star limit of a sequence of polynomials on φ if and only if $h \circ \varphi^{-1}$ is the restriction of a function belonging to $H^\infty(\widetilde{G})$.*

Sarason in [120] has obtained a characterization of weak generators, adapting the statement of Farrell's theorem to a certain more general setting. This is a reason to give here a simple overview of his results. Also we believe that the notion of order of a simply connected domain introduced by Sarason may be regarded as a further generalization of the concept of a Carathéodory domain.

Take $\varphi \in H^\infty$. Denote by M^0 the set of polynomials in φ, that is $M^0 = \{P \circ \varphi : \varphi \in \mathcal{P}\}$. Furthermore, let M^1 denote the set of all weak-star limits of sequences of functions in M^0. If α is a countable ordinal we define M^α inductively to be the linear manifold of H^∞ consisting of all functions which are weak-star limits of sequences of functions on $\bigcup_{\beta<\alpha} M^\beta$. By a property of weak topologies, see [7, pages 124, 213], there exists a least countable ordinal α' such that $M^{\alpha'} = M^\beta$ if $\beta > \alpha'$. Moreover, $M^{\alpha'}$ is the weak closure of M^0. We say that φ is a generator of H^∞ of order α' if $M^{\alpha'} = H^\infty$.

In order to understand the definition of the order of a simply connected domain we need the following definition.

Definition 4.53. Let G be a bounded domain in $\mathbb{C}$, and let Ω be a simply connected domain such that $G \subset \Omega$. The relative hull of G in Ω (or, for brevity, Ω-hull) is the set

$$\operatorname{Int}\Big\{w \in \Omega : |f(w)| \leqslant \sup_{z\in G} |f(z)| \text{ for every } f \in H^\infty(\Omega)\Big\}.$$

One crucial step in Sarason's papers is to show that if $G \subset \mathbb{D}$, then the $\mathbb{D}$-hull of G is G^*. Also a geometric description of the Ω-hull of G is given by the next result.

Proposition 4.54. *Let Ω and G be as before and let Y be the closure in Ω of the Ω-hull of G. Then, $\Omega \setminus Y$ consists of those points of Ω that can be separated from G by a cross cut of Ω. Moreover, the Ω-hull of G is the interior of Y.*

With these tools the following generalization of Farrell's result is readily followed.

Theorem 4.55. *Let Ω be a domain and let G be a bounded simply connected domain such that $G \subset \Omega$. Denote by G_Ω the component of the Ω-hull of G that contains G. Let $f \in H^\infty(G)$. Then, a sequence of bounded holomorphic functions in Ω which is uniformly bounded in G and converges to f at the every point of G exists if and only if f is the restriction of some function $f_1 \in H^\infty(G_\Omega)$.*

Let G be a simply connected domain. For every countable ordinal α let us define inductively a domain G^α containing G as follows. For $\alpha = 1$ we put G^1 as the component of G^* that contains G. If α has an immediate predecessor we define G^α as the component of the $G^{\alpha-1}$-hull of G that contains G. If α has no immediate predecessor we define G^α to be the component of the interior of $\bigcap_{\beta<\alpha} G^\beta$ that contains G. Then, G^α is simply connected and, moreover, there exists a least countable ordinal γ such that $G^{\gamma} = G^{\gamma+1}$. So, $G^{\gamma} = G^{\omega}$ for $\omega > \gamma$. This γ is called the order of G. For better understanding this notion the reader can see that both domains in Figure 3 have order 1. Next result generalizes Proposition 4.52.

Proposition 4.56. $M^{\alpha} = \{h \in H^{\infty} : h \circ \varphi^{-1} = F|_G$ *for some* $F \in H^{\infty}(G^{\alpha})\}$.

Corollary 4.57. *Now, the characterization of generators of* H^{∞} *obtained by Sarason can be stated in the following form.*

(1) *If* φ *is a generator of* H^{∞} *of order* γ *then the domain* $G = \varphi(\mathbb{D})$ *has order* γ *and* $G^{\gamma} = G$. *Conversely, if a given domain* G *has order* γ *and* $G^{\gamma} = G$, *then every conformal mapping* φ *from* $\mathbb{D}$ *onto* G *is a generator of* H^{∞} *of order* γ.

(2) *The function* $\varphi \in H^{\infty}$ *fails to be a generator of* H^{∞} *if and only if there exists a domain* Ω *which properly contains* G *and is such that* $\|f\|_{\Omega} = \|f\|_G$ *for every* $f \in H^{\infty}(\Omega)$.

(3) *If* φ *is a generator of* H^{∞}, *then* $G = \operatorname{Int}(\overline{G})$.

We refer to [120, Figures 1 and 2] to see domains which are the images of $\mathbb{D}$ under mapping by weak-star generators of order 2 and 3, respectively. The orders of these domains can be computed by using Proposition 4.54. In [121] the author was able to construct domains of arbitrary order using the fact that every countable well-ordered set can be realized as a subset of $\mathbb{R}$. We do not know whether it is possible to obtain some other type of characterization of weak-star generators avoiding, in particular, the usage of ordinals.

Finally, let us notice that the concept of domains of order 2 is underlying the result of [112, Theorem 4.1], so this theorem was the precursor of Sarason's studies.

Density of polynomials in $H^p(G)$

Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be a non-constant holomorphic function. The composition operator $C_{\varphi}\colon H(\mathbb{D}) \to H(\mathbb{D})$ is defined by the setting $C_{\varphi}(f) = f \circ \varphi$. If $\varphi(0) = 0$ then the Littlewood subordination theorem (see, for example, [42, Theorem 1.7]) implies

$$\|f \circ \varphi\|_{H^p} \leqslant \|f\|_{H^p}$$

for each $p \in (0, \infty)$. If $\varphi(0) \neq 0$ then $\|f \circ \varphi\|_{H^p} \leqslant M\|f\|_{H^p}$, where M is some constant depending only on $|\varphi(0)|$. Thus, $C_{\varphi}\colon H^p \to H^p$ is a bounded operator for each p, $1 \leqslant p \leqslant \infty$. A lot of efforts were applied for studying of such operators. In particular, in [31] the problem when $C_{\varphi}(H^p)$ is dense in H^p were considered.

We have the following clear facts.

Lemma 4.58. *If* $C_{\varphi}(H^p)$ *is dense in* H^p *for some* $0 < p < \infty$, *then* φ *is univalent in* $\mathbb{D}$.

Proof. Indeed, if $\varphi(z) = \varphi(w)$ for some $z \neq w$ in $\mathbb{D}$, then $f(\varphi(z)) = f(\varphi(w))$ for each $f \in H^p$. So, the function j does not belong to the closure of $C_\varphi(H^p)$ in H^p. ■

Lemma 4.59. *Let φ maps conformally $\mathbb{D}$ onto some domain $G \subset \mathbb{D}$, while $0 < p < \infty$. Then, $C_\varphi(H^p)$ is dense in H^p if and only if the set of polynomials is dense in $H^p(G)$.*

Proof. If $f \in H^p(G)$, then $f \circ \varphi \in H^p$. Therefore, if $C_\varphi(H^p)$ is dense in H^p, then Lemma 4.40 yields that there exists a sequence (P_n) of polynomials such that $P_n \circ \varphi \to f \circ \varphi$ in H^p, which imply that $P_n \to f$ in H^p. The converse is clear. ■

The following theorem was proved in [31].

Theorem 4.60 (Caughran). *Let p be such that $1 \leqslant p \leqslant \infty$. If G is a Carathéodory domain, then the set of polynomials is dense in $H^p(G)$. Conversely, if polynomials are dense in $H^p(G)$ and $\varphi \in C(\overline{\mathbb{D}})$, where φ is some conformal map from $\mathbb{D}$ onto G, then G is a Jordan domain.*

The Caughran's original proof, was made for $p = 2$ and it used the ideas of proving the sufficiency in Theorem 3.25. J. Caughran has mentioned that the given proof is valid, if interpreted properly, for H^p with $1 \leqslant p < \infty$. The following result is an immediate corollary of Caughran's theorem.

Corollary 4.61. *If φ maps $\mathbb{D}$ conformally onto a Carathéodory domain, $\|\varphi\|_{H^\infty} \leqslant 1$, then $C_\varphi(H^p)$ is dense in H^p for each $1 \leqslant p < \infty$.*

Later on in [109] the next generalization of the results under consideration was obtained.

Theorem 4.62 (Roan). *Let φ a weak star generator of H^∞, then the set of polynomials is dense in $H^p(G)$, where $G = \varphi(\mathbb{D})$ and $0 < p < \infty$.*

Proof. Assume that φ is a weak-star generator of H^∞. Denote by M the subspace $\{P \circ \varphi : P \in \mathcal{P}\}$, by M^1 the subspace of all functions in H^∞ which are weak-star limits of sequences of functions in M. Let $h \in M^1$, then there exists a sequence (P_n) of polynomials which are uniformly bounded and $P_n(\varphi(z)) \to h(z)$ for each $z \in \mathbb{D}$. We need the following lemma which corresponds to Lemma 4.41 for H^p.

Lemma 4.63. *Let $0 < p < \infty$, and let (f_n) be a bounded sequence in H^p. Assume that $f_n(z) \to f(z)$ for each $z \in \mathbb{D}$. Then, $f_n \to f$ in the weak topology in H^p.*

Notice that for $p \geqslant 1$ the proof of this lemma is essentially the same as it was done in [120, Lemma 1]. For the case $0 < p < 1$, it follows from [40], where it was proved that the point evaluation belongs to $(H^p)^*$ and the principle of uniform boundedness and the closed graph theorem remain valid for H^p.

By the lemma just mentioned we know that $P_n \circ \varphi \to h$ weakly in H^p. So, $h \in \mathrm{Clos}_{w;H^p}(M)$, the weak closure of M in H^p. Then, $M^1 \subset \mathrm{Clos}_{w;H^p}(M)$. One has

$$\mathrm{Clos}_{w;H^p}(M) = \mathrm{Clos}_{H^p}(M), \tag{4.30}$$

where the right-hand side of (4.30) is the closure of M in the original topology of H^p. Equality (4.30) follows from [114, Theorem 3.12] in the case when $1 \leqslant p < \infty$. In the case $p < 1$, (4.30) follows from [40, Lemma 8]. Then, $M^1 \subset \mathrm{Clos}_{H^p}(M)$. Now, inductively $M^\sigma \subset \mathrm{Clos}_{H^p}(M)$ for every countable ordinal number σ. Since φ is a weak-star generator of H^∞ there exists a countable ordinal τ such that $M^\tau = H^\infty$. Then, $H^\infty = \mathrm{Clos}_{H^p}(M)$, and hence M is dense in H^p. But the density of M in H^p exactly means the density of polynomials in $H^p(G)$. ■

The proof of above theorem is quite simple. The crucial reason why this theorem implies Theorem 4.60 is the fact, given by Proposition 4.51, that a Carathéodory domain is the image of some sequential generator of H^∞.

In view of the Lemma 4.59, Theorem 4.62 can be reformulated as follows.

Theorem 4.64 (Roan). *Let $0 < p < \infty$, and let φ be a weak-star generator of H^∞ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. Then, the range of C_φ is dense in H^p.*

We end this section by mentioning some results obtained in [17] and related with Bergman spaces.

Theorem 4.65 (Bourdon). *Let φ be a weak-star generator of H^∞ and let $G = \varphi(\mathbb{D})$. Then, the polynomials are dense in $A^2(G)$.*

Because there are many weak-star generators of H^∞ which map $\mathbb{D}$ onto non-Carathéodory domains, this result is more general (for $p = 2$) than Theorem 4.1.

The proof of Theorem 4.65 use a theorem of Hedberg that says that if G is a simply connected domain of finite area, then $H^\infty(G)$ is dense in $A^2(G)$ and certain properties of cyclic vectors of multiplication operators acting in $A^2(G)$ and $H^2(G)$, see [17] for the detailed explanation. Then, one has yet another proof of Theorem 4.60 for $p = 2$.

A key idea of work [17] is to relate the approximation by polynomials in $H^2(G)$ with the approximation also by polynomials in some weighted Bergman spaces.

Proposition 4.66. *Let φ map $\mathbb{D}$ conformably onto G. Then, the polynomials are dense in $A^2(G, (1 - |\varphi^{-1}(w)|^2)\, dA)$ if and only if the polynomials in φ are dense in H^2.*

Sketched proof. If $f \in H^2$ and $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$, then

$$\|f\|_{H^2}^2 = \sum_{n=0}^{\infty} |\hat{f}(n)|^2.$$

But this norm is equivalent to

$$\|f\|^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(w)|^2 (1 - |w|^2)\, dA(w),$$

as it can directly verified by the considering of the corresponding Taylor expansion. From now on, one can proceed as follows. If the polynomials are dense in $A^2(G, (1 - \varphi^{-1}(w))\, dA)$, then the set $\{(P \circ \varphi)\varphi' : P \in \mathcal{P}\}$ is dense in $A^2(\mathbb{D}, (1 - |z|^2)\, dA)$, but this implies (via the integration) that the set $\{P \circ \varphi : P \in \mathcal{P}\}$ is dense in $H^2(\mathbb{D})$. The converse may be verified by differentiation. ■

Corollary 4.67. *Let φ map $\mathbb{D}$ conformally onto G.*

(1) *The density of polynomials in $A^2(G)$ or in $A^2(\mathbb{D}, (1-|\varphi^{-1}(w)|^2) dA)$ implies the density of polynomials in $H^2(G)$.*

(2) *If polynomials are dense in $A^2(G)$ or in $A^2(\mathbb{D}, (1 - |\varphi^{-1}(w)|^2)\, dA)$, then φ is univalent almost everywhere on $\mathbb{T}$.*

Note that the part (1) of Corollary 4.67 says that Theorem 4.1 for $p = 2$ together with Proposition 4.66 give a direct proof of Theorem 4.60 in the case $p = 2$. Note also that the part (2) of Corollary 4.67 is the analogue for $A_a(G)$ of Proposition 4.47. Also it seems that Bourdon's techniques are only appropriate for $H^2(G)$ and not for $p \neq 2$.

Let us mention the paper [37], where another proof of Theorem 4.62 was given. It seems the author was unaware of Roan's, Caughran's and Bourdon's papers.

4.5 Approximation by polynomials on boundaries of domains

Let A be an uniform algebra on some compact Hausdorff space X, let $\phi \in M_A$ and assume that there exist a unique representative positive measure σ for ϕ (recall that this assumption is needed here, because, in general, such measure is not unique). Put $M = \ker \phi$ and denote by $M^+(X)$ the set of finite positive Borel measures on X. Let us recall the following result.

Theorem 4.68 (Szegö, Kolmogorov, Krein). *Let $\mu \in M^+(X)$, and suppose that $\mu = w \cdot \sigma + \nu$ is the Lebesgue decomposition of μ with respect to σ, where $w = d\mu/d\sigma \in L^1(\sigma)$ is the Radon–Nikodym derivative of μ and ν is singular with respect to σ. Let $0 < q < +\infty$. Then,*

$$\inf_{f \in M} \int |1 - f|^q\, d\mu = \inf_{f \in M} \int |1 - f|^q w\, d\sigma = \exp \int \log w\, d\sigma.$$

Szegö has proved this theorem when $A = A(\overline{\mathbb{D}})$, $M = \{P \in \mathcal{P} : P(0) = 0\}$, $\mu \ll dt$, $q = 2$ and $\phi(f) = f(0)$, $f \in A$. Later Kolmogorov and Krein showed that the infimum depends only on the absolutely continuous part of μ. A complete proof in the case $A(\overline{\mathbb{D}})$, $1 \leqslant q < +\infty$ and $M = \{P \in \mathcal{P} : P(0) = 0\}$ is given in [77, Chapter vii]. The proof of the general version may be found in [56, Chapter v] or in [18, page 236]. Observe that, by Jensen inequality, one has

$$\exp \int \log w \, d\sigma \leqslant \int w \, d\sigma < +\infty.$$

So, always,

$$-\infty \leqslant \int \log w \, d\sigma \leqslant \text{Const}\,.$$

Now, we will use the notation and result from Sections 3.2 and 3.4. Let G be a Carathéodory domain with the boundary Γ, then $P(\Gamma)$ is a Dirichlet algebra, and for each point $a \in G^*$ (the Carathéodory hull of G) the measure $\omega(a, \cdot, G)$ is the unique representative measure on the Shilov boundary Γ of the element of the spectrum of $P(\Gamma)$ defined by $P \mapsto P(a)$. In this context, given $\mu \in M^+(\Gamma)$, Theorem 4.68 can be applied, and this is the most general setting (in some sense) that the previous theorem can be applied. For example, if $G = \mathbb{D}$ we have the following.

Corollary 4.69. *Let be $\mu \in M^+(\mathbb{T})$ and let $0 < q < +\infty$. The set $\{P \in \mathcal{P} : P(0) = 0\}$ is dense in $L^q(\mu)$ if and only if*

$$\int_{\mathbb{T}} \log(\frac{d\mu}{dt}) \, dt = -\infty.$$

Proof. First note that

$$\inf_{p \in \mathcal{P}} \int \left| \frac{1}{z} - p \right|^q d\mu = \inf_{P : P(0) = 0} \int_{\mathbb{T}} |1 - P|^q \, d\mu = 0,$$

where the equality to zero is obtained applying Theorem 4.68. Then, z and $1/z$ are limits in $L^q(\mu)$ of polynomials. But each $f \in C(\mathbb{T})$ can be uniform approximated by a sequence of polynomials in z and $\bar{z}$ and $C(\mathbb{T})$ is dense in $L^q(\mu)$ for each $0 < q < +\infty$. ∎

Abdullaev and Dovgoshei in [3] have provided an interesting study of the question on how to generalize Corollary 4.69 for other domains. It turns out that the notion of Carathéodory domains plays a central role in this question. Before discussing their results let us fix yet more notation. Let $z_0, z_1, z_2, \ldots$ be a collection of points such that it contains only one point from each component of G^*. Moreover, we assume that $z_0 \in G_0 = G$ and denote by G_j, $j \geqslant 1$, the other bounded components of $\mathbb{C} \setminus \overline{G}$ (if they exist). Let $\omega_j = \omega(z_j, \cdot, G)$, for each $j \geqslant 0$. We know that each ω_j is supported on ∂G_j. Given $\mu \in M^+(\Gamma)$ let us denote by $P^q(\mu)$ the closure in $L^q(\mu)$ of the set of polynomials, and by $P^q(\mu, z_0)$ the closure in $L^q(\mu)$ of the set of polynomials that vanishes at z_0. With this notation we can state the result.

Theorem 4.70. *Assume that G is a bounded simply connected domain, $z_0 \in G$, and $0 < q < +\infty$. Then, the following assertions hold.*

(1) *Let G be a Carathéodory domain. Then,*

$$\begin{aligned} &\{\mu \in M^+(\Gamma) : P^q(\mu, z_0) = P^q(\Gamma)\} \\ &= \left\{\mu \in M^+(\Gamma) : \int_\Gamma \log\left(\frac{d\mu}{d\omega_0}\right) d\omega_0 = -\infty\right\}. \end{aligned} \tag{4.31}$$

(2) *Conversely, if the sets defined on (1) are equal, then G is a Carathédodory domain.*

(3) *Let G be a Carathéodory domain. Then,*

$$\begin{aligned} &\{\mu \in M^+(\Gamma) : L^q(\Gamma) = P^q(\Gamma)\} \\ &= \left\{\mu \in M^+(\Gamma) : \int \log\left(\frac{d\mu}{d\omega_j}\right) d\omega_j = -\infty \textit{ for all } j\right\}. \end{aligned}$$

(4) *In order to have that G is a Carathéodory domain that does not separate the plane it is necessary and sufficient that*

$$\begin{aligned} &\{\mu \in M^+(\Gamma) : L^q(\Gamma) = P^q(\Gamma)\} = \{\mu \in M^+(\Gamma) : L^q(\mu) = P^q(\mu, z_0)\} \\ &= \left\{\mu \in M^+(\Gamma) : \int_\Gamma \log\left(\frac{d\mu}{d\omega_0}\right) d\omega_0 = -\infty\right\}. \end{aligned}$$

Sketch of the proof. (1) Always $P^q(\mu, z_0) \subset P^q(\mu)$. Theorem 4.68 may be applied in our case to give that $\int \log(d\mu/d\omega_0)\, d\omega_0 = -\infty$ if and only if there exists a sequence of polynomials (P_n) such that $P_n \to 1$ in $L^q(\mu)$. Then, if $h \in \mathcal{P}$ then $hP_n \to h$ in $L^q(\mu)$.

(2) Assume that G is not a Carathéodory domain. Then, we need to find a measure that shows that both sets in (4.31) are different. Let Ω be the component of G^* that contains z_0 and let take $\widetilde{\omega} = \omega(z_0, \cdot, \Omega)$. Then, $\Omega \supset G$ and $\widetilde{\omega}$ is a positive measure on Γ but it is supported on $\partial\Omega$. Since G is not a Carathéodory domain, we know that $L := \Gamma \setminus \partial\Omega \neq \emptyset$, and even more, $\omega_0(L) > 0$. Since $\widetilde{\omega}$ vanishes on L, one has $\frac{d\widetilde{\omega}}{d\omega_0}(z) = 0$ for almost all points z in L. Then, $\int \log(d\widetilde{\omega}/\omega_0)\, d\omega_0 = -\infty$, so $\widetilde{\omega}$ belongs to the set in the right-hand side of (4.31). Because Ω is a Carathéodory domain, we can apply the result just proved in (1). Since $\int \log(d\widetilde{\omega}/d\widetilde{\omega})\, d\widetilde{\omega} = 0$, we know that $P^q(\partial\Omega, z_0) \neq P^q(\partial\Omega)$. So, $P^q(\Gamma, z_0) \neq P^q(\Gamma)$.

(3) Let $\mu \in M^+(\Gamma)$. If $L^q(\Gamma) = P^q(\Gamma)$ then, for each z_j, $j \geqslant 0$, one has $L^q(\mu|_{\partial G_j}) = P^q(\mu|_{\partial G_j})$. Since each G_j is a Carathéodory domain, the result of part (1) may be used to obtain that $\int_{\partial G_j} \log \frac{d\mu}{d\omega_j}\, d\omega_j = -\infty$ for each j. Assume now

$$\int_{\partial G_j} \log \frac{d\mu}{d\omega_j}\, d\omega_j = -\infty \tag{4.32}$$

for all $j \geqslant 0$. The important fact now is the assumptions in (4.32) do not depend on the point selected in each component. In other words, if $\omega'_j = \omega(b_j, \cdot, G)$ for other points $b_j \in G_j$, $j \geqslant 0$, then, by (3.4) and (4.32) remains true if we replace ω_j with ω'_j. By Corollary 3.11 one has $R(\Gamma) = C(\Gamma)$. We know that $C(\Gamma)$ is dense in $L^q(\Gamma)$ for each $0 < q < +\infty$. Then, it is enough to prove that for each fixed $b \in G^* \setminus G$ the function $z \to (z-b)^{-1}$ can be approximated in $L^q(d\mu)$ by polynomials. Applying Theorem 4.68 to $P(\Gamma)$ and to $M = \{P : P(b) = 0\}$ we obtain $\inf\{\int |1-P|^q\, d\mu = 0 : P \in \mathcal{P},\ P(b) = 0\} = 0$. If $P(b) = 0$ then $P(z) = (z-b)P_1(z)$ and hence one has

$$\left|\frac{1}{z-b} - P_1(z)\right| \asymp |1-(z-b)P_1(z)|$$

for each $z \in \Gamma$ and $P_1 \in \mathcal{P}$. Then,

$$\inf\left\{\int \left|\frac{1}{z-b} - P_1\right|^q d\mu = 0 : P_1 \in \mathcal{P}\right\} = 0.$$

So, $1/(z-b) \in P^q(\mu)$.

This is a consequence of (1) and (3). In particular, if the set $\mathbb{C} \setminus \bar{G}$ has some bounded component G_j, then the assumption that $\int_{\partial G_j} \log \frac{d\mu}{d\omega_j}\, d\omega_j = -\infty$ cannot be dropped. ∎

The next result gives a sufficient condition for approximation.

Corollary 4.71. *Let G be a Carathéodory domain such that $\bar{G}$ does not separate the plane, and let $0 < p < +\infty$. Assume that $\mu \in M^+(\partial G)$ is such that* $\operatorname{Supp}\mu \neq \partial G$. *Then, $P^q(\mu) = L^q(\mu)$.*

Theorem 2 in [3] gives yet other characterization of Carathéodory domains, however it is a bit technical in a nature and hence we do not state it here, but only mentioned for the interested reader.