Chapter 5

Miscellaneous results about Carathéodory sets

In this chapter we briefly mention some results, where the concept of a Carathéodory sets plays a certain role, but which cannot be placed appropriately into any of the above chapters and sections.

Approximation by polynomials of controlled degree

In this section we present one result which is formally related with Carathéodory sets (at least the corresponding assumption was made in it formulation), but actually it is independent on this concept.

Let K be a continuum and let Φ be the conformal map from $\Omega'_\infty(K)$ onto $\{w \in \mathbb{C}_\infty : |z| > \rho\}$ with the normalization $\Phi(\infty) = \infty$ and $\Phi'(\infty) = 1$, where $\rho > 0$ is determined uniquely by this normalization of Φ. Recall that the Taylor series of Φ at ∞ has the form

$$\Phi(z) = z + a_0 + \frac{a_1}{z} + \cdots, \quad |z| > R_1$$

for some $R_1 > 0$. Then, for each $n \geqslant 1$, one has

$$\Phi^n(z) = z^n + a_{n-1}^{(n)} z^{n-1} + \cdots + a_0^{(n)} + \frac{a_{-1}^{(n)}}{z} + \cdots, \quad |z| > R_1.$$

The polynomial

$$\Phi_n(z) = z^n + a_{n-1}^{(n)} z^{n-1} + \cdots + a_0^{(n)}$$

is called the *n-Faber polynomial* with respect to K. For each $r > 1$ let

$$C_r = \Phi^{-1}(\{z : |z| = r\}).$$

The question on studies of approximation of functions by polynomials of degree at most n was posed already in the thesis of S. N. Bernstein. Here, we mention two results.

Theorem 5.1 (Bernstein theorem). *Let K be a continuum and let $f \in C(K)$. Then, for every $\varepsilon > 0$ and $0 < q < 1$ there exists a sequence of polynomials (P_n) such that* $\deg P_n \leqslant n$ *and*

$$|f(z) - P_n(z)| \leqslant C(\varepsilon)(q + \varepsilon)^n, \quad n = 0, 1, 2, \ldots, \ z \in K,$$

if and only if f has an analytic extension $\tilde{f}$ to $D(C_{\rho/q})$. In the case that there exists such extension, the sequence (P_n) converges to $\tilde{f}$ locally uniformly in $D(C_{\rho/q})$.

Now, we will use the following notation. Let $1 \leqslant p \leqslant \infty$. If $f \in A^p(G)$ then define

$$E^p_{n,G}(f) = \inf\{\|f - P\|_{p,G} : P \in \mathcal{P},\ \deg P \leqslant n\}.$$

Theorem 5.2 (Bernstéin–Walsh theorem). *Let $K \subset \mathbb{C}$ be a compact set such that $\mathbb{C} \setminus K$ is connected. If $f \in H(K)$, then*

$$\limsup_{n\to\infty} E^\infty_{n,K}(f)^{1/n} \leqslant \theta < 1,$$

where $\theta = 0$ if K has logarithmic capacity zero, while θ is a positive number (related with the Green function of $\mathbb{C} \setminus K$) if capacity of K is positive.

Note that previous result is a quantitative version of Runge's theorem. A proof can found in [107, page 170]. Let us revert to Theorem 5.1. A proof can be seen in [85]. In the case that f has the continuous extension, the key point is to show that

$$f(z) = \sum_{n=0}^{\infty} a_n \Phi_n(z)$$

uniformly on K and one can take $P_m = \sum_{n=0}^{m} a_n \Phi_n$. In the proof the following estimate is obtained

$$|f(z) - P_n(z)| \leqslant \frac{3}{2}\bar{M}(f,r)\frac{(r'/r)^{n+1}}{1-(r'/r)}, \tag{5.1}$$

for each $z \in K$, where $r > r' > \rho$ and $\bar{M}(f,r) = \sup\{|f(z)| : z \in C_r\}$.

In a sequence of papers, see the references in [70], the following problem was studied: *whether the condition* $\lim(E^p_{n,G}(f))^{1/n} = 0$ *does imply that f is an entire function*. In [70] two results of such kind were obtained under the assumption that the domain G under consideration is a Carathéodory domain. However, it seems that this assumption is not relevant to the problem under consideration and it is not needed in the first of the aforementioned results. Let us reformulate and proof the corresponding statement.

Theorem¶ 5.3. *Let $1 \leqslant p \leqslant \infty$.*

(a) *Let $f \in H(\mathbb{C})$. Then, for each bounded domain G it holds*

$$\lim_{n\to\infty} E^p_{n,G}(f)^{1/n} = 0.$$

(b) *Let U be an open set in $\mathbb{C}$ and let $f \in H(U)$. If there exists an open disc D with $\bar{D} \subset U$, such that*

$$\lim_{n\to\infty} E^p_{n,D}(f)^{1/n} = 0,$$

then there exists $F \in H(\mathbb{C})$ such that $F|_U = f$.

Sketch of the proof. Put $K := \widehat{G}$, take the corresponding Φ for such K (see the beginning of this subsection), and put $r' = 2\rho$. If P_n is a polynomial which satisfies (5.1), then

$$E^p_{n,G}(f) \leqslant \|f - P_{n-1}\|_{p,K} \leqslant \sqrt[p]{\text{Area}(K)}\, \|f - P_n\|_K.$$

Using now the estimate (5.1) we obtain

$$E^p_{n,G}(f) \leqslant \text{Const}\, \bar{M}(f, r) \frac{(\rho/r)^{n+1}}{1 - (\rho/r)}$$

whenever $r > r' = 2\rho$. Then, $\limsup_{n\to\infty} E^p_{n,G}(f)^{1/n} \leqslant \rho/r$ and letting $r \to \infty$, the conclusion follows.

Let us prove the statement (b). Let $D = D(z_0, R)$. We know $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ uniformly on $\bar{D}$. Assume that P_{n-1} is a polynomial of degree at most $n-1$, then

$$\frac{\pi R^{2n+2}}{n+1} a_n = \int_D f(z)\, (\bar{z} - \overline{z_0})^n\, dA(z) = \int_D (f(z) - P_{n-1}(z))\, (\bar{z} - \overline{z_0})^n\, dA(z),$$

which in ones turn gives that

$$\frac{\pi R^{2n+2}}{n+1} |a_n| \leqslant R^n \|f - p_{n-1}\|_{1,D} \leqslant (\pi R^2)^{\frac{1}{q}} R^n \|f - P_{n-1}\|_{p,D},$$

where q is the conjugate exponent for p if $p > 1$. Then,

$$|a_n|^{1/n} \leqslant \mathrm{C}(n+1)^{1/n} E^p_{D,n-1}(f)^{1/n} \to 0$$

by the initial assumptions. So, $f \in H(\mathbb{C})$.

In the case of $p = \infty$ the theorem is essentially due to Winiarski, [137]. ∎

Dualities between $\mathcal{A}^{-\infty}(G)$ and $\mathcal{A}^{\infty}(\mathbb{C}_\infty \setminus G)$

The problem of characterization of the dual of the Fréchet space $H(G)$ for an open set G is a classical problem studied in several papers in the 1950s. It comes that there is an isomorphism from $H(G)^*$ onto $H_0(\mathbb{C}_\infty \setminus G)$. This is called the Main duality theorem. We recommend to the interested reader to look at the proof of this result and some related topics in [81, Chapter 8]. With this background he will understand perfectly the germinal ideas on this short section on dualities between spaces of analytic functions.

Let B be a bounded domain in $\mathbb{C}$. Consider the space $\mathcal{A}^{-\infty}(B)$ consisting of all holomorphic functions in B with polynomial growth near ∂B, so that

$$\mathcal{A}^{-\infty}(B) = \bigcup_{k=0}^{\infty} \{f \in H(B) : \|f\|_{(k)} = \sup_{z\in B} |f(z)|\, \text{dist}(z, \partial B)^k < \infty\},$$

and the space $\mathcal{A}^{\infty}(\mathbb{C}_{\infty} \setminus B)$ consisting of all C^{∞}-functions defined on $\mathbb{C}_{\infty} \setminus B$ vanishing at ∞ and holomorphic in the interior of $\mathbb{C}_{\infty} \setminus B$. We are going to mention the recent result of the paper [2], where it was show that the Cauchy transformation of functionals establish a mutual duality between these introduced spaces in the case when B is a Carathéodory domain. Let us recall, that the Cauchy transformation of functionals is the mapping

$$L \mapsto L\left(\frac{1}{a - \jmath}\right),$$

where $a \in B$ is some point and $\jmath$ stands for the identity function $\jmath(z) = z$.

Let us denote by S^* the dual space of a locally convex topological space S endowed with the strong topology. The aforementioned result of [2] is as follows (see Theorems 4.3 and 4.5 in this paper).

Theorem 5.4. *The following statements hold.*

(1) *Let G be a Carathéodory domain. Then, the Cauchy transformation of functionals is an isomorphism from $\mathcal{A}^{-\infty}(G)^*$ onto $\mathcal{A}^{\infty}(\mathbb{C}_{\infty} \setminus G)$.*

(2) *Let B be a bounded domain in $\mathbb{C}$ with rectifiable boundary possessing the property $B = \operatorname{Int}(\overline{B})$, then the Cauchy transformation of measures is an isomorphism from $\mathcal{A}^{\infty}(\mathbb{C}_{\infty} \setminus B)^*$ onto $\mathcal{A}^{-\infty}(B)$.*

Let us give an outline of the proof of Theorem 5.4, part (1). In order to prove the theorem it is enough (in view of [2, Proposition 4.1]) to verify that the system of Cauchy kernels $\{\frac{1}{z-a} : a \in \mathbb{C}_{\infty} \setminus G\}$ is complete in $\mathcal{A}^{-\infty}(G)$. In order to prove this completeness property we need to prove first that the set of all polynomials is dense in $\mathcal{A}^{-\infty}$. This fact is the consequence of Hedberg's theorem (see Theorem 4.34 above) applied for the weights $\mathbf{w} = \operatorname{dist}(\cdot, \partial\Omega)^k$, $k \in \mathbb{N}_0$. The final step is to approximate each polynomial by respective Cauchy kernels in the topology of the space $\mathcal{A}^{-\infty}(G)$. ■

As it was mentioned above (see Propositions 1.5 and 1.6) any Carathéodory domain G is simply connected and possesses the property $G = \operatorname{Int}(\overline{G})$. Moreover, the latter condition is equivalent to the Carathéodory one whenever G is a bounded simply connected domain whose closure does not separate the plane.

As a corollary of Theorem 5.4 in [2] (see Corollary 4.6 of the paper cited) it was stated the following result: If G is a Carathéodory domain with rectifiable boundary, then the Cauchy transformation of measures establishes a mutual duality between the spaces $\mathcal{A}^{-\infty}(G)$ and $\mathcal{A}^{\infty}(\mathbb{C}_{\infty} \setminus G)$.

In this connection it is worth to recall Corollary 2.13 which says that any Carathéodory domain with rectifiable boundary is a Jordan domain. Thus, the mutual duality between the spaces $\mathcal{A}^{-\infty}(G)$ and $\mathcal{A}^{\infty}(\mathbb{C}_{\infty} \setminus G)$ is actually established only for the class of Jordan domains.

The same remark holds for the result of [2, Theorem 5.7] which we do not state explicitly because it goes too far from our main line of considerations.

Analytic balayage of measures supported in Carathéodory domains

Let us briefly discuss one topic concerning the structure of measures that are orthogonal to rational functions, in which the concept of a Carathéodory domain plays a certain role.

Let G be a Jordan domain with rectifiable boundary, and let μ be a measure with $\operatorname{Supp}(\mu) \subset G$. Then, by [28, Lemma 4.1], the measure $\mu + \widehat{\mu}\, d\zeta|_{\partial G}$ is orthogonal to $\mathcal{P}$ (as before, the symbol $\widehat{\mu}$ denotes the Cauchy transform of μ). In view of the term $\widehat{\mu}\, d\zeta|_{\partial G}$ it is not clear how to extend this observation to a wider class of domains. The following result was proved in [26, Proposition 3].

Proposition¶ 5.5. *Let G be a Carathéodory domain in $\mathbb{C}$, let f be a conformal map from $\mathbb{D}$ onto G, and ω be the corresponding complex harmonic measure on ∂G.*

(1) *Let μ be a measure with* $\operatorname{Supp}(\mu) \subset G$. *Then, the measure*

$$\mu^* = \mu + (\widehat{\eta} \circ f^{-1})\, \omega, \quad \text{where } \eta = f^{-1}(\mu),$$

is orthogonal to $A(\overline{G})$.

(2) *Let $K \subset G$ be a compact set, and σ be a measure on $K \cup \partial G$ with $\sigma \perp R(\overline{G})$. Then, there exists a function $h \in H^1$ such that $\sigma = (\sigma|_K)^* + (h \circ f^{-1})\, \omega$.*

Proof. We start with the proof of the first assertion. Put $E := \operatorname{Supp}(\eta)$. Since $\widehat{\eta}$ is holomorphic outside E, then $\widehat{\eta}\, d\zeta$ is a well-defined measure on $\mathbb{T}$ and $\nu = f(\widehat{\eta}\, d\zeta)$ is a measure on ∂G. Take a function $g \in A(\overline{G})$, so that $g \circ f \in H^\infty$. Using Fubini and Cauchy theorems and the definition of $\widehat{\eta}$ we have

$$\int g\, d\mu^* = \int g\, d\mu + \int g\, d\nu = \int g\, d\mu + \int_{\mathbb{T}} g(f(\zeta))\, \widehat{\eta}(\zeta)\, d\zeta$$
$$= \int g d\mu + \int_E \left[\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{g(f(\zeta)) d\zeta}{w - \zeta}\right] d\eta(w) = \int g d\mu - \int_E g(f(w)) d\eta(w) = 0.$$

In order to prove the second assertion we need to observe that $\sigma - (\sigma|_K)^*$ is a measure on ∂G orthogonal to $R(\overline{G})$. It remains to apply (3.21). ∎

The representation of orthogonal measures obtained in this proposition has an interesting connection with the notion of an analytic balayage of measures, which was introduced by D. Khavinson [74], and which turned out to be a useful tool in approximations by analytic functions.

Definition 5.6. Let X be a compact set in $\mathbb{C}$, and let μ be a measure such that $\operatorname{Supp}(\mu) \subset X^\circ$. The measure ν on ∂X is called an *analytic balayage* of μ if $\mu - \nu \perp R(X)$, and for any measure $\widetilde{\nu}$ on ∂X such that $\mu - \widetilde{\nu} \perp R(X)$, the inequality $\|\widetilde{\nu}\| \geqslant \|\nu\|$ holds.

In all cases considered below, the analytic balayage of a given measure is uniquely determined. Having this remark in mind, we will denote the analytic balayage of μ by $\alpha(\mu) = \alpha(\mu, \partial X)$.

The presented definition of an analytic balayage of measures was given in [74, Definition 2] for finitely connected compact sets with piecewise analytic boundaries, but it also makes sense for general compact sets. For measures μ supported on X (but not only on X°), the analytic balayage was defined in another way by means of a special implicit construction, namely, a weak-star limit of analytic balayages of the initial measure to (piecewise analytic) boundaries of certain finitely connected compact sets approaching X (see [74, Definition 3]).

Let us see what an analytic balayage looks like in a simple case. Let G be a Jordan domain with piecewise analytic boundary Γ, and let μ be a measure such that $\operatorname{Supp}(\mu) \subset G$. As was shown in [74, Proposition 2]

$$\alpha(\mu) = g^*\, dz|_\Gamma - \widehat{\mu}\, dz|_\Gamma, \tag{5.2}$$

where $g^* \in R(\overline{G})$ is such that

$$\|\widehat{\mu} - g^*\|_{L^1(\Gamma)} = \inf \|\widehat{\mu} - g\|_{L^1(\Gamma)},$$

the infimum being taken over all functions $g \in R(\overline{G})$, and the Lebesgue space $L_1(\Gamma)$ is considered with respect to the measure $|dz|$ on Γ. The fact that the analytic balayage of μ is uniquely determined in this case is the consequence of [74, Proposition 3]. The formula (5.2) highlights the role of the term $\widehat{\mu}\, dz|_\Gamma$ which have appeared also in the part (1) of Proposition 5.5.

Since the explicit expression for analytic balayage is known only for finitely connected compact sets with piecewise analytic boundaries, it would be interesting to find such formulae for a wider class of compact sets. The class of Carathéodory compact sets fits this problem most naturally. This is mainly due to the structural properties of orthogonal measures stated in Proposition 5.5 which hold for the class of Carathéodory domains but not for any other known wider class of domains in $\mathbb{C}$.

The next result which was obtained in [52], see also [1], gives the desired expression for analytic balayage in the case of Carathéodory domains. We recall, that $H^1 = H^1(\mathbb{D})$ and the space $L^1 = L^1(\mathbb{T})$ is considered with respect to the measure $m_\mathbb{T}$.

Theorem¶ 5.7. *Let G be a Carathéodory domain and μ be a measure with $\operatorname{Supp}(\mu) \subset G$. Then, $\alpha(\mu, \partial G)$ is concentrated on $\partial_a G$ and has the form*

$$\alpha(\mu, \partial G) = (h^* \circ f^{-1})\, \omega - (\widehat{\eta} \circ f^{-1})\, \omega,$$

where f is a conformal map from $\mathbb{D}$ onto G, the measure η is defined as $\eta = f^{-1}(\mu)$, and the function $h^* \in H^1$ is the solution of the extremal problem

$$\|\widehat{\eta} - h^*\|_{L^1} = \inf_{h \in H^1} \|\widehat{\eta} - h\|_{L^1}. \tag{5.3}$$

It follows from this theorem that the analytic balayage of μ in the case under consideration is uniquely determined (notice that the extremal problem (5.3) has a unique solution, see [58, Chapter iv, Section 1.2]).

Sketch of the proof of Theorem 5.7. By part (1) of Proposition 5.5 one has

$$\mu^* = \mu + (\widehat{\eta} \circ f^{-1})\, \omega \perp R(\overline{G}).$$

Let now $\widetilde{\nu}$ be an arbitrary measure on ∂G such that $\mu - \widetilde{\nu} \perp R(\overline{G})$. Then,

$$\widetilde{\nu} + (\widehat{\eta} \circ f^{-1})\, \omega = \widetilde{\nu} + (\mu^* - \mu) = \mu^* - (\mu - \widetilde{\nu}) \perp R(\overline{G}).$$

Since $\widetilde{\nu} + (\widehat{\eta} \circ f^{-1})\, \omega$ is a measure on ∂G, there exists some function $h \in H^1$ such that

$$\widetilde{\nu} + (\widehat{\eta} \circ f^{-1})\, \omega = (h \circ f^{-1})\, \omega,$$

and hence

$$\widetilde{\nu} = (h \circ f^{-1})\, \omega - (\widehat{\eta} \circ f^{-1})\, \omega.$$

It remains to observe that the measure $\nu = \alpha(\mu, \partial G)$ is the measure among $\widetilde{\nu}$ that has the minimum norm. ∎

The formula for analytic balayage given in Theorem 5.7, has a simpler form in the case when the solution h^* of the extremal problem (5.3) is zero. Let us describe the measures for which it is the case. The following result was proved in [1].

Proposition 5.8. *Let G be a Carathéodory domain in $\mathbb{C}$, and let f, μ and η be as in Theorem 5.7. Then, $\alpha(\mu, \partial G) = -(\widehat{\eta} \circ f^{-1})\, \omega$ if and only if μ is a finite sum of point-mass measures, one of which is supported at the point $f(0)$.*

In order to verify this result we need to use the concept of badly approximable functions in L^1. The function $\varphi \in L^1$ is called *badly approximable*, if only the function $g^* \equiv 0$ solves the extremal problem

$$\|\varphi - g^*\|_{L^1} = \inf_{g \in H^1} \|\varphi - g\|_{L^1}.$$

It follows from [58, Theorem 1.2, Chapter iv] that the solution of this extremal problem is unique. The class of badly approximable functions admits the following description.

Proposition 5.9. *A function $\varphi \in L^1$ is badly approximable if and only if it has the form $\varphi = \overline{\Theta}\, \Phi$, where Θ is an inner function (i.e., $\Theta \in H^\infty$ and $|\Theta(\zeta)| = 1$ for a.a. $\zeta \in \mathbb{T}$), $\Theta(0) = 0$, and $\Phi \in L^1$ is such that $\Phi \geqslant 0$.*

This result may be found in [75, Theorem 1], where it was obtained in a slightly different terms, and the proof provided was lengthly and technically involved. A new readable proof of this fact was given in [1].

Outline of the proof of Proposition 5.8. Let K be a compact subset of $\mathbb{D}$, and let $\varphi \in H(\mathbb{C}_\infty \setminus K)$. It follows from Proposition 5.9 that φ is badly approximable if and only if $\varphi = c\overline{B}$ on $\mathbb{T}$, where c is a positive constant, and B is a finite Blaschke product with $B(0) = 0$.

This fact yields that the solution h^* of the extremal problem (5.3) is zero if and only if the function $\widehat{\eta}$ coincides on $\mathbb{T}$ with the conjugation of some finite Blaschke product vanishing at 0. It means that η (and hence μ) is a finite sum of point-mass measures, one of which is supported at the origin (at the point $f(0)$, respectively). ∎

Harmonic reflection over boundaries of Carathéodory domains

Recently, interest has intensified in the problems on reflection of harmonic functions over boundaries of domains in the plane and in space and in the problems on preservation of the smoothness properties of functions under such reflection. It is known several different approaches to define the harmonic reflection. Many of them are based on constructions of point-to-point reflection related with different variations of the symmetry principle for harmonic functions. At the same time in [53, 100–102] the construction was studied that was based on usage of the Dirichlet problems for harmonic functions in a given domain and in its complement. This construction is closely connected with the notion of a Carathéodory domain. In the rest of this section let $k \in \mathbb{N}$ and $k > 1$.

Definition 5.10. A nonempty bounded domain $G \subset \mathbb{R}^k$ is called a simple Carathéodory domain if it possesses the following properties:

(1) the set $\Omega = \mathbb{R}^k \setminus \overline{G}$ is a domain;

(2) $\partial G = \partial \Omega$;

(3) if $k \geqslant 3$ then both domains G and Ω are regular with respect to the classical Dirichlet problem for harmonic functions.

In fact, a simple Carathéodory domain in $\mathbb{R}^2$ is a Carathéodory domain, whose closure does not separate the plane. Notice that the third property in Definition 5.10 is assumed only for $N \geqslant 3$, since any Carathéodory domain in $\mathbb{R}^2$ is simply connected (see Proposition 1.5), and hence it is regular with respect to the Dirichlet problem for harmonic functions (see, e.g., Section 3.2).

Recall that for $m \in (0, 1]$ and for a closed set $X \subset \mathbb{R}^k$ (containing at least two points) the Lipschitz–Hölder space of order m is defined as follows:

$$\mathrm{Lip}^m(X) = \left\{ h \in C(X) : \|h\|'_{X,m} := \sup \frac{|h(\boldsymbol{x}) - h(\boldsymbol{y})|}{|\boldsymbol{x} - \boldsymbol{y}|^m} < +\infty \right\},$$

where sup is taken over all couples of points $\boldsymbol{x}, \boldsymbol{y} \in X$ with $\boldsymbol{x} \neq \boldsymbol{y}$. The norm of a function $h \in \mathrm{Lip}^m(X)$ is defined as follows: $\|h\|_{X,m} := \max\{\|h\|'_{X,m}, \|h\|_X\}$.

Furthermore, for $m \in (0, 1)$ we put

$$C^m(X) = \left\{ h \in \mathrm{Lip}^m(X) : \lim_{\delta \to 0} \sup_{0 < |\boldsymbol{x} - \boldsymbol{y}| < \delta} \frac{|h(\boldsymbol{x}) - h(\boldsymbol{y})|}{|\boldsymbol{x} - \boldsymbol{y}|^m} = 0 \right\}.$$

Notice that using the Whitney extension theorem and a regularization operator, it can be readily verified that for compact sets $X \subset \mathbb{R}^k$ the space $C^m(X)$ for $m \in (0, 1)$ coincides with the closure in $\mathrm{Lip}^m(X)$ of the subspace $C^\infty(\mathbb{R}^k)|_X$.

Denote by $C_{\mathbb{R}}(X)$ the space of all real-valued continuous functions on a given closed set $X \subset \mathbb{R}^k$. Put

$$C_H(X) := C_{\mathbb{R}}(X) \cap \mathrm{Har}(\mathrm{Int}(X))$$

if X contains no punctured neighborhood of ∞, or, otherwise,

$$C_H(X) := \{h \in C_{\mathbb{R}}(X) \cap \mathrm{Har}(\mathrm{Int}(X)) : h(\boldsymbol{x}) = O_{|\boldsymbol{x}| \to \infty}(|\boldsymbol{x}|^{2-k})\}.$$

Take a simple Carathéodory domain $G \subset \mathbb{R}^k$, and let $\Omega = \mathbb{R}^k \setminus \overline{G}$, so that Ω is a domain and $\partial\Omega = \partial G$.

Let us define two operators, related with the Dirichlet problem for harmonic functions in G and in Ω. The first one is the *Poisson operator* $\boldsymbol{P}_G$ which maps a given function $\varphi \in C_{\mathbb{R}}(\partial G)$ to the function $f \in C_H(\overline{G})$ such that $f|_{\partial G} = \varphi$. The Poission operator $\boldsymbol{P}_\Omega$ is defined by the same way. The second one is the *harmonic reflection operator* $\boldsymbol{R}_G$ that acting from the space $C_H(\overline{G})$ to the space $C_H(\overline{\Omega})$ and that maps a given function $f \in C_H(\overline{G})$ to the function $g \in C_H(\overline{\Omega})$ such that $g|_{\partial\Omega} = f|_{\partial G}$.

Let us consider the question what conditions on G are necessary and sufficient in order that the operators $\boldsymbol{P}_G$ or $\boldsymbol{R}_G$ preserve smoothness properties of functions, when smoothness is understood in the sense of Lip^m-spaces for $0 < m \leqslant 1$ and C^m-spaces for $0 < m < 1$. This question is interesting both in its own, and in connection with problems on C^m-extension and C^m-approximation for harmonic and subharmonic functions.

Let now m and m' be such that $0 < m' \leqslant m \leqslant 1$. One says that the operator $\boldsymbol{P}_G$ is (m, m')-continuous, if it is continuous as an operator from the space $\mathrm{Lip}^m(\partial G)$ to the space $\mathrm{Lip}^{m'}(\overline{G}) \cap \mathrm{Har}(G)$. Respectively, one says that the operator $\boldsymbol{R}_G$ is (m, m')-continuous, if it is continuous as an operator from $\mathrm{Lip}^m(\overline{G}) \cap \mathrm{Har}(G)$ to $\mathrm{Lip}^{m'}(\overline{\Omega}) \cap C_H(\overline{\Omega})$.

Similarly, for m and m' such that $0 < m' \leqslant m < 1$, the operator $\boldsymbol{P}_G$ is called $C(m, m')$-continuous, if it is continuous operator from $C^m(\partial G)$ to $C^{m'}(\overline{G}) \cap \mathrm{Har}(G)$, while the operator $\boldsymbol{R}_G$ is called $C(m, m')$-continuous if it is continuous operator from $C^m(\overline{G}) \cap \mathrm{Har}(G)$ to $C^{m'}(\overline{\Omega}) \cap C_H(\overline{\Omega})$.

Finally, one says that $\boldsymbol{P}_\Omega$ is (m, m')-continuous, if it is continuous operator from $\mathrm{Lip}^m(\partial\Omega)$ to $\mathrm{Lip}^{m'}(\overline{\Omega}) \cap C_H(\overline{\Omega})$. Respectively, $\boldsymbol{P}_\Omega$ is $C(m, m')$-continuous, if it is continuous from $C^m(\partial\Omega)$ to $C^{m'}(\overline{\Omega}) \cap C_H(\overline{\Omega})$.

The next proposition combines the results obtained in [100] and [53].

Theorem 5.11. *The following holds.*

(1) *For any Jordan Lyapunov–Dini domain G in $\mathbb{R}^k$ the operator $\boldsymbol{R}_D$ is $(1,1)$-continuous; but there exist Jordan domains G with C^1-smooth boundaries for which it is not the case.*

(2) *For every simple Carathéodory domain $D \subset \mathbb{R}^k$ both operators $\boldsymbol{P}_D$ and $\boldsymbol{P}_\Omega$ are not $(1,1)$-continuous.*

We are not going here to give a precise definitions of a Jordan Lyapunov–Dini domain, but we mention that it is a Jordan domain with C^1-smooth boundary, whose boundary satisfies additional Dini-type continuity condition on inner normal vector.

Theorem 5.11 shows that the problem on (m,m')-continuity for operators $\boldsymbol{P}_G$ and $\boldsymbol{R}_G$ are independent in the general case. At the same time, in many instances the problems on (m,m')- and $C(m,m')$-continuity of the operator $\boldsymbol{R}_G$ can be reduced to the corresponding problems for the operator $\boldsymbol{P}_\Omega$. Notice that the domain Ω is unbounded, and assume, without loss of generality, that the initial domain G contains the origin. Using the classical Kelvin transform we can further reduce the problems on (m,m')- and $C(m,m')$-continuity of the operator $\boldsymbol{P}_\Omega$ to the corresponding problems for the operator $\boldsymbol{P}_{B(\Omega)}$, where $B(\Omega) = \{\boldsymbol{x} \in \mathbb{R}^k : \boldsymbol{x}/|\boldsymbol{x}|^2 \in \Omega\}$. Let us recall that the Kelvin transform maps a given function $h(\boldsymbol{x})$ to the function $|\boldsymbol{x}|^{2-k} h(\boldsymbol{x}/|\boldsymbol{x}|^2)$; this mapping is an isomorphism of the spaces $Q^m(\overline{B(\Omega)}) \cap \operatorname{Har}(B(\Omega))$ and $Q^m(\overline{\Omega}) \cap C_H(\overline{\Omega})$, where $Q^m(\cdot)$ stands for both $\operatorname{Lip}^m(\cdot)$ and $C^m(\cdot)$.

Theorem 2 in [53] gives the following criterion for Lip^m-continuity of the Poisson operator.

Theorem 5.12. *Let G, with $\operatorname{diam}(G) \leqslant 1$, be a simple Carathéodory domain in $\mathbb{R}^k$, and let $0 < m' \leqslant m \leqslant 1$. The following conditions are equivalent:*

(a) *the operator $\boldsymbol{P}_G$ is (m,m')-continuous;*

(b) *there exists $A > 0$ such that for each point $\boldsymbol{b} \in \partial D$ and for $\varphi(\boldsymbol{x}) = |\boldsymbol{x} - \boldsymbol{b}|^m$ one has*
$$\boldsymbol{P}_G(\varphi) \in \operatorname{Lip}^{m'}(\overline{G}) \quad \text{and} \quad \|\boldsymbol{P}_G(\varphi)\|_{\overline{D},m'} \leqslant A;$$

(c) *there exists $A > 0$ such that for every point $\boldsymbol{a} \in G$ and for each point $\boldsymbol{a}' \in \partial G$ with the condition $\delta = |\boldsymbol{a} - \boldsymbol{a}'| = \operatorname{dist}(\boldsymbol{a}, \partial G)$ the following estimate is satisfied:*
$$\sum_{n=1}^{N} (n\delta)^m \omega(\boldsymbol{a}, E_n, G) \leqslant A\delta^{m'},$$
where we set
$$E_0 = \{\boldsymbol{x} \in \partial G : |\boldsymbol{x} - \boldsymbol{a}'| \leqslant \delta\},$$
$$E_n = \{\boldsymbol{x} \in \partial G : n\delta < |\boldsymbol{x} - \boldsymbol{a}'| \leqslant (n+1)\delta\}, \quad n \geqslant 1,$$
and, where N is the maximal integer such that $E_N \neq \emptyset$.

Using this theorem one can show that for every simple Carathéodory domain G in $\mathbb{R}^k$ there exists a number $m_G \in [0,1]$ possessing the following properties: the operator $\boldsymbol{P}_G$ is (m,m)-continuous for all $m \in (0, m_G)$, but it is not the case for all $m \in (m_G, 1]$. Moreover, the operator $\boldsymbol{P}_G$ is (m,m')-continuous for all (m,m') such that $0 < m' < m_G$ and $m' \leqslant m \leqslant 1$.

Theorem 1 in [102] says that a similar picture holds in the case of C^m-continuity of the operator $\boldsymbol{P}_G$.

Theorem 5.13. *Let G be a simple Carathéodory domain in $\mathbb{R}^k$, and let m_G be the number defined in the previous statement. The operator $\boldsymbol{P}_G$ is $C(m,m)$-continuous for all $m \in (0, m_G)$, but it is not the case for all $m \in (m_G, 1)$. Moreover, the operator $\boldsymbol{P}_G$ is $C(m,m')$-continuous for all (m,m') such that $0 < m' < m_G$ and $m' \leqslant m < 1$.*

It follows from [53, Corollaries 3, 8, and 9] that for any simple Carathéodory domain $G \subset \mathbb{R}^2$ we have $m_D \in [1/2, 1]$, while in the case that $k \geqslant 3$ the number m_D may take any value from the segment $[0,1]$ in the general case. Let us now clarify what the numbers m_G and m_Ω are equal to in the case when G and Ω satisfy certain special geometrical conditions.

Given $\nu \in (0,1)$ and $r \geqslant 0$ let us define (closed spherical) *truncated cone* (closed sector in the two-dimensional case) $K(\nu, r)$ in $\mathbb{R}^k$ as follows:

$$K(\nu, r) = \{\boldsymbol{x} \in \mathbb{R}^k : 0 < |\boldsymbol{x}| \leqslant r,\ \theta_{\boldsymbol{x}} \leqslant \nu\pi\} \cup \{0\},$$

where

$$\theta_{\boldsymbol{x}} = \arccos(x_1/|\boldsymbol{x}|)$$

stands for the angle between the vector $\boldsymbol{x} = (x_1, \dots, x_k)$ and the direction of the axes Ox_1.

One says that a simple Carathéodory domain $G \subset \mathbb{R}^k$ satisfies the *external truncated cone condition* with parameters (α, r), where $\alpha \in (0,1]$ and $r > 0$, if for every point $\boldsymbol{a} \in \partial G$ there exists a truncated cone $K_{\boldsymbol{a}}$ congruent to $K(\alpha/2, r)$ with the vertex $\boldsymbol{a}$, and such that $K_{\boldsymbol{a}} \cap G = \emptyset$. The *internal truncated cone condition* is defined by the same way.

For further considerations we need one auxiliary construction, see [82, Section 1] and [92, Section 2]. For every $k \geqslant 2$ and $\lambda > 0$ there exists a unique function $g_{k,\lambda} \in C^2([0,\pi))$ such that

$$g''_{k,\lambda}(t) + (k-2)\cot(t) g'_{k,\lambda}(t) + \lambda(\lambda + k - 2) g_{k,\lambda}(t) = 0, \quad t \in (0, \pi),$$

with $g_{k,\lambda}(0) = 1$ and $g'_{k,\lambda}(0) = 0$. Moreover, the function $g_{k,\lambda}$ has its first (with respect to the increasing order) positive zero $\theta_k(\lambda)$ in the interval $(0,\pi)$; the function $\theta_k(\cdot) \colon (0, +\infty) \to (0, \pi)$ is continuous and strictly decreasing; the corresponding inverse function $\lambda_k(\cdot) \colon (0,\pi) \to (0, +\infty)$ is also continuous and injective.

Both functions $\theta_k(\cdot)$ and $\lambda_k(\cdot)$ may be found in an explicit form for $k = 2, 4$. Thus, in the case $k = 2$ the corresponding equation for the function $g_{k,\lambda}$ has a very simple form $g''_{2,\lambda} + \lambda^2 g_{2,\lambda} = 0$, so that $g_{2,\lambda}(t) = \cos(\lambda t)$, $\theta_2(\lambda) = \pi/(2\lambda)$ and $\lambda_2(\theta) = \pi/(2\theta)$. For $k = 4$ it can be shown, that $\theta_4(\lambda) = \pi/(\lambda + 1)$ and $\lambda_4(\theta) = -1 + \pi/\theta$, respectively.

For $\alpha \in (0, 1]$ we can define the number $m_{k,\alpha} := \lambda_k(\pi - \alpha\pi/2)$ so that, in particular, $m_{2,\alpha} = 1/(2 - \alpha)$. It was shown in [53] that for a simple Carathéodory domain $G \subset \mathbb{R}^k$ satisfying the external truncated cone condition with parameters (α, r) for some $\alpha \in (0, 1)$ and $r > 0$, it holds $m_G \geqslant m_{k,\alpha}$.

In several cases when a given simple Carathéodory domain G satisfies certain additional conditions stated in terms of external (or internal) truncated cone conditions (with some α), we have that $m_G = m_{k,\alpha}$ (or, respectively, $m_\Omega = m_{k,\alpha}$). In the latter case the operator $\boldsymbol{R}_G$ is (m, m')-continuous for $0 < m' < m_\Omega$ with $m' \leqslant m \leqslant 1$, and it is not the case for all m, m' such that $m_\Omega < m' \leqslant m \leqslant 1$. Moreover, the operator $\boldsymbol{R}_G$ in this case is $C(m, m')$-continuous for all m and m' such that $0 < m' < m_\Omega$ and $m' \leqslant m < 1$, but it is not the case for all m and m' with $m_\Omega < m' \leqslant m < 1$. These results and related discussions may be found in [53, Section 3] and [102, Section 2].

Carathéodory domains and invariant subspace problem

We end this chapter and the whole survey by stating one result showing the application of Carathéodory domains to the invariant subspace problem. The respective result was recently obtained in [76]. It states as follows.

Theorem 5.14. *Let T be a bounded linear operator on a separable infinite-dimensional Hilbert space $\mathcal{H}$ with the spectrum $\sigma(T)$. Assume that*

(i) *T is such that $\|P(T)\| \leqslant \|P\|_{\sigma(T)}$ for every $P \in \mathcal{P}$, and*

(ii) *$\widehat{\sigma(T)}$ is the closure of a Carathéodory domain such that for every $\zeta \in \partial_a G$ there exists a rectifiable arc $\Upsilon \subset \partial G$ containing ζ.*

Then, T has a nontrivial invariant subspace $\mathcal{H}_0$ (so that $T\mathcal{H}_0 \subset \mathcal{H}_0$).

As the corollary of this theorem, in [76] the existence of nontrivial invariant subspace for a certain subclass of hyponormal operators was proved.