Preface

This book is an introduction to the theory of Brownian motions and heat kernels on matrix Lie groups and manifolds with an emphasis on explicit computations related to a special class of Brownian motion functionals. These functionals, known as stochastic area functionals and winding processes, are studied in the context of Riemannian and semi-Riemannian fibrations with totally geodesic fibers.

The base and total spaces of these fibrations are defined over the complex and quaternionic fields, respectively, and admit principal bundle structures associated with abelian and non-abelian unitary groups, respectively. The base spaces are symmetric while the total spaces are only homogeneous. After a review of the general theory, the book deals with rank-one symmetric spaces, namely hyperbolic and projective spaces, and then moves on to compact and non-compact Grassmannian spaces.

A substantial part of the book's computations rely on stochastic analysis on manifolds and Lie groups. Appendix A provides all necessary background for readers who are not familiar with stochastic calculus. Another key ingredient in the book's study is the notion of horizontal lift of Brownian motions which yields the notion of horizontal Brownian motion. Almost all formulas for the generators and semigroups of the corresponding Brownian motions are known for rank-one symmetric spaces, but less is known for higher ranks. Appendices B and C collect various existing formulas for the sake of self-containedness.

The study of stochastic area functionals and winding processes in the curved geometrical models alluded to above reveals the effect of the curvature and of the choice of the base fields on their long-time behaviors. In particular, the reader will notice the different speeds of convergence and limiting distributions depending on compactness or non-compactness of the underlying models.

Another new aspect of the book's approach is the connection it makes between radial parts of sub-Laplacians arising from anti–de Sitter fibrations and radial parts of magnetic Laplacians with constant magnetic field (referred to as Maass Laplacians when the base field is complex). Since this connection nowadays seems to be known only to a few people, the book has tried to make it explicit and further exploits it to derive formulas for subelliptic heat kernels.

As to Grassmannian spaces, the book's approach provides new constructions of matrix-valued (spherical and hyperbolic) Jacobi diffusions starting from Brownian motions on unitary (definite and indefinite) groups.

Overall, this book provides a comprehensive and self-contained introduction to the theory of heat kernels and Brownian motions on manifolds and to the study of stochastic area functionals and winding processes of Brownian motions in curved

geometrical models. It is intended for readers with some background in probability theory, stochastic processes, and differential geometry, but no prior knowledge of stochastic analysis on manifolds or Lie groups is required.

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