Introduction

By its simplicity, the number of its far-reaching applications, and its connections to many areas of mathematics, Paul Lévy's stochastic area formula is undoubtedly among the most important and beautiful formulas in stochastic calculus.

Let $Z(t) = B_1(t) + iB_2(t)$, $t \geq 0$, be a Brownian motion in the complex plane such that $Z(0) = 0$. The algebraic area swept out by the path of Z up to time t is given by half of the value

$$S(t) = \int_{Z[0,t]} (x\,dy - y\,dx) = \int_0^t (B_1(s)\,dB_2(s) - B_2(s)\,dB_1(s)),$$

where the stochastic integral is an Itô integral, or equivalently a Stratonovich integral, since the quadratic covariation between B_1 and B_2 is 0. Lévy's area formula

$$\mathbb{E}\big(e^{i\lambda S(t)} \mid Z(t) = z\big) = \frac{\lambda t}{\sinh \lambda t} e^{-\frac{|z|^2}{2t}(\lambda t \coth \lambda t - 1)}$$

was originally proved in [136] by using a series expansion of Z. The formula nowadays admits many different proofs. A particularly elegant probabilistic approach is due to Yor [190] (see also [186]). The first observation is that, due to the invariance by rotations of Z, one has for every $\lambda \in \mathbb{R}$,

$$\mathbb{E}\big(e^{i\lambda S(t)} \mid Z(t) = z\big) = \mathbb{E}\big(e^{-\frac{\lambda^2}{2}\int_0^t |Z(s)|^2\,ds} \mid |Z(t)| = |z|\big).$$

One considers then the new probability measure

$$\mathbb{P}^{\lambda}_{/\mathcal{F}_t} = \exp\left(\frac{\lambda}{2}(|Z(t)|^2 - 2t) - \frac{\lambda^2}{2}\int_0^t |Z(s)|^2\,ds\right)\mathbb{P}_{/\mathcal{F}_t}$$

under which, thanks to Girsanov's theorem, $(Z(t))_{t\geq 0}$ is a Gaussian process (an Ornstein–Uhlenbeck process). The stochastic area formula then easily follows from standard computations on Gaussian measures.

Somewhat surprisingly, Lévy's stochastic area formula and the stochastic area process $(S(t))_{t\geq 0}$ appear in many different contexts, for instance the following:

- Watanabe [184] points out the connection between the stochastic area formula and the differential of the exponential map in Lie groups.
- The stochastic area formula also appears in works by Bismut [54, 55], where probability methods are used to prove index theorems. Variations of these methods allow one to construct explicit parametrics for the heat equation on vector bundles; see [22].

- The Mellin transform of $S(t)$ is closely related to analytic number theory and in particular to the Riemann zeta function; see the survey paper [52] by Biane, Pitman, and Yor. We also point out some connections with algebraic geometry as explained by Hara and Ikeda [118].
- The stochastic area process is a central character in the construction of Terry Lyons' rough paths theory; see the monograph by Friz and Victoir [102].
- The stochastic area formula also appears as an important tool in Malliavin calculus; see in particular Watanabe [183].
- The stochastic area process is also intimately connected to sub-Riemannian geometry. More precisely, Gaveau [106] actually observed that the 3-dimensional process $(B_1(t), B_2(t), S(t))_{t\geq 0}$ is a horizontal Brownian motion on the Heisenberg group. As a consequence, the stochastic area formula yields an expression for the heat kernel of the sub-Laplacian on the Heisenberg group; see also Hulanicki [126].
- The stochastic area formula is also relevant in mathematical physics as pointed out in Duplantier [84] and Yor [191] (see also the references therein).

The stochastic area process or the Lévy area formula can be generalized in many different directions. For instance, there exist analogues for Gaussian processes as in Ikeda, Kusuoka, and Manabe [128] (see also Coutin and Victoir [75]) or, for free Brownian motion, as in Capitaine and Donati-Martin [66] or Victoir [176].

In the present monograph we present generalizations of both the area process and the Lévy area formula for Brownian motions on manifolds. As we will show, natural generalizations of the stochastic area for a Brownian motion $(X(t))_{t\geq 0}$ on a manifold $\mathbb{M}$ are functionals defined as

$$S(t) = \int_{X[0,t]} \alpha,$$

where α is a 1-form with some geometric significance taking values in a Lie algebra (or one of its quotients). To ensure the existence of an explicit expression for the stochastic area formula, we will need that $\mathbb{M}$ is a Riemannian homogeneous space and that α is a form on $\mathbb{M}$ coming from the connection form of a homogeneous bundle over $\mathbb{M}$; for a somewhat general setting see Example 3.1.16 and Section 3.5. For instance, in the simplest case where α is $\mathbb{R}$-valued, the *area* 1*-form* will arise from a Kähler structure on $\mathbb{M}$. By definition, a Kähler form on a complex manifold $\mathbb{M}$ is a closed 2-form ω that induces the metric on $\mathbb{M}$ in the sense that $\omega(X, JY)$ is the Riemannian metric on $\mathbb{M}$, where J is the almost complex structure. It is a classical result in complex analysis that such a 2-form can (at least locally) be written as $\omega = i\partial\bar{\partial}\Phi$, where ∂ and $\bar{\partial}$ are the Dolbeault operators and Φ is a smooth function.

The (at least locally defined) real 1-form

$$\alpha = \frac{1}{2i}(\partial - \bar{\partial})\Phi$$

is then a natural *area 1-form* on $\mathbb{M}$ which satisfies $d\alpha = (\partial + \bar{\partial})\alpha = \omega$. When $\mathbb{M}$ is compact, the homogeneous bundle over $\mathbb{M}$ we mentioned before is an $\mathbb{S}^1$-bundle referred to as the Boothby–Wang fibration (see the discussion in Section 3.5.2). It is worth noting that the pull-back to that bundle of the form α then yields a *winding 1-form*: integrating this form against a path describes the $\mathbb{S}^1$ fiber component of that path. This interplay between stochastic areas and windings is at the heart of a lot of our analysis.

We will also consider more general 1-forms α like $\mathfrak{su}(2)$-valued 1-forms and associated bundles; however, we stress that our goal in the monograph is not to develop a general and abstract theory of stochastic-area-type functionals associated with homogeneous bundles. We will rather focus on specific relevant examples for which very concrete calculations can be done. Covering specific examples in great detail will give us the opportunity to explore several topics of independent interest related to the study of stochastic area functionals. We will in particular focus our attention on connections with the theory of Riemannian submersions and associated horizontal Brownian motions, the theory of complex and quaternionic projective and hyperbolic spaces, the theory of hypoelliptic heat kernels, and the theory of random matrices.

Organization of the monograph

In Chapter 1 we introduce the stochastic area functionals for Euclidean Brownian motions and the associated Lévy area formulas. We show how the geometry of such functionals is intimately related to the theory of nilpotent Lie groups. We also study some Brownian functionals which belong to the same family as the stochastic areas: the Brownian winding functionals.

Chapter 2 is an introduction to the theory of Brownian motions on Lie groups and Riemannian manifolds as is needed in the monograph.

Chapter 3 presents the general theory of Riemannian submersions and associated horizontal Brownian motions. A special emphasis is put on examples and on the situation where the submersion is the projection map of a principal bundle; it is then shown in that case that stochastic area functionals appear as the fiber motion of horizontal Brownian motion of that bundle.

Chapters 4 through 7 are devoted to the study of stochastic areas and of their distributions on complex projective spaces and complex hyperbolic spaces, on the quaternionic projective spaces, on the quaternionic hyperbolic spaces, and on the octonionic projective and hyperbolic spaces, respectively.

In Chapter 8, using the techniques of random matrix theory, we study Brownian motions and associated eigenvalue processes on complex Grassmannian spaces. We then define stochastic area functionals in that setting by means of the Stiefel fibration. The distribution of these area functionals is computed and limit theorems are proved. Finally, in Chapter 9 we study Brownian motions and associated eigenvalue processes on the complex hyperbolic Grassmannian spaces. Stochastic area functionals in that setting can be defined by means of the hyperbolic Stiefel fibration. The distribution of these area functionals is computed and limit theorems are proved.

Each chapter ends with a short “Notes and references” section which suggests references for further reading on some of the materials.

At the end of the monograph, we included three appendices. The first one is a short summary of stochastic calculus results used throughout the monograph. In particular, in Theorem A.9.1 we present Yor’s transform method, which is the powerful method due to Yor that is used throughout the monograph to compute conditional Laplace transforms of additive functionals of diffusion processes. The second appendix gives some useful formulas concerning the special diffusion operators appearing in the monograph. Finally, the last appendix lists the formulas for the radial parts of the Laplace–Beltrami operator on some rank-one symmetric spaces.