Preface

One of the major challenges of AI is the task of making machines (robots) capable of moving autonomously. A machine, such as a self-driving car, receives an order expressed in a natural language (such as English) to get to a specific state, and it autonomously decides the motion and implements it taking into account the external conditions. A motion algorithm which makes it possible for a robot to move autonomously is, mathematically, a function of the current and the target states which outputs the robot's motion. About 20 years ago it was observed that the complexity of robot motion algorithms depends on the topology of their configuration spaces. This observation was developed further by the mathematics research community of topological roboticists.

The present book is designed as a panorama of the current achievements of topological robotics, with each chapter surveying a different theme or technique. The book is written by leading researchers and represents up-to-date results including some very recent achievements.

Another major goal of this publication is to promote the subject and its potential within the scientific and engineering communities. We believe that geometric and topological methods and techniques presented in the book deliver great promise for engineering applications in a broader context of AI, especially where a machine is required to make decisions which amount to choices out of a continuum of possibilities.

The authors of the volume state many open problems, a feature which will be helpful for young researchers who intend to join the field.

Chapter 1, *Autonomous Robots, Motion Algorithms and Topology*, written by Michael Farber, explains how methods and concepts of algebraic topology can assist the task of programming a robot to move autonomously. This chapter starts with the main motivations and basic results and then progresses to cover some recent developments. In particular, this chapter introduces the fundamental notion of topological complexity, or TC, on which a significant part of the interactions between topology and robotics is based. Examples of calculations of this invariant are given as well as relations with other classical invariants. This chapter also discusses the Rationality Conjecture and the theory of sequential parametrized motion algorithms. Furthermore, the chapter includes detailed analysis of the problem of controlling multiple robots in the presence of many obstacles in Euclidean space and describes an explicit motion algorithm (inspired by topological ideas) for controlling swarms of robots avoiding collisions with multiple obstacles.

Chapter 2, *Equivalent Environments and Covering Spaces for Robots*, written by Vadim K. Weinstein and Steven M. LaValle, formally defines a robot system, including its sensing and actuation components and its surrounding environment, as a general, topological dynamical system. The focus is on determining conditions under which various environments where the robot can be placed are indistinguishable from the viewpoint or experience of the robot. The overall framework is applied to unify previously identified phenomena in robotics and related fields, in which moving agents with sensors must make inferences about their environments based on limited data.

Chapter 3, *Some geometric and topological data-driven methods in robot motion planning*, written by Boris Goldfarb, surveys the recent progress and promising future research directions in data-driven geometric and topological methods for path-planning in robotics with emphasis on the use of discrete Morse theory. The recent advances in topological data analysis and related metric geometry, topology and combinatorics have provided new tools to address these engineering tasks.

Chapter 4, *Towards control, learning and intelligence in reconfigurable systems*, written by Dan P. Guralnik, describes configuration spaces of reconfigurable systems as non-positively curved cubical complexes and discusses practical questions of how one can exploit geometric ideas for the control of such systems. The chapter reviews the approaches to navigation in CAT(0) cubical complexes by Owen and Ardila, and compares them with the alternatives developed by Guralnik and Koditschek. The author then proceeds to review more recent efforts to obtain formal results on learning of navigation tasks in certain classes of reconfigurable systems relevant to autonomy and AI.

Chapter 5, *Geometric and topological properties of manifolds in robot motion planning*, written by Stephan Mescher, gives an overview of results on topological complexities of manifolds and explains how their geometric structures influence their topological complexities. This chapter also studies geodesic motion planning in Riemannian manifolds (see also Chapter 11) when the motion produced by an algorithm is required to be a shortest length geodesic. Other themes of this chapter include the method of navigation functions and the theory of spherical complexities.

Chapter 6, *Equivariant topological complexities*, written by Mark Grant, discusses situations when the configuration space of a system admits symmetry. Several variations of topological complexity have emerged that take symmetry into account in different ways, either by asking that the motion planners themselves admit compatible symmetries, or by exploiting the symmetry to motion plan between functionally equivalent configurations. The chapter surveys the main definitions due to Colman–Grant, Lubawski–Marzantowicz, Błaszczyk–Kaluba and Dranishnikov, and some related notions.

Chapter 7, *Computing with the* TC*-canonical class*, by Lucile Vandembroucq, presents techniques which are effective for studying the maximality of the topological complexity especially in the case of closed manifolds. The chapter gives maximality and non-maximality results for manifolds with abelian fundamental group.

Chapter 8, *Motion planning in real projective spaces*, written by Jesús González, surveys a surprisingly close connection between the manifold structure of the real projective space P^n and the path planning problem for an autonomous system whose space of states is homotopy equivalent to P^n. This relates the classical differential topology invariants of P^n, such as its immersion and embedding dimensions, with properties of motion algorithms over such manifolds.

Chapter 9, *Generalized topological complexity and its monoidal version*, written by José Manuel García-Calcines, surveys important results showing that in many practically relevant cases the topological complexity can be realized by partitions into arbitrary subsets, and not necessarily open subsets. The chapter also includes the most significant implications of this result as well as the current status of the Iwase–Sakai conjecture.

Chapter 10, *Robotics and the fundamental group*, written by John Oprea, reveals the influence of the fundamental group of the configuration space of the system on TC. In particular, this chapter discusses in detail the case when the configuration space is aspherical and illustrates deep relations with objects of equivariant topology such as classifying spaces for families of subgroups.

Chapter 11, *Geodesic complexity*, written by Donald M. Davis, surveys the results on geodesic complexity that arises by requiring the motion paths to be minimal geodesics. Some important examples have been thoroughly analysed including the cube and the 2-point ordered configuration spaces of star graphs.

Chapter 12, *Farber's conjecture and beyond*, written by Ben Knudsen, surveys the state-of-the-art knowledge about sequential topological complexity of configuration spaces of graphs and gives a unifying, elementary, and self-contained account of the major results and methods.

Chapter 13, *Topological complexity of a map*, written by Petar Pavešić, surveys recent results on versions of topological complexity associated with a continuous map. The author discusses similarities and differences between alternative definitions as well as relations with the classical invariants (such as the Lusternik–Schnirelmann category and the topological complexity of spaces). This chapter discusses applications of topological complexity of maps to the manipulation problems of robotics, and some other applications.

Chapter 14, *Distributional topological complexity and LS-category*, written by Alexander Dranishnikov and Ekansh Jauhari, introduces a totally new probabilistic version of topological complexity of a space, denoted dTC, which can be relevant to some motion planning problems of robotics. The authors also define a distributional version of the Lusternik–Schnirelmann category dcat and do explicit computations and estimates for both $\mathsf{dTC}(X)$ and $\mathsf{dcat}(X)$.

Finally, we warmly thank our colleagues and friends, the authors of this volume. We are also thankful to the reviewers for rigorously reading and commenting on the chapters.

We believe that the book will be of interest for a variety of readers and will inspire further use of topological tools in engineering and AI.

April 2024 Michael Farber and Jesús González