Introduction

In 1959, Ferdinand Veldkamp's dissertation was published, entitled "Polar geometry." In this work he unifies the substructures of projective spaces determined by the absolute points of orthogonal, Hermitian and symplectic polarities. Although these structures are quite different within projective space, he observes that they are intrinsically the same. More precisely, if you consider only the system of all subspaces formed by the absolute points under such a polarity, you get geometries with the same properties. He then puts these properties into an axiomatic system. Simultaneously, Jacques Tits investigated geometries of exceptional type, looking for a unified axiomatic approach to the geometries of all groups of Lie type. This would eventually lead to the notion of a building. However, in the 1950s, Jacques Tits was still thinking of point-line geometries, and he concentrated on the exceptional types because the classical ones – the projective spaces and the polar spaces – were already completely known to him. In particular, he had four axioms to describe all polar spaces and these we will use in Chapter 1. They actually yield more types of polar spaces than in Veldkamp's approach. However, these axioms presuppose that we start with a family of projective spaces. An alternative and more elementary axiom system, using only points and lines, is given later by Francis Buekenhout and Ernie Shult. We will prove in Chapter 2 that it is equivalent to the system of Jacques Tits.

Just as projective spaces have a *dimension*, polar spaces have a *rank*. This is one more than the maximum dimension that occurs in its projective subspaces. The polar spaces of rank 2 are also known as *generalised quadrangles*, and they are to polar spaces what axiomatic projective planes are to projective spaces. Indeed, if a projective space has dimension at least 3, then we know that it comes from a vector space over a skew field, while there is no classification for the projective planes. Polar spaces of rank at least 3 can also be classified (although the situation here is slightly more complicated), but the generalised quadrangles cannot be classified and form a rich theory in itself, which we will only consider occasionally.

The above already indicates that there are more polar spaces than the classes of examples that gave them their names. Thus we will see that *not* for every polar space there exists a projective space whose point set naturally contains that of the polar space. In Chapter 5 we discuss two classes of such *non-embeddable* polar spaces of rank 3, because it will turn out that all polar spaces of rank at least 4 are embeddable. A polar space that is *embeddable in projective space* can always be described by a reflexive form on a vector space over a skew field, except when the characteristic of the underlying skew field is 2; in the latter case we need the more general concept of a *pseudo-quadratic form*. We discuss this in detail in Chapter 4. This generalises the concept of polarity in projective geometry, and forms the basis of the classification of polar spaces of rank at least 3, as obtained by Jacques Tits. The connection with polarities and reflexive forms is explained in Chapter 3.

The driving force behind the study of polar spaces is the fact that these structures are closely related to some classical simple groups. By this we mean the special linear groups $\mathsf{SL}(n)$, the orthogonal groups $\mathsf{O}(n)$, the symplectic groups $\mathsf{Sp}(2n)$ and the unitary groups $\mathsf{U}(n)$. The ones of the first type are related to projective spaces of dimension $n-1$; the three other ones to polar spaces. They fall into the broader class of *groups of Lie type*, and they are classical in the sense that they exist for any rank n. The other Lie-type groups exist only for specific (low) ranks and are called *exceptional*; the prominent examples of these are those of type G_2, F_4, E_6, E_7 and E_8.

In the 1950s and 1960s, Tits introduced his famous *theory of buildings*. His goal was to attach a geometry (called a *spherical building*) to each of the above groups (particularly the exceptional ones), on which that group would then act as an automorphism group. This idea works spectacularly well. Tits obtained a particularly general axiomatic definition of these combinatorial objects, of which the projective geometries and the polar spaces form important classes. In fact, it allows us to study these geometries without any knowledge of the corresponding group of Lie type. We discuss some important automorphisms of polar spaces in Chapter 7.

Moreover, if you consider the buildings attached to the exceptional groups of type F_4, E_6, E_7 and E_8 as a certain point-line geometry, then these appear to be full of polar spaces. In the 1980s, the foundational work of Bruce Cooperstein gave rise to an axiomatic system for point-line geometries equipped with polar spaces, which precisely aimed to capture the behaviour of these exceptional geometries. This gave rise to *parapolar spaces*, which we will meet in Chapter 9, at the end of the course.

We do not have the ambition here to study polar spaces as buildings theoretically. Nevertheless, we will make use of some typical concepts from this theory. They allow certain theorems to be proved more elegantly, but above all they also increase insight and immediately open the door to generalisations to other buildings.

Finally, we note that generalised quadrangles can also be seen as special cases of *generalised polygons* [9], which are essentially the atoms of the buildings. The generalised triangles correspond exactly to the axiomatic projective planes. The generalised hexagons are closely related to the exceptional groups of type G_2 and are constructed using polar spaces of “type D_4” (which are actually hyperbolic quadrics of rank 4) and the *triality principle*. In Chapter 6 we will study this type of polar spaces, which we will call *top-thin* polar spaces, and classify them for general rank n. We focus on the special case $n = 4$ in Chapter 8, where our interest is mainly in the associated *trialities* (a generalisation of polarities in projective spaces).

Throughout these notes, especially in Chapters 3, 4, 5 and 8, certain algebras will play a special role, such as the (whether or not *split*) *octave or octonion algebras*. These algebras are 8-dimensional over a field and the special thing about them is that they are no longer associative (but still *alternative*). Appendix A is devoted to a study of these algebras and their relatives.

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We now comment in somewhat more detail on each chapter.

Chapter 1 contains all basic material that is needed to understand the theory of polar spaces, and to provide the proofs in the rest of these notes. Some proofs might not be the shortest possible ones, but I have also tried to take didactics into account. This chapter is not copied from anywhere, but similar results might be found in the literature, in particular in [1] and [3]. The proof that planes in polar spaces have the Moufang property is original in that I use a characterisation of Moufang planes that I proved myself (see [10]). The proofs in [1] and [8] are more involved (because they do not use [10]).

Chapter 2 is a chapter that cannot be skipped in any modern treatise on polar spaces. I based the proof on [1, Section 7.4]. I explicitly prove that the Buekenhout–Shult axioms imply the Tits axioms, including the finite rank. In most modern literature one includes the infinite rank polar spaces when stating the Buekenhout–Shult axioms, but usually only deals with the finite rank anyway. Here, my aim is to stick as closely as possible with the original set-up. It is already general enough.

Chapter 3 connects polar spaces with polarities of projective spaces (and that is where the name comes from), taking Tits' viewpoint [8] of generalised polarities. The algebraic counterpart is a reflexive form on a vector space. That is either a symmetric bilinear form or a sesquilinear form proportional to a Hermitian form (which follows from standard algebra and is not proved in the current notes). We derive a standard equation of polar spaces obtained from a polarity and discuss this in the case of some particular fields. All results of this chapter are quite standard. The properties of generalised polarities that we prove are the basic ones that we need further on. In particular, we prepare to prove that polar spaces of any rank exist that are embedded in projective space, but do not arise from a reflexive or quadratic form.

Chapter 4 introduces pseudo-quadratic forms. These are algebraic structures generalising reflexive forms. In characteristic 2 they can describe polar spaces that are not related to hermitian forms as in the previous chapter, and we explicitly prove this (being not aware of a similar proof in the literature). Chapters 3 and 4 used to be one chapter, and then we only treated pseudo-quadratic forms. This, however, was less transparant and more technical. The current approach only considers the most technical case of a pseudo-quadratic form where it is really needed. Note also that our exposition is based on lectures of Jacques Tits at the Collége de France and as such differs slightly from any literature. In particular, we do not need to worry about traced-valued forms, although we comment on it to connect with the existing literature. The rank 2 case can be found in [9, Chapter 2].

Chapter 5 defines the non-embeddable polar spaces of rank 3. The top-thin case is well known and standard. We do not go into too much detail here. You will not find an elementary description of the thick case in the existing literature; the latter usually provides references to Tits [8] using algebraic groups, or a paper by Bernhard Mühlherr [5] proving that such a space arises as fixed point structure of a building of type E_7. I worked out an elementary description on the occasion of the first year of teaching the course on polar spaces in 2011 and later published it together with Bart De Bruyn [4]. The proof in the current notes is still the original one before publication of the paper.

Roughly speaking, Chapters 1 and 2 approach polar spaces in a synthetic way, whereas Chapters 3, 4 and 5 approach them in an analytic way. This situation can be best compared with the classical distinction between synthetic affine and Euclidean geometry and the analytic approach using coordinates. General properties are best handled in a synthetic way using the axioms, whereas properties of specific polar spaces are better proved using the associated algebraic structures.

Chapter 6 classifies top-thin polar spaces. These polar spaces admit a description as so-called oriflamme geometry. This is best explained and put in a broader picture by introducing diagrams and geometries of type M. The classification itself is based on the fact that singular subspaces are Pappian (defined over a field) as soon as the rank is at least 4 (and this is proved using geometry: we construct a generalised quadrangle inside a projective space which can only exist in the commutative case) and that a cone over a hyperbolic quadric supports a unique hyperbolic quadric of one rank larger. The main result of this chapter proves the equivalence of the synthetic and the analytic approach for top-thin polar spaces.

Chapter 7 studies certain collineations of polar spaces. There is so much to say about the automorphism groups of polar spaces, because they are classical groups, and have a long history. However, since the emphasis in the current notes is on the geometry of the polar spaces, we limit ourselves to studying the long root elations, here under the more specific names of central and axial collineations. They have a close connection to geometry via the theory of long root subgroup geometries. We have no intention to touch that theory, but these kind of collineations have nice geometric properties, and this chapter should show this.

Chapter 8 is devoted to a very beautiful and rare phenomenon called triality. It is an exceptional symmetry of the oriflamme geometry corresponding to a top-thin polar space of rank 4. As usual in these lecture notes, we do not study the algebraic side of triality, but instead prove in detail what is called in the literature "geometric triality." We explain both the synthetic and analytic approach. For the latter, we use the algebraic tool of split Cayley algebra. This approach is inspired by work of Springer and Veldkamp [7]. However, we avoid the use of results of "algebraic triality" (as happens in [7]) and provide a complete elementary treatise (which we did not read elsewhere).

The proof that trialities of order 3 provide examples of generalised hexagons is taken from [9], with a simplification in one of the arguments.

Chapter 9 provides an introduction to Cooperstein's theory. This theory is responsible for the creation of geometries called parapolar spaces. The building blocks of such geometries are polar spaces. Polar spaces themselves yield parapolar spaces by considering their Grassmannians. The true motivation for introducing parapolar spaces are the exceptional groups and geometries, and we give a taste of that by mentioning an axiom system for parapolar spaces related to the exceptional groups of type E_6. This chapter opens the door to the point-line approach of spherical buildings, which is the main theme of Shult's book [6], where further reading can be found. This chapter is entirely synthetic.

Appendix A provides some more background on one of the most prominent algebraic structures related to polar spaces, namely the alternative quadratic algebras. Properties used throughout these lecture notes are proved in this appendix, but we provide much more background. The bulk of this appendix is inspired by online notes of McCrimmon. It permits to define the class of Moufang projective planes that are not Desarguesian.

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We have included a few exercises in these lecture notes. They all concern exercises that were given during the course at Ghent University. Some of them served as exam problems. In general, the students worked in small groups of two or three persons and had to report in writing on the solutions individually.

It is perhaps worthwhile to note that the exercises differ from exercises in other books (like [1] and [6]) in that they do not explore deeper theory and further developments, but instead have the purpose to really train the students in finding arguments and reasonings using the given properties and theory. We seldom define new notions in the theory and certainly do not use properties mentioned in the exercises in further theory.

The level of the exercises can differ; it varies from very easy to quite difficult.

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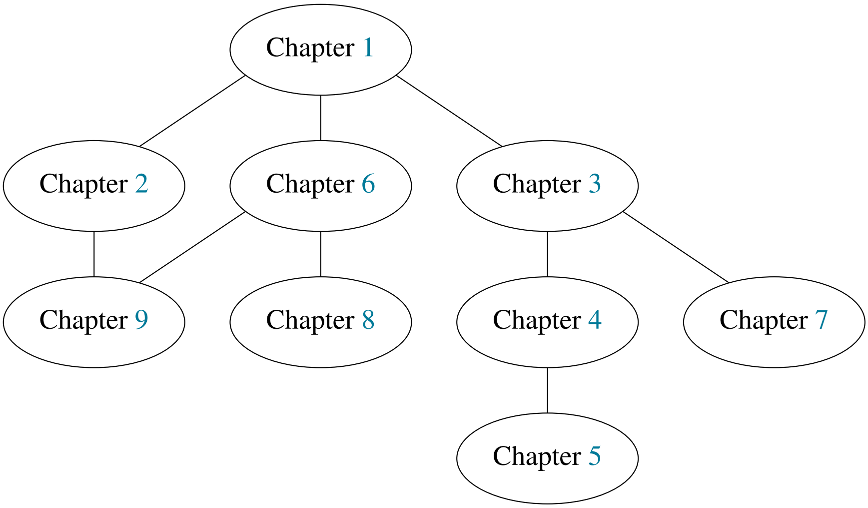
Leitfaden. We recommend to start with Chapter 1. From there, all following chapters can be read quite independently, if one accepts the equivalent Buekenhout–Shult axiom system without proof (although it is best to read that chapter before embarking on Chapter 9). Also, Chapter 4 is best read after Chapter 3 and Chapter 8 uses some results of Chapter 6. Chapter 7, finally, uses terminology and results from Chapter 3; it also refers to Chapter 5, but only in an non-essential way.


Chapter 1
Chapter 2
Chapter 6
Chapter 3
Chapter 9
Chapter 8
Chapter 4
Chapter 7
Chapter 5