Preface

This is the final volume of a three-volume set, which comprehensively studies the tractability of multivariate problems. The subjects treated in the three volumes can be briefly characterized as follows.

- Volume I [81]: we primarily study multivariate problems specified by *linear operators* and algorithms that use *linear information* Λ^{all} given by arbitrary *continuous linear functionals*.
- Volume II [82]: we study multivariate problems specified by *linear functionals* and a few selected *nonlinear functionals*, and algorithms that use *standard information* Λ^{std} given by *function values*.
- Volume III: we study multivariate problems specified by *linear operators* and a few *nonlinear operators*, and algorithms that use mainly *standard* information Λ^{std}.

As already mentioned in the previous two volumes, the class Λ^{std} is often the only class of information operations that is allowed for many computational problems. The class Λ^{all} is usually too general to be of practical importance. But there are several reasons why Λ^{all} is very useful from a theoretical point of view. First of all, Λ^{all} is usually much easier to analyze than Λ^{std}. Secondly, all negative results for Λ^{all} also hold for Λ^{std}. In particular, all lower bounds for Λ^{all} presented in Volume I are also applicable for Λ^{std}. Thirdly, and this is the most important property, the power of Λ^{std} is often the same or roughly the same as the power of Λ^{all}. This allows us to use the positive results for Λ^{all} also for Λ^{std}. Relations between the classes Λ^{std} and Λ^{all} are one of the central topics of Volume III.

The problems studied in the three volumes are defined on spaces of d-variate functions. In computational practice, d is often very large, perhaps even arbitrarily large. Such a d-variate problem is tractable if we can approximate it with error at most ε, using a number of linear functionals from Λ^{std} or Λ^{all} that is *not* exponential either in d or in ε^{-1}. Tractability has been studied since the 1990s, see [156], [157]. There are several notions of tractability, such as

- strong polynomial,
- polynomial,
- quasi-polynomial,
- weak, and
- T-tractability.

Moreover, we study tractability in different settings for the classes Λ^{std} and Λ^{all} and for the absolute, normalized and relative error criteria. Each setting (worst case, average case, probabilistic, randomized) is specified by the definition of the error and the cost of an algorithm.

In this volume we present tractability results in the worst case, average case and randomized settings. We do this for the absolute and normalized error criteria. We do not cover the probabilistic setting and the relative error criterion, leaving these subjects for future research.

Many multivariate problems suffer from the *curse of dimensionality*. This means that even the best possible algorithm must use exponentially many in d function values (for the class Λ^{std}) or linear functionals (for the class Λ^{all}) to have error at most ε. The *curse of dimensionality* is usually present for problems defined over standard unweighted spaces. In this case all variables and groups of variables play the same role. The curse of dimensionality can often be vanquished if we switch to *weighted* spaces, in which we monitor the importance of all variables and groups of variables by sufficiently decaying weights.

In all volumes of this work, one of our main goals has been to present conditions on the weights that are necessary and sufficient to obtain the various notions of tractability. In Volume I, we studied linear and a few selected nonlinear operators using Λ^{all}. In Volume II we studied linear and a few selected nonlinear functionals using Λ^{std}. In the current Volume III, we once again study linear and a few selected nonlinear operators, but now using Λ^{std}. We explain the approach taken in this volume to prove tractability of linear problems for the class Λ^{std}.

We first analyze *multivariate approximation*, i.e., the embedding operator

$$\text{APP}_d : F_d \to L_2$$

for a reproducing kernel Hilbert space F_d of d-variate functions which is continuously embedded in a weighted L_2 space. Then we approximate $\text{APP}_d f = f$ in the randomized, average case and worst case settings. We verify whether the power of the class Λ^{std} is the same (or roughly the same) as the power of the class Λ^{all}. We compare the classes Λ^{std} and Λ^{all} in terms of the rate of convergence of the nth minimal errors, and in terms of the particular notion of tractability. If the power of the classes Λ^{std} and Λ^{all} is roughly the same then necessary and sufficient conditions on weights to get various notions of tractability for the class Λ^{all} presented in Volume I are also applicable for the class Λ^{std}.

For multivariate approximation the power of the class Λ^{std} is roughly the same as the power of the class Λ^{all} in the randomized and average case settings for the normalized error criterion, and also for the absolute error criterion under certain assumptions on the behavior of the initial errors. This means that the rates of convergence are the same for the classes Λ^{std} and Λ^{all}, as well as that various kinds of tractability are equivalent for the classes Λ^{std} and Λ^{all} and hold under the same conditions on weights.

In the worst case setting, the situation is more complicated. The powers of Λ^{std} and Λ^{all} depend on the trace $\text{tr}(W_d)$ of the operator $W_d = \text{APP}_d^* \text{APP}_d : F_d \to F_d$. If

$$\text{tr}(W_d) = \infty$$

then there is *no* general relation between the class Λ^{std} and Λ^{all}. The nth minimal errors for the class Λ^{all} can be proportional to $n^{-\alpha}$ with $\alpha \in (0, \frac{1}{2}]$, and the nth minimal errors for the class Λ^{std} can go to zero arbitrarily slowly (e.g., they can go like $1/\ln\ln n$, or even slower). There is also no general relation between tractability for these two classes. We may have strong polynomial tractability for the class Λ^{all}, but the curse of dimensionality for the class Λ^{std}.

The situation is better when we assume that

$$\operatorname{tr}(W_d) < \infty.$$

Then the rates of convergence for the classes Λ^{std} and Λ^{all} are related. More precisely, if the nth minimal errors for the class Λ^{all} go to zero as n^{-r} for some $r > \frac{1}{2}$ then the nth minimal errors for the class Λ^{std} go to zero at least as fast as n^{-r_1}, where

$$r_1 = r - \frac{r}{2r+1}.$$

Note that $r - r_1 \le \frac{1}{2}$. Moreover, for large r, we have $r/r_1 \approx 1$, and so the minimal errors are roughly equal for Λ^{all} and Λ^{std}.

When the trace is finite, tractability results for the classes Λ^{std} and Λ^{all} depend on the behavior of the trace $\operatorname{tr}(W_d)$ as a function of d. Almost anything can happen. For instance, there are problems that are strongly polynomially tractable for the class Λ^{all}, but suffer from the curse of dimensionality for the class Λ^{std}. On the other hand, there are problems having the same kind of tractability for both classes Λ^{std} and Λ^{all}.

The results on multivariate approximation in the randomized, average case and worst case settings follow from results obtained in many papers by various authors as well as from some additional results presented in this volume. We give a proper credit in the successive sections and chapters where multivariate approximation is studied.

Knowing the results for multivariate approximation, we turn to study general linear problems which are specified by linear multivariate operators. We approximate linear operators by

- modified algorithms for multivariate approximation or
- modifications of Smolyak/sparse grid algorithms which we already discussed in Volume II for approximation of linear functionals.

It is remarkable that the results on multivariate approximation can be successfully applied not only for general linear problems but also for some nonlinear problems as will be shown in the corresponding chapter. These relations justify the central role of multivariate approximation in Volume III. Under appropriate assumptions, we show how tractability of multivariate approximation implies tractability of other linear operators.

Algorithms based on Smolyak/sparse grid constructions need not be restricted to multivariate approximation. Smolyak/sparse grid can be used for any linear problem having a tensor product structure, assuming that we know how to approximate the univariate operators efficiently. We study such algorithms mostly for finite-order and product weights. One of the main results is that, under mild assumptions, finite-order weights imply strong polynomial or polynomial tractability for linear operators. A similar result holds for product weights that decay sufficiently quickly.

One chapter deals with a few selected nonlinear problems. As has been often stated, nonlinearity is not a property; it is the lack of a property. As a result, we cannot develop tractability theory for arbitrary nonlinear problems; rather, we need to treat each nonlinear problem on its own. We restrict our attention to the worst case setting; extending the analysis of nonlinear problems to other settings (such as the average case and randomized settings) is a challenging and difficult problem. As we shall see, the worst case analysis of nonlinear problems will be done by showing various relations to multivariate approximation. Quite often tractability results for multivariate approximation imply tractability results for the nonlinear problems studied here.

Volume III consists of nine chapters numbered from 21 to 29 since Volumes I and II have the first twenty chapters. We briefly comment on their contents.

Chapter 21 presents four examples of multivariate approximation which, as already mentioned, plays the major role in Volume III. We present results in the worst case setting. The first two examples are for classes of infinitely differentiable functions. The unbounded smoothness yields the excellent rate of convergence of the nth minimal worst case errors. This means that multivariate approximation is asymptotically easy in ε^{-1}. However, it is not a priori clear how long we have to wait to benefit from this excellent convergence. It turns out that we still have the curse of dimensionality for the first example and only quasi-polynomial tractability for the second example when we consider the unweighted case with the normalized error criterion. The next two examples of multivariate approximation are for classes of monotone and convex functions. In both cases we have the curse of dimensionality.

Chapter 22 deals with the randomized setting for multivariate approximation. We approximate functions from a reproducing Hilbert space that is continuously embedded in the weighted space L_2. We already reported in Volume I that in this case randomization for the class of arbitrary linear functionals does not help, giving us basically the same results as for the worst case setting. The main question studied in this chapter is what happens for standard information Λ^{std}. It turns out that the power of Λ^{std} in the randomized setting is the same as the power of Λ^{all} in the randomized and worst case settings. Furthermore, the proofs are constructive. That is, we know algorithms using function values at randomly chosen points, enjoying the same property as the optimal error algorithms that use linear functionals in the worst case setting. In particular, this also means that tractability results for the class Λ^{all} in the worst cases setting presented in Volume I can be readily applied for the class Λ^{std} in the randomized setting.

Chapter 23 deals with linear problems in the randomized setting. Since quasi-polynomial tractability was introduced in [33] in 2011 after the publication of Volume II, one of the sections of this chapter is devoted to quasi-polynomial tractability. We then study linear problems over a weighted L_2 space. In this case we have more or less a complete analysis of the randomized setting. We also study linear problems defined over general Hilbert spaces. One of the sections is devoted to multivariate integration, where we report a surprising result of Hinrichs [50] on optimal importance sampling.

Chapter 24 deals with multivariate approximation in the average case setting. We consider Banach spaces equipped with zero-mean Gaussian measures. As in the randomized setting, it turns out that the power of $\Lambda^{\rm std}$ is the same as the power of $\Lambda^{\rm all}$. However, the proofs are now *not* constructive. That is, we know that there are algorithms using function values at some deterministic points enjoying the same tractability properties as algorithms that use optimally chosen linear functionals, but we do not know how to construct them. There is a "semi-construction" of such algorithms but we do not explain here what we mean by semi-construction.

Chapter 25 deals with linear problems in the average case setting. Again we show how the results on multivariate approximation can be also used for linear problems. We also present algorithms for approximation of linear problems whose domain is equipped with finite-order weights.

Chapter 26 deals again with multivariate approximation, but now in the worst case setting. We study relations between the classes $\Lambda^{\rm std}$ and $\Lambda^{\rm all}$. We first consider the case when the trace of the operator $W_d = \mathrm{APP}_d^* \mathrm{APP}_d$ is infinite. Then the class $\Lambda^{\rm std}$ is very weak and it is not related to the class $\Lambda^{\rm all}$. As already mentioned, the rate of convergence of the nth minimal errors in $\Lambda^{\rm std}$ can be arbitrarily bad and there is no relation between tractability for $\Lambda^{\rm std}$ and $\Lambda^{\rm all}$. We then study the case when W_d has a finite trace and show that the classes $\Lambda^{\rm std}$ and $\Lambda^{\rm all}$ are related. However, we still do not know whether the power of $\Lambda^{\rm std}$ is the same as the power of $\Lambda^{\rm all}$.

Chapter 27 deals with linear problems in the worst case setting. Again we show relations between linear problems and multivariate approximation. There are also sections dealing with finite-order weights and weighted tensor products algorithms. Under mild assumptions, finite-order weights imply strong polynomial or polynomial tractability of linear problems.

Chapter 28 deals with a few nonlinear problems in the worst case setting. We study quasi-linear problems. Examples of such problems include the Poisson equation with the Dirichlet or Neumann boundary conditions. We also study Fredholm equations of the second kind. We briefly mention also the heat and Helmholtz equations as well as variants of multivariate approximation for non-convex classes and classes of ridge functions.

Chapter 29 is our final chapter, in which we summarize the results concerning multivariate approximation for both classes $\Lambda^{\rm std}$ and $\Lambda^{\rm all}$ in different settings. We compare the powers of $\Lambda^{\rm std}$ and $\Lambda^{\rm all}$ for arbitrary Hilbert and Banach spaces, under the hypothesis that multivariate approximation is well defined. This summary is done in terms of the rate of convergence of the nth minimal errors, along with tractability results for $\Lambda^{\rm std}$ and $\Lambda^{\rm all}$. We define the *power* function as a quantitative measure of the powers of $\Lambda^{\rm std}$ and $\Lambda^{\rm all}$ and study its properties.

As in the first two volumes, many specific results presented in Volume III have been already published and we tried to carefully report the authors of these results in each chapter and additionally in the Notes and Remarks of each chapter. In fact, each chapter is based on one or more papers, although in many cases we needed to generalize, synthesize or modify the existing results. There are also many new results.

Again all this is described in the Notes and Remarks.

In the course of this book we present a number of open problems. In Volume I we have 30 open problems, and in Volume II we have 61 open problems. That is why we started the count of new open problems in Volume III from 92. The last open problem has the number 149 so there are 58 open problems in Volume III. The list of all open problems is presented in Appendix F. We call it Appendix F since there are five appendices A, B, C, D and E in Volumes I and II. Appendix G presents a number of mistakes or typos our colleagues or we noticed in Volume I and II.

We are especially pleased that some of the open problems from Volumes I and II have been already solved. These results are also reported in Appendix F. We hope that the open problems will be of interest to a general audience of mathematicians and many of them will be soon solved. In this way, research on tractability will be further intensified.

As in the previous volumes, we decided to be repetitious in a number of places. This is the case with some notation as well as with some assumptions in the successive theorems or lemmas. We do this so that the reader will not have to flip too many pages to look for the meaning of the specific notation or the specific assumption. We believe that our approach will be particularly useful after the first reading of the book when the reader wants to return to some specific result without having to remember too much about the hidden assumptions and notation used in the book. At the expense of some repetition, we tried to write each chapter to be as independent of the other chapters as possible. We hope that the reader may study Chapter n without knowing the previous $n-1$ chapters.

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Having finally completed this series of books, let us informally summarize and characterize what it means for a multivariate problem to be tractable. We begin with the worst case setting. Modulo a very few important counterexamples (see the star discrepancy in Volume II), tractability holds only when the space of input functions has some special *structure*. Such structure can be provided by

- weights,
- increasing smoothness with respect to successive variables,
- some other conditions as explained in Section 28.4.4.

Otherwise we usually have the curse of dimensionality in the worst case setting. We think that one of the most challenging tractability issues is to fully characterize structures of spaces for which tractability of multivariate problems holds in the worst case setting. We hope that the list above is only the beginning of many such structures.

The other way to break the curse of dimensionality in the worst case setting is to switch to a more lenient setting such as the randomized or average case setting. The switch to the randomized setting can be very powerful for some multivariate problems. Again, we may here mention the result of Hinrichs [50] on importance sampling for multivariate integration. For this problem, we may have the curse of dimensionality in the worst case setting and strong polynomial tractability in the randomized setting. The switch to the average case setting requires the use of a probability measure on the class of input functions. The choice of such a measure is a delicate issue and we usually choose a Gaussian measure. As we know from Section 24.9 of Chapter 24, for some Gaussian measures we may have strong polynomial tractability for multivariate approximation in the average case setting whereas the same problem in the worst case setting is not even solvable. What is still open and what seems like a major theoretical challenge is the characterization of multivariate problems which suffer the curse of dimensionality in the worst case setting and which enjoy some kind of tractability in the randomized or average case setting.

We would like to add a few personal words. We started this project in 2006 and hoped to write one relatively short volume on tractability of multivariate problems summarizing the known results and maybe adding a few new results. It was soon quite clear to us that the tractability project is not so simple, requiring many more pages and much more time to cover it completely. After almost six years, we have finally finished this project, the fruit of this endeavor consisting of three volumes, totalling more than 1600 pages. It may be hard to convince the reader that there are many topics that we did not cover; we have intentionally left out many important problems in tractability theory, which we hope that others will study in the future. The list of 149 open problems is a good indicator of what else needs to be done in the future. We hope that tractability of multivariate problems will be extensively studied by many people and we will be extremely pleased to see new research results in years to come.

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