Preface

This is the second volume of a three-volume set comprising a comprehensive study of the tractability of multivariate problems. The subjects treated in the three volumes can be briefly characterized as follows.

- Volume I [222]: we primarily studied multivariate problems specified by *linear operators* and algorithms that use arbitrary linear information given by arbitrary *linear functionals*.

- Volume II: we study multivariate problems specified by *linear functionals* and a few selected *nonlinear functionals*, and algorithms that use *standard* information given by *function values*.

- Volume III: we will study multivariate problems specified by *linear operators* and a few *nonlinear operators*, and algorithms that use *standard* information given by *function values*.

The problems studied in the three volumes are defined on spaces of d-variate functions. It often happens in computational practice that d is very large, perhaps even arbitrarily large. By tractability we mean that we can approximate the d-variate problem with error at most ε and with cost that is *not* exponential either in d or in ε^{-1}. Tractability has been studied since the 1990s, see [348], [349].

We study tractability in different settings. Each setting is specified by the definition of the error and the cost of an algorithm. We present tractability results in the worst case, average case, probabilistic and randomized settings. We do this for the absolute, normalized and relative error criteria.

There are many ways of measuring the lack of exponential dependence; therefore, we have various notions of tractability. Examples include polynomial tractability, T-tractability and weak tractability. The reader is referred to the first two chapters of Volume I for an overview and motivation of tractability studies.

Many multivariate problems specified by linear functionals suffer from the *curse of dimensionality*. This means that even the best possible algorithm must use exponentially many (in d) function values to have error at most ε. The *curse of dimensionality* is usually present for linear functionals defined over standard (unweighted) spaces. In this case all variables and groups of variables play the same role. The curse of dimensionality can often be vanquished if we switch to *weighted* spaces, in which we monitor the importance of all variables and groups of variables by sufficiently decaying weights. As in Volume I, we want to find necessary and sufficient conditions on weights to get various kinds of tractability.

Only standard information (or its analogues) makes sense for the approximation of linear functionals. Standard information was not systematically studied in Volume I.

Therefore the tractability results presented in Volume I for linear information are irrelevant for linear functionals.

The proof techniques for linear and standard information are quite different. For linear information and linear operators defined over Hilbert spaces, tractability depends on the singular values of a corresponding problem. For standard information the situation is much more complex and tractability results depend very much on the specific spaces and linear functionals. It is relatively easy to establish upper error bounds; however, many of these bounds are obtained by *non-constructive* arguments. It is especially hard to establish meaningful lower error bounds. Here, the concept of *decomposable* reproducing kernels is helpful, allowing us to find matching lower and upper error bounds for some linear functionals. We can then conclude tractability results from such error bounds.

Tractability results for linear functionals are very rich in possibilities, and almost anything can happen for linear functionals. They can be *trivial*, since there are Hilbert spaces of infinite dimension for which all linear functionals can be solved with arbitrary small error by using just one function value. They can be *very hard*, since there are Hilbert spaces for which all non-trivial linear functionals suffer from the curse of dimensionality. The last two properties hold for rather esoteric Hilbert spaces. For "typical" Hilbert spaces some linear functionals are easy and some are hard. One of the main challenges is to characterize which linear functionals are tractable and which are not.

Volume II consists of twelve chapters numbered from 9 to 20 since Volume I has the first eight chapters. We comment on their order and contents. We decided to start with a chapter on discrepancy and integration. The notion of discrepancy is simple and beautiful with a clear geometrical meaning. It is striking in how many areas of mathematics discrepancy plays an important role. In particular, discrepancy is intimately related to integration and we thoroughly explain these relations in Chapter 9. Many people would claim that integration is the most important multivariate problem among linear functionals since it appears in many applications such as finance, physics, chemistry, economic, and statistics. There is a tremendous need to compute high dimensional integrals with d in the hundreds and thousands, see Traub and Werschulz [306]. If so, discrepancy is also very important. Moreover, it connects us to many other mathematical areas. That is why we decided to start from discrepancy and explain its relation to integration from the very beginning. We hope the reader will appreciate our decision.

The order of the next chapters is as follows. We begin with the worst case setting. In Chapter 10 we study general linear functionals, whereas in Chapter 11 we study linear functionals specified by tensor products. In Chapter 11 we explain the idea of decomposable kernels and present lower error bounds. We use these lower bounds to obtain necessary conditions on tractability, as well as to show that the curse of dimensionality indeed occurs for many linear functionals. This also serves as a motivation for switching to weighted spaces in Chapter 12. We present a number of necessary tractability conditions in terms of the behavior of weights. As in Volume I, we concentrate on product and finite-order weights.

In Chapter 13 we analyze the average case setting. For linear functionals, there is a pleasing relation between the average case and worst case settings. Knowing this relation, we can translate all tractability results from the worst case setting to the average case setting. So there is no need to do additional work in the average case setting, and so this chapter is relatively short. We stress that such a relation is only present for linear functionals; we will not be so lucky in Volume III with linear operators.

In Chapter 14 we study the probabilistic setting. Here we also have a surprising and pleasing relation to the average case setting. Since the average case is related to the worst case, we conclude that the probabilistic setting is also related to the worst case setting. This means that, with some care, we can translate all tractability results from the worst case to the probabilistic setting. We also study the relative error. We show that only negative results hold for the relative error in the worst case and average case settings, and positive results are only possible in the probabilistic setting.

The relations between the average case, probabilistic and worst case settings mean that it is enough to study tractability in the worst case setting. That is why in the next two long chapters, Chapter 15 and Chapter 16, we return to the worst case setting. Our emphasis is on constructive results, since many tractability results presented so far have been based on non-constructive arguments. Chapter 15 is on the Smolyak/sparse grid algorithms for unweighted and weighted tensor product linear functionals. These algorithms are very popular. Many people have been analyzing error bounds of the Smolyak/sparse grid algorithms with the emphasis on the best order of convergence. The dependence of the error bounds on the number d of variables has been addressed only in a few papers. Of course, this dependence is crucial for tractability which is our emphasis in this chapter.

In Chapter 16 we return to multivariate integration. This problem was analyzed earlier in this book. However, this was usually done as an illustration or specification of general tractability results. We analyze multivariate integration for the Korobov spaces of smooth and periodic functions in the first part of Chapter 16. Our emphasis is on constructive lattice rules. We present the beautiful CBC (component-by-component) algorithm that efficiently computes a generating vector of the lattice rule. The history of this algorithm is reported in the introduction of Chapter 16. In the second part of Chapter 16 we exhibit relations between Korobov and Sobolev spaces. We show how the shifted lattice rules with the generating vectors computed by the CBC algorithm can be used for non-periodic functions from the Sobolev space.

In Chapter 17 we turn our attention to the randomized setting. We first report when the standard Monte Carlo algorithm for multivariate integration leads to tractability. It is well known that the rate of convergence of the standard Monte Carlo algorithm is independent of d. However, it is often overlooked that since the randomized error of the standard Monte Carlo algorithm depends on the variance of a function and the variance may depend on d, then tractability is not necessarily achieved. Indeed, this is the case for a number of standard spaces. Again, weighted spaces can help. Tractability conditions on weights for the standard Monte Carlo algorithm are usually more lenient than for the best algorithms in the worst case setting. We also discuss how importance sampling can help to relax tractability conditions or even guarantee tractability for unweighted spaces. In the final section, we discuss how we can approximate the local solution of the Laplace equation by algorithms based on a random walk.

In Chapter 18 we study tractability of a few selected nonlinear functionals in the worst case and randomized settings. We study a nonlinear integration problem where we integrate with respect to a partially known density, the local solution of Fredholm integral equations, global optimization, and computation of fixed points and computation of volumes, primarily of convex bodies.

Finally in Chapter 19, we briefly mention two generalizations of the material of the previous chapters. The first generalization is when we switch from d-variate problems with finite (and maybe arbitrarily large) d to problems for which $d = \infty$. We illustrate this point for path integration and integration over a Sobolev space of functions depending on infinitely many variables. The second generalization is when we switch from computations performed on a classical computer to computations performed on a (future) *quantum* computer and check how tractability study can be done in the quantum setting.

In Chapter 20 we present a summary of tractability results for multivariate integration defined over three standard weighted Sobolev spaces. We cover four settings:

- worst case,
- average case,
- probabilistic,
- randomized,

and three error criteria:

- absolute,
- normalized,
- relative.

Many specific results presented in this volume have been already published and we tried to carefully report the authors of these results in each chapter and additionally in the Notes and Remarks of each chapter. In fact, each chapter is based on a single paper or a few papers although in many cases we needed to generalize, synthesize or modify the existing results. There are also many new results. Again all this is described in the Notes and Remarks.

In the course of this volume we present a number of open problems. In Volume I we have 30 open problems, and so we started the count of new problems in Volume II from 31. The last open problem has the number 91 so there are 61 open problems in Volume II. The list of open problems is in Appendix D. We call it Appendix D since there are three appendices A, B and C in Volume I. We hope that the open problems will be of interest to a general audience of mathematicians and many of them will be solved soon. In this way, research on tractability will be further intensified.

We realize that this volume is very long since it has more than 650 pages. Despite this book's length, we did not cover some issues and tractability of linear functionals will be continued in Volume III. The reason for this is that there are interesting relations between linear functionals and some linear operators. These relations allow us to apply tractability results for linear operators to linear functionals. Since linear operators for standard information will be thoroughly studied in Volume III, we have to wait to present relations between linear functionals and operators, as well as the corresponding tractability results, till Volume III.

We decided to be repetitious in a number of places. This is the case with some notation as well as with some assumptions in the successive theorems or lemmas. This was done to help the reader who will not have to flip too many pages and look for the meaning of the specific notation or the specific assumption. We believe that our approach will be particularly useful after the first reading of the book when the reader wants to return to some specific result without remembering too much about the hidden assumptions and notation used in the book.

At the expense of some repetitions, we tried to write each chapter as much independent as possible of the other chapters. We hope that the reader may study Chapter n without knowing the previous $n-1$ chapters. We also think that even the last chapter with the summary of tractability results should be understood with only some knowledge of terminology already presented in Volume I.

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