

Contents

Preface	vii
9 Discrepancy and Integration	1
9.1 Introduction	1
9.2 L_2 Discrepancy Anchored at the Origin	7
9.2.1 Bounds for the L_2 Discrepancy	8
9.2.2 The Exponent of the L_2 Discrepancy	11
9.2.3 Normalized L_2 Discrepancy	12
9.3 Weighted L_2 Discrepancy	13
9.3.1 Normalized Weighted L_2 Discrepancy	17
9.4 Multivariate Integration	18
9.5 Multivariate Integration and L_2 Discrepancy	23
9.5.1 Discrepancy Anchored at the Origin	23
9.5.2 Discrepancy and Wiener Sheet Measure	25
9.5.3 Discrepancy Anchored at α	27
9.5.4 Quadrant Discrepancy at α	31
9.5.5 Extreme or Unanchored Discrepancy	33
9.6 Are They Always Related?	36
9.7 Tractability	44
9.8 L_p Discrepancy	52
9.8.1 L_p Discrepancy Anchored at the Origin	52
9.8.2 Centered L_p Discrepancy	57
9.8.3 Spaces Without Boundary Values	59
9.9 Star Discrepancy	61
9.10 Notes and Remarks	68
10 Worst Case: General Linear Functionals	71
10.1 Introduction	71
10.2 Linear Functionals	73
10.3 One Function Value	75
10.4 Bad or Good Convergence	79
10.4.1 Example: Kernel for Non-Separable Space	81
10.4.2 Example: Unbounded Kernel	83
10.4.3 Example: Sobolev Space Anchored at 0	85
10.5 Non-negative Kernels and Algorithms	87
10.5.1 Example: Kernel for Non-Separable Space (Continued)	90
10.5.2 Example: Unbounded Kernel (Continued)	91
10.5.3 Example: Kernels Related to Discrepancies	92
10.5.4 Example: Polynomials of Degree Two	94
10.6 Power of Negative Coefficients	96

10.7	Multivariate Integration	98
10.7.1	QMC Algorithms	99
10.7.2	Example: Tensor Product Problems	103
10.7.3	Example: Modified Sobolev Space	105
10.7.4	Example: Korobov Space with Varying Smoothness	107
10.7.5	Example: Korobov Space with Fixed Smoothness	110
10.7.6	Properly Normalized QMC Algorithms	111
10.7.7	Example: Tensor Product Problems (Continued)	115
10.7.8	Example: Modified Sobolev Space (Continued)	116
10.7.9	Algorithms with Arbitrary Coefficients	117
10.7.10	Example: Tensor Product Problems (Continued)	121
10.7.11	Example: Another Modified Sobolev Space (Continued)	122
10.7.12	Example: Unbounded Kernel (Continued)	124
10.8	The Operator W_d	126
10.9	Relations to Multivariate Integration	130
10.9.1	Example: Linear Functionals in Korobov Space	132
10.10	General Case	135
10.10.1	Example: Tensor Product Problems (Continued)	141
10.11	Notes and Remarks	142
11	Worst Case: Tensor Products and Decomposable Kernels	146
11.1	Introduction	146
11.2	Linear Tensor Product Functionals	147
11.3	Preliminary Error Estimates	148
11.4	Decomposable Kernels	161
11.4.1	Example: Weighted Integration	169
11.4.2	Example: Uniform Integration	173
11.4.3	Example: Centered Discrepancy	175
11.4.4	Example: Two Function Values	178
11.5	Non-decomposable Kernels	179
11.5.1	Example: Weighted Integration (Continued)	187
11.5.2	Example: Uniform Integration	188
11.5.3	Example: Centered Discrepancy (Continued)	189
11.5.4	Example: Sobolev Space Anchored at 0	191
11.6	Which Linear Tensor Product Functionals Are Tractable?	195
11.6.1	Example: Sobolev Spaces with $r = 1$ over $[0, 1]^d$	198
11.6.2	Example: Sobolev Space with $r \geq 1$ over \mathbb{R}^d	202
11.7	Notes and Remarks	206
12	Worst Case: Linear Functionals on Weighted Spaces	208
12.1	Introduction	208
12.2	Weighted Linear Functionals	210
12.3	Lower Bounds	214
12.3.1	Example: Constant and Almost Constant Weights	219

12.3.2	Example: Three Function Values	220
12.3.3	Example: Order-Dependent Weights	222
12.4	Product Weights	224
12.4.1	Example: Unbounded Weights	231
12.4.2	Example: Weighted Integration (Continued)	232
12.5	Further Lower Bounds	236
12.5.1	Example: Weighted Integration (Continued)	244
12.6	Upper Bounds for Multivariate Integration	248
12.6.1	Example: Weighted Integration (Continued)	260
12.6.2	Example: Uniform Integration (Continued)	260
12.6.3	Example: Bounds on Weighted L_2 Discrepancy	265
12.6.4	Upper Bounds Based on Theorem 10.10	266
12.6.5	Example: Integration and Unbounded Kernel	275
12.7	Upper Bounds for the General Case	277
12.8	Notes and Remarks	286
13	Average Case Setting	288
13.1	Introduction	288
13.2	Basics of the Average Case Setting	288
13.2.1	Example: Wiener Measure	290
13.3	Linear Functionals	290
13.4	Relations to the Worst Case Setting	291
13.5	Notes and Remarks	296
14	Probabilistic Setting	297
14.1	Introduction	297
14.2	Tractability in the Probabilistic Setting	298
14.3	Relations to the Worst Case Setting	300
14.3.1	Absolute Error Criterion	304
14.3.2	Normalized Error Criterion	309
14.3.3	Summary	310
14.4	Relative Error	311
14.5	Notes and Remarks	319
15	Smolyak/Sparse Grid Algorithms	320
15.1	Introduction	320
15.2	Unweighted Case: Algorithms	322
15.2.1	Explicit Form	324
15.2.2	Explicit Error Bound	326
15.2.3	Explicit Cost Bound	333
15.2.4	ε -Cost Bound	335
15.2.5	Tractability	339
15.2.6	Example: Integration of Smooth Periodic Functions	345
15.2.7	Example: Integration of Non-Periodic Functions	348

15.2.8	Example: Discrepancy	351
15.2.9	Implementation Issues	353
15.3	Weighted Case: Algorithms	356
15.3.1	Explicit Form	363
15.3.2	Explicit Error Bound	364
15.3.3	Explicit Cost Bound	366
15.3.4	ε -Cost Bound	367
15.3.5	Tractability for Finite-Order Weights	370
15.3.6	The First Class of WTP Algorithms	371
15.3.7	The Second Class of WTP Algorithms	375
15.3.8	Example: Perturbed Coulomb Potential	380
15.3.9	Open Problems for Finite-Order Weights	382
15.3.10	Tractability for Product Weights	383
15.3.11	Example: Uniform Integration (Continued)	387
15.3.12	Example: Weights for Cobb Douglas Functions	389
15.3.13	Example: Integration of Smooth Functions	391
15.3.14	Robustness of WTP Algorithms	392
15.4	Notes and Remarks	395
16	Multivariate Integration for Korobov and Related Spaces	398
16.1	Introduction	398
16.2	Weighted Korobov Spaces	402
16.3	Multivariate Integration	404
16.4	Lattice Rules	406
16.4.1	The Existence of Good Lattice Rules	407
16.4.2	Tractability for General Weights	415
16.4.3	Tractability for Product Weights	421
16.4.4	Tractability for Weights Independent of d	427
16.4.5	Tractability for Finite-Order and Order-Dependent Weights	429
16.5	Worst Case Error for α Approaching to $1/2$	432
16.6	CBC Algorithm	433
16.6.1	Cost of the CBC Algorithm	438
16.6.2	Cost for Product Weights	439
16.6.3	Cost for Finite-Order and Finite-Diameter Weights	441
16.6.4	Cost for Order-Dependent Weights	442
16.7	Weighted Korobov and Sobolev Spaces	444
16.7.1	Kernels and Shift-Invariant Kernels	444
16.7.2	Relations to Weighted Sobolev Space	450
16.8	Tractability for Weighted Korobov Spaces	457
16.9	Weighted Sobolev Spaces	463
16.9.1	Shifted Lattice Rules	466
16.9.2	Tractability for Finite-Order Weights	471
16.9.3	Shifted Lattice Rules for Finite-Order Weights	477
16.9.4	Tractability Using Low Discrepancy Sequences	480

16.10	Notes and Remarks	486
17	Randomized Setting	487
17.1	Monte Carlo for Multivariate Integration	488
17.1.1	Uniform Integration	494
17.1.2	Gaussian Integration	510
17.1.3	Periodicity May Help	513
17.1.4	Periodicity May Hurt	514
17.1.5	Monte Carlo May Lose	516
17.1.6	Can Monte Carlo Be Improved?	517
17.2	Importance Sampling	526
17.2.1	Results for Sobolev spaces	528
17.3	Local Solution of the Laplace Equation	532
17.4	Notes and Remarks	538
18	Nonlinear Functionals	539
18.1	Integration with Unknown Density	540
18.1.1	Deterministic Algorithms and Quasi-Linearity	540
18.1.2	Randomized Algorithms	542
18.1.3	Analysis for $\mathcal{F}_C(\Omega)$	543
18.1.4	Log Concave Densities	548
18.1.5	Explicit Error Bounds for MCMC	549
18.2	Integral Equations	562
18.2.1	Worst Case Setting	563
18.2.2	Restricted Randomized Setting	564
18.3	Computation of Fixed Points	571
18.4	Global Optimization	574
18.5	Computation of the Volume	576
18.6	Notes and Remarks	579
19	Further Topics	580
19.1	Path Integration	580
19.2	Weighted Sobolev Space with $d = \infty$	585
19.3	The Result of Sobol	589
19.4	Quantum Computation	590
19.4.1	Model of Quantum Computation	591
19.4.2	Grover's Search Algorithm	594
19.4.3	Computation of Sums and Integrals	595
19.4.4	Solution of PDEs	596
19.4.5	Tractability in the Quantum Setting	597
19.5	Notes and Remarks	598

20 Summary: Uniform Integration for Three Sobolev Spaces	600
20.1 Introduction	600
20.2 Preliminaries	601
20.3 First Sobolev Space	605
20.4 Second Sobolev Space	615
20.5 Third Sobolev Space	624
20.6 Notes and Remarks	626
Appendices	627
D List of Open Problems	627
E Errata for Volume I	632
Bibliography	635
Index	655