Introduction

The present volume is dedicated to our friend and colleague Vladimir Turaev, whose work has a major impact on contemporary mathematics. The book consists of a collection of essays that reflect Turaev's broad interests, and at the same time, the richness of the developments that took place in topology and geometry in the last 45 years, in particular on invariants of knots and links and 3-manifolds, and in the field that now bears the name quantum topology.

The book contains, after the present introduction, 28 chapters and a postface. Because these various chapters cover a wide variety of topics, it was difficult to organise them in sections, but I have tried to arrange these contributions according to a certain general scheme. The first chapter is a mathematical biography of Turaev and where I give a rather detailed account of his work. After this chapter, I have included the papers on topology, then geometric group theory, then geometry, then philosophy of mathematics. The topics that are treated in the topology part include invariants of knots, links and 3-manifolds, TQFTs, the Turaev co-bracket and its various ramifications, mapping class group representations, and iterated integrals. The part on geometry contains essays on geometric structures on knot and link complements, Kleinian groups, non-commutative geometry, Galois group representations, and a differential geometric approach to black holes. The last two essays, on the philosophy of mathematics, together with the postface, complement the purely mathematical papers by opening up a broader perspective to our reflections on our work as mathematicians.

All along the book, several open problems are formulated by various authors. The fact that so many problems remain open on the topics discussed is an indication of the fact that these subjects are still extremely active.

The chapter written by the author of these lines, VLADIMIR TURAEV, FRIEND AND COLLEAGUE (pp. 15–43), is a present-day account of Turaev's life and work. Several ideas and techniques he developed and their impact on mathematics and mathematicians are reviewed in a mathematical and historical perspective. At the same time, skimming this overview of Turaev's works will give the reader an idea of the remarkable activity that took place over the last four and a half decades in low-dimensional topology and of which Turaev was one of the important actors.

The next five chapters deal with classical problems in topology: the differential topology of curves in the plane, unknotting knots, linking numbers in the 3-sphere, the theory of knots in the 3-dimensional projective space, and generalizations of the Alexander polynomial.

In Chapter 2, titled TRIANGLES ON PLANAR JORDAN C^1-CURVES AND DIFFERENTIAL TOPOLOGY (pp. 45–58), Jean-Claude Hausmann starts by proving that given any triangle and any C^1-curve in the plane, one can transform the triangle using translations and homotheties (no rotations are needed) such that the resulting triangle has its three vertices on the given curve. The proof uses fundamental results in differential and algebraic topology, involving the configuration space of planar triangles modulo translation and homothety. The author also shows that the result does not hold if the curve is not everywhere C^1 (he gives a counterexample where the curve is a lemniscate). In the last part of the paper, he considers a generalization to higher dimensions. More precisely, Hausmann proves that for $n \neq 4$, any C^1-Jordan sphere in $\mathbb{R}^n$ can be transformed into a simplex whose vertices lie on this sphere using a translation and a homothety with positive ratio. The proof fails for $n = 4$ because the smooth Schoenflies conjecture is still open in that dimension.

Chapter 3, titled NULL-HOMOLOGOUS UNKNOTTINGS (pp. 59–68), by Charles Livingston, focuses on a question of unknotting knots in the 3-sphere. A famous result in this area, obtained by Ohyama (1994), says that every knot can be unknotted using two twists along a set of parallel strands. Livingston gives an alternative view on this theorem and uses it to prove that every knot can be unknotted using $2g$ null-homologous twists, that is, twists that have linking number zero. Furthermore, he shows that the result is optimal in the sense that there exist genus g knots that cannot be unknotted with fewer than $2g$ null-homologous twists.

The next chapter, by Norbert A'Campo, is titled A QUESTION OF TURAEV ABOUT TRIPLE HIGHER MILNOR LINKING NUMBERS OF DIVIDE LINKS (pp. 69–74). The author starts by recalling the notion of *divide*, a relative, generic immersion of finitely many copies of the unit interval of $\mathbb{R}$ into the unit disk in $\mathbb{R}^2$. By a classical construction of A'Campo involving the consideration of the tangent vectors to a divide, such an object defines a knot or a link in the 3-sphere, called a divide knot or divide link. In a series of papers starting in 1975, A'Campo, working in the setting of singularities of complex curves associated with divide links in the 3-sphere, developed the theory of divide knots and links, as a method to present knots or links using real plane curves.

The question which was asked by Turaev, and on which the author reports in this chapter, is the following: *Let L be a link in the 3-sphere, consisting of n knots with given n divides. How can one compute the Milnor higher linking numbers from the system of divides?*

A'Campo gives an answer to this question for the triple higher linking numbers. At the same time, he provides an introduction to the theory of linking numbers and divide knots and links, and a new set of open questions.

Chapter 5, by Julia and Oleg Viro, is titled FUNDAMENTAL GROUPS IN PROJECTIVE KNOT THEORY (pp. 75–92). The authors study knots and links in the 3-dimensional real projective space, developing a theory of projective link diagrams which leads to analogues of the classical knot and link invariants in the 3-sphere in this new setting. As in the classical theory, a central object of study is the fundamental group of the knot complement (in projective space). The authors propose a simple algorithm which gives a presentation of this group in terms of a link diagram. Some basic notions of projective geometry play a central role in this theory. The main results presented are the following:

1. A link is isotopic to a projective line if and only if the fundamental group of its complement is $\mathbb{Z}$;
2. a link is isotopic to an affine circle if and only if the fundamental group of its complement is the free product of $\mathbb{Z} * \mathbb{Z}/2$;
3. a link is isotopic to a link that is disjoint from a projective plane if and only if the fundamental group of its complement contains a non-trivial element of order two.

Louis Kauffman's chapter, titled THE AFFINE INDEX POLYNOMIAL AND THE SAWOLLEK POLYNOMIAL (pp. 93–108), contains a proof of a relationship between two knot polynomials: The first polynomial is called by Kauffman the Affine Index Polynomial. It is an elementary combinatorial invariant of virtual knots which he introduced in a paper published in 2013 titled *An Affine Index Polynomial invariant of virtual knots*. This polynomial is a concordance invariant for virtual knots. The second polynomial is known under the name Sawollek Polynomial. It was introduced by Jörg Sawollek in an unpublished paper titled *On Alexander–Conway polynomials for virtual knots and links* (1999), and it is a generalized Alexander polynomial. The relationship between the two polynomials was discovered by Blake Mellor, who, in his paper *Alexander and writhe polynomials for virtual knots* (2016), extracted the Affine Index Polynomial from the Sawollek Polynomial. The proof of this result that Kauffman gives in Chapter 6 is concise, and it constitutes the basis of a new approach on the relationship between the two polynomials.

The next 11 chapters belong to the field of quantum topology, that is, broadly speaking, topology influenced by ideas from quantum physics.

Chapter 7, by Julien Marché, is titled INTRODUCTION TO QUANTUM REPRESENTATIONS OF MAPPING CLASS GROUPS (pp. 109–130). The first such representations were introduced by Witten and by Reshetikhin–Turaev in the 1990s, in the setting of topological quantum field theory, and they became the main applications of this theory in low-dimensional topology. Marché presents a detailed self-contained construction of such representations. The exposition is based on ideas that are now classical, namely, skein theory (the Kauffman bracket) and more particularly the reduced skein modules studied independently by Roberts and Sikora in the 1990s. Using these tools, the author describes a construction of finite-dimensional projective representations of mapping class groups over the cyclotomic field of order $4r$. In the language of Witten and Reshetikhin–Turaev, this corresponds to an SU_2 TQFT of level $r-2$. Marché also investigates the Hermitian structures and the integral versions (where one takes the ring of integers of the base field) associated with these representations. In particular, he shows that when r is an odd prime, the representations are irreducible and that in this case their images lie in arithmetic groups.

In Chapter 8, titled ON SYMMETRIC MATRICES ASSOCIATED WITH ORIENTED LINK DIAGRAMS (pp. 131–145), Rinat Kashaev presents a construction of an invariant of oriented links in the 3-sphere which has the form of an equivalence class of matrices indexed by the set of regions of an unoriented link diagram, with coefficients in the ring of polynomials in one variable with integer coefficients. This construction is a modification of the notion of S-equivalence introduced by Trotter and Murasugi in the 1960s. Kashaev proposes a conjecture that makes a relationship between this invariant and the Tristram–Levine signature function on links. This work is motivated by an attempt to understand the metaplectic invariant introduced by Goldsmith and Jones in 1989, generalizing the cyclotomic invariants of Kobayashi–Murakami–Murakami.

The Chapter 9, by Jun Murakami, is titled ON CHEN–YANG'S VOLUME CONJECTURE FOR VARIOUS QUANTUM INVARIANTS (pp. 147–159). The author starts by recalling the family of Turaev–Viro type quantum invariants of 3-manifolds with boundary that were introduced by Chen and Yang in a paper which appeared in 2018, titled *A volume conjecture for a family of Turaev–Viro type invariants of hyperbolic 3-manifolds with boundary*. The definition of these invariants uses a decomposition of the manifold by ideal or truncated tetrahedra. Based on numerical computations, Chen and Yang formulated a conjecture concerning the relation between a certain limit of this family of invariants and the Gromov simplicial volume of the manifold. The conjecture was proved in special cases in a paper by Detcherry, Kalfagianni and Yang that appeared in 2018. In the present chapter, motivated by the work of Chen and Yang, Murakami formulates more general conjectures that concern quantum invariants of closed manifolds and cone manifolds. The conjectures are supported by diagrams, numerical computations and values of coloured Jones invariants, of Witten–Reshetikhin–Turaev invariants and of Kirillov–Reshetikhin–invariants in specific cases, and by tables of volumes.

In a paper published in 2008, in which he studies invariants of integral homology spheres, Habiro introduced a polynomial which he used to establish a class of unified invariants of homology 3-spheres that dominate all the known Witten–Reshetikhin–Turaev invariants (which is the common name for the Reshetikhin–Turaev quantum invariants of framed links when they are turned into invariants of 3-manifolds by making them invariant under Kirby moves). This polynomial is known as *Habiro's series*, or *Habiro's cyclotomic expansion of the coloured Jones polynomial*. Habiro's work was used by Beliakova, Chen and Le in a paper published in 2014, titled *On the integrality of Witten–Reshetikhin–Turaev 3-manifold invariants*, in which these authors prove the integrality of the Witten–Reshetikhin–Turaev invariants at all roots of unity.

In Chapter 10, titled NON-SEMISIMPLE INVARIANTS AND HABIRO'S SERIES (pp. 161–174), Anna Beliakova and Kazuhiro Hikami establish relationships between Habiro's series and the non-semisimple invariants of the double twist knots, introduced by Akutsu, Deguchi and Ohtsuki in a paper published in 1992. At the same time, they make connections between the Witten–Reshetikhin–Turaev invariants of knots and invariants due to Costantino–Geer–Patureau of 3-manifolds obtained by 0-surgery on these knots. They also propose several related open questions.

In the next chapter, titled MODULAR CATEGORIES AND TQFTS BEYOND SEMISIMPLICITY (pp. 175–208), Christian Blanchet and Marco De Renzi survey the recent developments in quantum topology that led to a generalized notion of modular category, extending Turaev's initial TQFT theory to a setting of non-semisimple generalized modular categories. After an overview of the classical work of Turaev on this subject, several recent works that contributed to the non-semisimple generalization are reviewed, including those of Geer–Patureau–Turaev, Hennings, Lyubashenko, Blanchet–Constantino–Geer–Patureau, and others.

Chapter 12, by Louis-Hadrien Robert and Emmanuel Wagner, is titled STATE SUMS FOR SOME SUPER LINK INVARIANTS (pp. 209–245). The authors study link invariants through representations of quantum groups associated with various Lie algebras. This topic was inaugurated by Reshetikhin and Turaev in their interpretation of the Jones polynomial using the quantum group $U_q(\mathfrak{sl}_2)$. The authors present a combinatorial formulation (in the form of a state sum) of the quantum link invariants associated with exterior power representations of the super quantum group $U_q(\mathfrak{gl}_{N|M})$, where N and M are non-negative integers. This work is motivated by a combinatorial description of link invariants associated with the exterior power representations of $\mathfrak{sl}_N$, obtained by Murakami–Ohtsuki–Yamada in 1998. This gives in particular a new approach to the relationship between the $U_q(\mathfrak{gl}_{N|M})$ link invariants and the Kashaev invariants. The chapter also contains an overview of the representation-theoretic background, in particular, of the maps that lead to the Reshetikhin–Turaev functors.

In Chapter 13, titled BRANE TOPOLOGICAL FIELD THEORY AND HURWITZ NUMBERS FOR CW-COMPLEXES (pp. 247–256), Sergey Natanzon develops a so-called Brane Topological Field Theory (BTFT). This is a TQFT constructed using some particular CW-complexes called brane complexes. It has an associated infinite-dimensional Frobenius algebra graded by CW-complexes of lower dimensions. Natanzon introduces the notions of general and regular Hurwitz numbers of brane complexes and he explains how they generate BTFTs. The notion of Frobenius algebra originates in algebraic geometry, where it is generated by the classical Hurwitz numbers of complex algebraic curves that appear by counting the number of branched coverings of closed surfaces with prescribed types of critical values. In the present setting, the Frobenius algebra corresponding to general Hurwitz numbers is an algebra of coverings of lower dimensions. For regular Hurwitz numbers, the Frobenius algebra is an algebra of families of subgroups of finite groups.

Chapter 14, by Stavros Garoufalidis and Rinat Kashaev, is titled RESURGENCE OF FADDEEV'S QUANTUM DILOGARITHM (pp. 257–271). Resurgence is a technique used for dealing with divergent series that appear as germs of analytic functions via a summation called Borel summation, in sectors of the complex plane. This technique is applied here to the quantum dilogarithm function of Faddeev, a special function with remarkable analytic and functional properties which finds applications in mathematical physics, quantum topology and cluster algebras. This function played a role in the theory of quantum invariants of knots and 3-manifolds, in quantum Teichmüller theory and in the setting of exact and perturbative invariants of Chern–Simons theory.

Garoufalidis and Kashaev show that the Faddeev quantum dilogarithm function is obtained as the Borel summation of a divergent formal power series solution of a linear difference equation. At the same time, they mention relations of resurgence theory with a strengthening of the famous volume conjecture in quantum topology which asserts that the Kashaev invariant of a hyperbolic knot grows exponentially at a rate proportional to the volume of the knot. This strengthening involves taking asymptotics of all orders. These connections with resurgent functions were previously conjectured by Garoufalidis in a paper published in 2008. The authors also mention relationships with parametric resurgence phenomena of non-linear equations and with wall-crossing formulae obtained by Kontsevich and Soibelman.

Chapter 15, by Louis Funar, is titled ON MAPPING CLASS GROUP QUOTIENTS BY POWERS OF DEHN TWISTS AND THEIR REPRESENTATIONS (pp. 273–308). It is an exposition of the relatively recent developments of the subject announced in the title, in relation with finite-dimensional representations of mapping class groups. As the quantum representations of mapping class groups in semisimple groups have Zariski dense images, they admit many finite quotients. The theory relies on the so-called Fibonacci TQFT representation, which the author shows to be a specialization of the Jones representation of the mapping class group of a closed oriented surface of genus 2 constructed in 1987. The main ideas originate in the notion of modular tensor category which originate in the works of Witten on Chern–Simons theory and of Reshetikhin–Turaev on invariants of 3-manifolds. The author also presents an algebraic/geometric method of constructing large families of mapping class group representations which is due to Long and Moody. This method was set originally for representations of braid groups, and in the present chapter, it is generalized to general automorphism groups. All along the exposition, Funar formulates a large number of open questions and conjectures.

In Chapter 16, titled HIGHER HOLONOMY AND ITERATED INTEGRALS (pp. 309–327), Toshitake Kohno develops a method of construction, for any positive integer n, of representations of the homotopy n-groupoid of a manifold as an n-category, using the so-called Chen formal homology connections. The latter were introduced by Chen in the 1970s, in the context of his theory of iterated path integrals of differential forms. In this setting, he gave a description of the homology group of the loop space of a manifold by the chain complex formed by the tensor algebra of this group.

Kohno constructs an n-holonomy functor from the homotopy n-groupoid to a category obtained from the tensor algebra over the homology of the manifold. He applies this theory to the study of complements of complex hyperplane arrangements in $\mathbb{C}^n$.

Chapter 17, titled SOME ALGEBRAIC ASPECTS OF THE TURAEV COBRACKET (pp. 329–356), by Nariya Kawazumi, is a survey of his results obtained in collaboration with Alekseev, Kuno, and Naef, centered about the algebraic aspects of Turaev's cobracket together with its framed variants and their ramifications. The exposition includes a review of the applications of the Goldman–Turaev bialgebra to surface mapping class groups (and in particular to the Johnson homomorphism), and an exposition of the relation between the formality of the Turaev cobracket and the higher-genus Kashiwara–Vergne problem. At the same time, Kawazumi provides an exposition of this problem and its generalizations. The original problem asks whether a certain set of equations in a Lie algebra, involving the so-called Campbell–Hausdorff series, admits a solution. Alekseev, Kuno, Naef and Kawazumi proved that the set of solutions of this problem is in one-to-one correspondence with the set of special expansions which induce the formal description of the framed Turaev cobracket. The original Kashiwara–Vergne problem involves genus 0 surfaces, and Kawazumi and his co-authors were led to the formulation of a positive-genus version of it.

In the same chapter, Kawazumi reports on a homological description of a cobracket recently obtained by Hain, who used it to prove that the framed cobracket is a morphism of mixed Hodge structures, deducing from it the existence of solutions to the Kashiwara–Vergne problem.

Chapter 18, by Yusuke Kuno, Gwénaël Massuyeau and Shunsuke Tsuji, is titled GENERALIZED DEHN TWISTS IN LOW-DIMENSIONAL TOPOLOGY (pp. 357–398). We recall that a Dehn twist along a simple closed curve on a surface is a homeomorphism of the surface which is well defined up to homotopy, whose effect may be described by the operation of cutting the surface along this curve and gluing it back after performing a complete twist. A famous result of Max Dehn says that Dehn twists along simple closed curves generate the mapping class group of the surface. Since the work of Dehn, such homeomorphisms were used in several ways in geometric topology. A "generalized Dehn twist" is a Dehn twist along a closed curve which is not simple (that is, which has self-intersection) and the first question is to give a geometric meaning to such a notion.

It is seen by experience that it is difficult to give a useful geometric definition of the notion of generalized Dehn twist along a non-simple curve. In their chapter, the authors review a set of works which propose an algebraic approach to these Dehn twists. The setting is that of a surface with boundary. The idea is to define a generalized Dehn twist as an automorphism of the Malcev completion of the fundamental group of the surface. This construction was first introduced by Kawazumi and Kuno in a paper published in 2014. When the curve is simple, the construction amounts to that of the action of the usual Dehn twist.

Kuno, Massuyeau and Tsuji examine several questions concerning generalized Dehn twists, including their realizability as diffeomorphisms of the underlying surface and the diagrammatic description of these twists using decorated trees whose leaves are coloured by the first homology group of the surface. They also review the Hopf-algebraic framework underlying the construction of generalized Dehn twists and the relation between these twists and 3-dimensional homology cobordisms over surfaces. They present two variants of a formula defining the generalized Dehn twists for the Kauffman bracket algebra and for the HOMPFY-PT skein algebra of the surface, and they establish relationships between the resulting skein version of generalized Dehn twists via a set of commutative diagrams. They review several applications of this theory to the construction of topological invariants of homology 3-spheres. They highlight the relations between the works done on the subject over several years by Kawazumi–Kuno, Massuyau–Turaev and Tsuji.

Chapter 19, by Valentin Poénaru, is titled ON GEOMETRIC GROUP THEORY (pp. 399–431). It consists of a survey of a variety of notions and results from geometric group theory, with a stress on the relationship between this field and the topology of 3-manifolds. The topics considered include simple-connectivity at infinity, quasi-simply-filtered (QSF) sets, Dehn's condition on 2-disks embedded in 3-manifolds, ends of groups, the Whitehead manifold, Dehn-exhaustibility, geometric simple connectivity, the author's notion of REPRESENTATION of a group (written always in capital letters, to distinguish it from the usual representation theories) and its equivariant version, zipping, the Whitehead nightmare, and the notion of easy group.

At the same time, Poénaru offers a glimpse into his own contribution to 3-dimensional topology, mentioning in particular his work on the fundamental group at infinity of 3-manifolds and presenting what he calls his Mantra: *discrete symmetry with compact fundamental domain*. He comments on his result stating that all finitely presented groups are QSF, announced in three long papers written between 2013 and 2015.

Furthermore, Poénaru makes several conjectures, mentioning possible connections between geometric simple connectivity and non-commutative geometry.

The next 7 chapters of this volume have a more geometric taste.

The topic studied in Chapter 20, titled GEOMETRY OF KNOTS AND LINKS (pp. 433–453), by Nikolay Abrosimov and Alexander Mednykh, is geometric structures on link complements. More precisely, the authors make an overview of the question of existence of hyperbolic, spherical or Euclidean structure on various cone manifolds whose underlying space is the three-dimensional sphere with singular set a given knot or link. The development of this theory is motivated by ideas of Thurston, which he communicated in the second half of the 1970s. Several specific examples of geometric structures on known link complements are mentioned, including the Borromean rings, the Hopf link, the trefoil knot and other toric knots and links, the figure eight knot, the Whitehead link, and other twist links. Furthermore, the authors present trigonometric identities involving the lengths of singular geodesics and cone angles of the cone manifolds, and they show how these identities imply integral formulae for the volumes of the corresponding cone manifolds.

Chapter 21, by Charalampos Charitos, is titled ESSENTIAL CLOSED SURFACES AND FINITE COVERINGS OF NEGATIVELY CURVED CUSPED 3-MANIFOLDS (pp. 455–476). It is inspired by Thurston's construction of hyperbolic structures on the complement of certain knots and links in the 3-sphere. Based on the realization of this complement as a finite union of hyperbolic ideal tetrahedra and by later works of Fuji and Kojima on triangulating the interior of compact hyperbolic manifolds by hyperbolic polyhedra, the author studies the existence of incompressible surfaces in finite coverings of a class of 3-manifolds that admit triangulations by topological ideal tetrahedra. Incompressible surfaces in 3-manifolds play the role of nontrivial simple closed curves on surfaces. They play an important role in 3-manifold topology. They can be used for decomposing the manifold into simpler pieces, and as such they are at the basis of induction arguments. Haken 3-manifolds constitute a class of 3-manifolds whose main characteristic is that they contain incompressible surfaces. Thurston proved at the beginning of the 1980s his hyperbolization theorem for Haken manifolds (it is considered that this is the result for which he was awarded the Fields medal). The famous Virtually Haken conjecture, stated by Waldhausen in 1968 and proved by Agol in 2012, states that any compact, irreducible 3-manifold with infinite fundamental group is Haken up to a finite cover.

Chapter 22, titled CONTINUOUS AND DISCONTINUOUS FUNCTIONS ON DEFORMATION SPACES OF KLEINIAN GROUPS (pp. 477–501), by Ken'ichi Ohshika, is an overview of spaces of Kleinian groups and functions defined on such spaces, with a special emphasis on continuity and discontinuity properties of these functions. The main idea conveyed is that in most cases, discontinuity phenomena are related to the fact that the spaces on which the functions are studied are endowed with more than one natural topology; in the present case, these are the algebraic and geometric topologies. Among the topics that are discussed in detail in this chapter are the properties of length functions on deformation spaces, end invariants as functions that can be used in the parameters for these spaces, and the deformation theory of the Cannon–Thurston maps.

In Chapter 23, titled A GENERALIZATION OF KING'S EQUATION VIA NONCOMMUTATIVE GEOMETRY (pp. 503–536), Gourab Bhattacharya and Maxim Kontsevich introduce a noncommutative geometry framework that leads to moment map equations on the space of connections on an arbitrary finitely-generated projective Hermitian module. They recover, as particular cases, a large number of equations in algebra (King's equations for representations of quivers, ADHM equations, noncommutative instantons and vortex equations), in classical gauge theory (Hermitian Yang–Mills equations, Hitchin equations, Bogomolny and Nahm equations for the gauge group $U(k)$, etc.), and others in the noncommutative gauge theory developed by Connes, Douglas and Schwarz. Bhattacharya and Kontsevich also discuss an idea of Nekrasov who proposed a reinterpretation of noncommutative instantons on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ as infinite-dimensional versions of King's equation.

Chapter 24, by Noémie Combe, Yuri Manin and Matilde Marcolli, is titled DESSINS FOR MODULAR OPERADS AND THE GROTHENDIECK–TEICHMÜLLER GROUP (pp. 537–560). The authors give a new point of view on the study of actions of the absolute Galois group of the field of algebraic numbers in the setting of moduli spaces of algebraic curves, following Grothendieck's program. Considering more especially stable algebraic curves of genus zero with marked points, they introduce techniques from quantum statistics. Their main idea is that dual graphs of such curves, which play the role of "modular dessins d'enfant", may be used in an appropriate operadic context. They construct an enriched cyclic modular operad of genus zero curves in relation with the action of the absolute Galois group on a class of combinatorial graphs they call modular dessins. They translate this setting of enriched genus zero operads into a quantum statistical mechanical system context in which one can use constructions from commutative Hopf algebras similar to the Connes–Kreimer Hopf algebra. The ideas introduced in this chapter establish connections between several beautiful geometrical notions.

In Chapter 25, titled ON THE INVOLUTION JIMM (pp. 561–577), Muhammed Uludağ studies an involution of the circle $\mathbb{R} \cup \{\infty\}$ called Jimm (the third letter of the Arabic alphabet). This involution was introduced by himself and Hakan Ayral in 2016. It is induced by an outer automorphism of the projective general linear group $\mathrm{PGL}(2, \mathbb{Z})$ discovered by Dyer in the 1970s. Jimm has beautiful properties. It is continuous at the irrationals and has jump discontinuities at the rationals. It satisfies a set of functional equations of modular type. It preserves the set of real quadratic irrationals, inducing a highly non-trivial mapping on this set and commuting with the action of the Galois group on it. It restricts to a non-trivial involution of the set of elements of norm $+1$ in real quadratic number fields. It conjugates the Gauss continued fraction map to the so-called Fibonacci map of the unit interval, establishing a relation between the dynamics of these two maps. It preserves setwise the orbits of the modular group, inducing an involution of the moduli space of real rank-two lattices. Uludağ and Ayral conjecture that Jimm sends algebraic numbers of degree at least three to transcendental numbers.

In Chapter 26, titled WHAT IS A BLACK HOLE? (pp. 579–598), Sumio Yamada considers the shape and evolution of black holes, and he explains how they appear in Einstein's theory. The approach is geometric and it leads the reader progressively through ideas from Newtonian mechanics to special relativity and then to the geometry of the Einstein equations.

The last two chapters of this book are invitations for us to think about our work as mathematicians.

Chapter 27, by Vassiliki Farmaki and Stelios Negrepontis, is titled THE PARADOXICAL NATURE OF MATHEMATICS (pp. 599–642). The authors, going deep into the essence of mathematics, develop a new and perhaps revolutionary idea, saying that the deductive power of this science lies in a paradoxical side of its nature, namely, in its closeness to the contradiction. This thesis is supported by a series of examples showing how infinity is used in the foundations of mathematics, in topology, in geometry, in number theory and in set theory. A central idea here is that of the "finitisation of the infinite". At the same time, the authors offer an original interpretation of the platonic idea of beauty in mathematics as being a philosophical analogue of this finitisation of the infinite, expressed in terms of periodic anthyphairesis (a general version of the notion of continued fractions, contained in Book VII Euclid's *Elements*). This leads Farmaki and Negrepontis gradually, based on a detailed argumentation, to the conclusion that the feeling of beauty in mathematics, as it is manifested, for mathematicians, by the sudden and deep understanding of an idea, or a problem or of a proof, is one of the manifestations of what they call the "paradoxical, near contradictory deductive power of mathematics". Among several conclusions, they express the fact that beauty in mathematics and Winger's famous "Unreasonable effectiveness of mathematics in the natural sciences" are equal manifestations of the same power.

The last chapter in this volume, by Arkady Plotnitsky, has also a philosophical taste. It is titled RETURNS OF GEOMETRY: FROM THE PYTHAGOREANS TO MATHEMATICAL MODERNISM AND BEYOND (pp. 643–682). The author presents his thoughts on the interaction between geometry and algebra, in what he calls "mathematical modernism," a period of the twentieth century which he characterizes by an algebraisation of mathematics, in particular of geometry and topology, stressing on the fact that geometrical thinking was persistent amidst this process of algebraisation. This leads the author to an exposition of his point of view on the question of continuity in the history of mathematics, while he discusses the evolution of a philosophical concept he calls the "unthinkable in thought" (*alogon*): "the rational containing the irrational within it". He explains that there is, in the twentieth-century, a form of thinking which is proper to mathematicians and physicists, towards this unthinkable in thought, and he also addresses the philosophical notion of existence and how it is affected by mathematical thought. Metaphysical questions of how and why, and that of the nature of the existent things, which are implied in this chapter, are generally not addressed by mathematicians, but they are essential for philosophers of mathematics, since antiquity, probably culminating in Plato's writings, and continued by Bernhard Riemann, Hermann Weyl, René Thom and other mathematicians from modern times.

Plotnitsky also comments on the differences between the creative processes of physics and mathematics, and his essay becomes at some places an excursion into the interaction between geometry on the one hand and quantum mechanics and quantum field theory on the other, topics which find their application in several chapters of the present volume. Amidst the quotes that punctuate Plotnitsky's essay, let me highlight one by Heidegger, arguing that modern physics, beginning with Decartes and Galileo, "is experimental because of its mathematical project". This quote, which may seem paradoxical, has a beautiful resonance in many of the articles contained in this volume.

Athanase Papadopoulos
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