Preface

In 1974 Drinfeld revolutionized the field of arithmetic over global function fields. He introduced a function field analogue of elliptic curves over number fields, which he called elliptic modules but are now eponymously named Drinfeld modules. For him and for many subsequent developments their main use was in the exploration of the global Langlands conjecture for automorphic forms over function fields. One of its predictions is a correspondence between automorphic forms and Galois representations. The deep insight of Drinfeld was that the moduli spaces of Drinfeld modules can be assembled in a certain tower such that the corresponding direct limit of the associated ℓ-adic cohomologies would be an automorphic representation which at the same time carries a Galois action. This would allow him to realize the correspondence conjectured by Langlands in geometry. Building on this, Drinfeld himself proved the global Langlands' correspondence for function fields for GL_2 and later L. Lafforgue obtained the result for all GL_n.

In a second direction, the analogy of Drinfeld modules with elliptic curves over number fields made them interesting objects to be studied on their own right. One could study torsion points and Galois representations, one could define cohomology theories such as de Rham or Betti cohomology and thus investigate their periods as well as transcendence questions. A main advance in this direction is the introduction of t-motives by Anderson. Passing from Drinfeld modules to t-motives may be compared to the passage from elliptic curves to abelian varieties. But more is true. The category of t-motives is also a simple function field analogue of Grothendieck's conjectured category of motives over number fields. It is this second direction which constitutes a main theme of the present volume, including advances on Galois representations, L-functions, transcendence results, Hodge structures and period domains.

Many exciting developments in the arithmetic of functions fields in recent decades have centered around the notion of a t-motive. Some of the most important ones are:

- New developments in the transcendence theory over function fields: For instance, it has been shown that the period matrix of a t-motive has transcendence degree equal to the dimension of its motivic Galois group, much in the same way that Grothendieck's conjecture predicts the transcendence degree of the period matrix of an abelian variety to be equal to the dimension of its Mumford–Tate group.
- Hodge structures for function fields: Defined by Pink in 1997, they allow him to define the analogue of the Mumford–Tate group of a t-motive and to formulate a Mumford–Tate conjecture for certain t-motives and to prove the conjecture for Drinfeld modules. Also Pink proved the Hodge-conjecture in this theory, stating, that the Mumford–Tate group is equal to the motivic Galois group.
- Period domains: More recently Pink's Hodge structures have been used extensively to lay foundations for period domains over function fields and state an analogue of Fontaine's theory of crystalline Galois representations.

- Galois representations: While the Tate-conjecture for a t-motive over a finitely generated field has been proved already in the early 90's, only recently results on the openness of the image of Galois (l-adically and adelically) have been obtained.
- Tannakian formalisms: Such have recently been described for t-motives in various contexts. They should ultimately link transcendence, Hodge structures and images of Galois representations.
- L-series: There are now cohomological approaches to L-series attached to t-motives. Moreover, recently new results on the zeroes of these L-series have emerged.

The above topics are tightly interwoven. The Tannakian formalism is used in transcendence theory as well as in a formulation of a Mumford–Tate conjecture based on function field Hodge structures (which is proved for Drinfeld modules). This in turn spurs the interest in Galois representations over function fields. All of the above topics have close relations to similar questions in number theory. The function field Hodge structures and analogues of Fontaine's theory have influenced questions on period spaces for number fields. Other developments such as the transcendence theory have gone far beyond comparable results in number theory.

The first part of this volume consists of survey articles on central topics in the arithmetic of function fields, the first three of which focus on properties of t-motives and Anderson t-modules. There is a brief introductory article on Drinfeld modules, t-modules, and t-motives by Brownawell and Papanikolas. The article by Hartl and Juschka on Pink's theory of Hodge structures provides an extensive view of the interconnectedness between cohomology theories, Hodge–Pink structures, t-motives, and Anderson t-modules. The article by Hartl and Kim investigates local shtukas connected to Hodge–Pink structures and Galois representations and provides a function field analogue of Fontaine's theory of p-adic crystalline Galois representations and Kisin's theory of crystalline Galois deformation rings.

There are three further survey articles on transcendence methods over function fields. Chang has provided an overview of techniques in transcendence theory arising from solutions of Frobenius difference equations and their t-motivic interpretations. Pellarin's article gives an overview of Mahler's method for deducing transcendence and algebraic independence in the context of function fields. Finally, Thakur has written a survey of how automata theory can be applied to transcendence problems.

The remaining three articles of the volume are research articles. The article by Bandini, Bars, and Longhi focuses on Iwasawa theory over function fields, and in particular they formulate a Main Conjecture for abelian varieties in $\mathbb{Z}_p^{\mathbb{N}}$-extensions of function fields. Taelman constructs and proves fundamental results for 1-t-motifs, which can be viewed as function field analogues of 1-motives over number fields. The article by Thakur reviews the theory of multizeta values over function fields and presents recent results on linear relations among them and their period interpretations.

The present volume grew out of the workshop, "t-motives: Hodge structures, transcendence and other motivic aspects," held at the Banff International Research Station on September 27–October 2, 2009, which brought together researchers from across the globe to discuss progress in function field arithmetic and related topics. The workshop page https://www.birs.ca/workshops/2009/09w5094/files/ contains further material for download that is not covered in the present volume.

Due to the long time it took for this collection to appear, it also seems appropriate to discuss some important further developments since the time of the workshop.

- Having introduced good notions of class module and unit group shortly before the Banff meeting, the search of Taelman for a class number formula for special values of L-functions proved to be successful. We refer to [3] for a first decisive theorem by Taelman in this direction, where some crucial steps are inspired by a trace formula of V. Lafforgue. This spurred much research afterwards by Taelman and many others. Recently M. Mornev has given a cohomological reformulation of some of Taelman's work. There is also recent work on special L-values by B. Anglès, C.-Y. Chang, C. Debry, F. Demeslay, A. El-Guindy, J. Fang, T. Ngo Dac, M. Papanikolas, F. Pellarin, and F. Tavares Ribeiro.
- In [1], F. Pellarin introduced a new kind of Drinfeld modular form along with a new kind of L-function. The forms are now called vectorial Drinfeld modular forms over the Tate algebra, and the L-functions are named after Pellarin. Both constructions have not yet found analogues over number fields but proved extremely useful in function field arithmetic. Recent research is also due to B. Anglès, Q. Gazda, O. Gezmiş, D. Goss, N. Green, A. Maurischat, T. Ngo Dac, M. Papanikolas, R. Perkins, and F. Tavares Ribeiro.
- The work started by Thakur [4] on multizeta values, on which his article in the present volume reports, proved to be extremely influential; perhaps also because of the motivic interpretation given jointly by him and Anderson. In particular many transcendence results on algebraic independence for the infinite or v-adic places have been proved in the meanwhile and many relations among such values are now understood. There has been much work in this area, particularly by B. Anglès, C.-Y. Chang, H.-J. Chen, N. Green, W.-C. Huang, Y.-L. Kuan, J. A. Lara Rodríguez, Y.-H. Lin, Y. Mishiba, T. Ngo Dac, M. Papanikolas, S. Shi, F. Tavares Ribeiro, G. Todd, and J. Yu.
- Breuer and Pink started a program to develop foundations of higher rank Drinfeld modular forms. Work of Pink [2] and of some of his students, led to a good understanding of the Satake compactification of Drinfeld modular varieties of higher rank, algebraically and analytically. Very recent work on this and on higher rank Drinfeld modular forms is also due to D. Basson, F. Breuer, E.-U. Gekeler, S. Häberli, M. Papikian, S. Schieder, and F.-T. Wei.

The above list is certainly not complete and we apologize for not mentioning the many other developments that took place in the last years in function field arithmetic and the future directions that have been started recently.

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23 May 2019

N.B. It is with great sorrow that we observe that David Goss passed away during the compilation of this volume. David was one of the early pioneers in modern approaches to function field arithmetic, and throughout recent decades he enthusiastically championed new developments in the subject and continually encouraged both junior and established researchers to reach for new discoveries. He was a wonderful colleague and friend.

References

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[4] D. S. Thakur, *Function Field Arithmetic*, World Scientific, River Edge, NJ, 2004.