# Introduction

## "Spectral structures and topological methods in Mathematics"

## Collaborative Research Centre 701 (2005–2017)

Over the 12 years of funding, the CRC covered a broad spectrum of research. Its participants were driven by the vision of reinforcing and building bridges between various branches of theoretical and applied mathematics. Many significant developments in mathematics are related to spectral structures and use topological methods. Frequently, they have their origins in applied fields, for example in new concepts of mathematical physics and fluid dynamics, crystallography and materials science.

The research pursued in the framework of the CRC may be attributed to one or several of the following broad mathematical topics:

- asymptotics and universality
- lattices
- representation theory
- harmonic and geometric methods
- deterministic and stochastic dynamics
- moduli spaces
- $p$-adic $L$-functions
- $p$-adic cohomology

In this volume, we would like to highlight some illustrating examples of this research, to embed it into a wider mathematical context, and to emphasise connections within and between these topic areas. Let us now give a synopsis of the various chapters.

*Asymptotic approximations* are a major tool for the analysis of distributions in different areas of mathematics. In Chapter 1 (Götze and Kösters), they are used to investigate the accuracy of *universal* statistical laws for local and global distribution of spectral values of random matrices and sums with independent parts. Here, *universality* for large times or for large complexity means that these distributions show an emergent collective behaviour in the limit which is independent of special properties of the model, such as the starting distribution, the details of the dynamics or the details of the distribution of the constituent parts.

The *asymptotic growth* of numbers of geometric or algebraic objects are a common theme of Chapter 9 (Baake, Gähler, Huck and Zeiner) as well as Chapter 15 (Voll). The first one considers the enumeration of particular lattices in Euclidean space, the second one centres around the enumeration of subrings and representations of unipotent group schemes. In both cases, these numbers are studied via an analytic encoding in zeta functions, which generalise those of Hecke and Tate.

Other generalisations of these zeta functions with applications to *Arthur–Selberg trace formulas* in the framework of the Langlands program are reviewed in Chapter 14 by Hoffmann. Their analytic continuation and functional equations are obtained by tools of harmonic analysis like Poisson summation.

Apart from enumeration, Chapter 9 (Baake, Gähler, Huck and Zeiner) also focuses on harmonic properties of *lattices and quasiperiodic sets* as well as on spectral implications of quasiperiodic tilings, their generation and connection to dynamical systems.

Similarly, spectral properties of nonlinear dispersive equations of Schrödinger type, studied in Chapter 7 (Herr), are closely tied to spectral sets of lattice points which are investigated by methods of *harmonic analysis* parallel to those used in analytic number theory and the *geometry of numbers* studied in Chapter 1 (Götze and Kösters).

*Stochastic dynamical systems* and their spectral properties represent another main topic of the CRC 701. In Chapter 5, Gentz studies *metastability* in parabolic SPDEs and other noise-induced phenomena in coupled dynamics by means of harmonic and stochastic analysis, *large-deviation methods* and random Poincaré maps.

*Equivariant dynamical systems* are investigated in Chapter 6 by Beyn and Otten, where the surprising stability of equivariant evolution equations and their relative equilibra is studied under numerical discretisation. The stability of waves can be analysed using holomorphic nonlinear eigenvalue problems.

In Chapter 8, Kassmann applies methods from partial differential equations and *nonlocal operators* in Euclidean spaces to study variational solutions of fractional Dirichlet problems and related Harnack inequalities.

In Chapter 4, Kondratiev, Kutovyi and Tkachov study *Markov birth and death processes* in spatial ecologies by means of evolutions of configuration sets and semigroup methods, passing from microscopic stochastic configuration processes to mesoscopic kinematic model equations.

In Chapter 2, Röckner discusses open problems and new approaches in solving *Fokker–Planck–Kolmogorov equations* in finite and in particular in infinite-dimensional spaces. He also reviews important results on the corresponding *stochastic differential equations* in this general infinite-dimensional setup and discusses applications to *stochastic differential equations* such as the stochastic porous media equation.

A panorama of views showing the interplay of different fields of mathematics is developed for the notion of a poset (or lattice) of *non-crossing partitions* in Chapter 11 (Baumeister, Bux, Götze, Kielak and Krause). Non-crossing partitions are fundamental in moment computations for *universal laws* of non-commutative convolutions in *free probability*, and the Kreweras complement in this poset allows to analyse multiplicative non-commutative convolutions. Similarly, it can be used to determine the *classifying spaces* of *braid groups* and to describe the Hurwitz action in finite Coxeter systems. Last but not least, non-crossing partitions lie at the heart of bijections between subcategories of thick and coreflective subcategories related to *crystallographic Coxeter systems* in *representation theory*. These fruitful connections have been the topic of several conferences hosted by the CRC 701 in the last years on free probability, quantum groups, algebraic combinatorics, buildings and representation theory.

Connections between the *algebraic geometry* of the projective line and its *cohomological localisations* in *representation theory* are reviewed in Chapter 12 by Krause and Stevenson. The example of the projective line shows that the classification of thick subcategories via non-crossing partitions that arises in representation theory is nicely complemented by the classification of thick tensor ideals arising in algebraic geometry.

*Harmonic analysis and stochastic dynamics* on Riemanian manifolds together with associate heat kernel bounds and escape rates are studied in Chapter 3 (Grigor'yan). Some of these results can be transferred to (ultra-)metric measure spaces by means of Dirichlet forms. For *discrete spaces* like graphs, the notions of Hochschild homology and other fundamental homology constructions like Künneth's formula can be partially extendend.

In a similar vein, *complexes for graphs* and surfaces are studied in Chapter 13 (Bux) together with Morse functions on *cell complexes* in order to access *higher finiteness properties* of braided version of certain groups.

Chapter 10 (Callies, Haydys) is devoted to the interplay of local and global geometry and harmonic analysis for special *affine Kähler structures*.

Finally, *p-adic analysis*, number theory and geometry are in the focus of Chapter 16 (Nickel), reporting evidence for conjectures by Gross and Stark on vanishing orders and leading terms of *p-adic $L$-functions* and complex $L$-functions at zero. The theory of displays and the classification of *p-divisible groups* and with its important recent applications are studied in Chapter 17 by Zink.

The research within the CRC 701 established viable connections at the interface between theoretical and applied mathematics: Algebraic geometry and dynamical systems, representation theory and probability theory, stochastic analysis and numerics, harmonic analysis connecting nonlinear partial differential equations, stochastics and analytic number theory. The special added value of the CRC 701 has been to realise the full potential of the mathematical theories around these interfaces, which motivated the newly recruited researchers to engage themselves into a coherent, stimulating research environment.

Establishing such a framework of interactions between different fields of mathematics might be viewed as the most important legacy of the CRC 701.

Friedrich Götze Michael Röckner Thomas Zink