Introduction

Classically, the expression "non-Euclidean geometry" refers to the two geometries of non-zero constant curvature, namely, the spherical and the hyperbolic.[1] Spherical geometry was developed in Greek antiquity and it attained there a high degree of matureness, especially in the work of Menelaus of Alexandria (1^{st}–2^{nd} centuries A.D.).[2] Hyperbolic geometry is a nineteenth century achievement. It was discovered by Lobachevsky, Bolyai and Gauss, with a few anticipated results obtained in the eighteenth century, especially in the work of J. H. Lambert who developed a geometry (which, from his point of view, was hypothetical), in which all the Euclidean axioms hold except the parallel postulate and where the latter is replaced by its negation. Although Lambert's goal was to find a contradiction in the consequences of this system of axioms (like many other prominent geometers, he thought that such a geometry cannot exist), the net result of his investigations is a collection of interesting theorems in hyperbolic geometry.[3]

In the present volume, the term "non-Euclidean geometry" is used in a broader sense, including geometries such as de Sitter, anti-de Sitter and others which can be developed, in analogy with the three classical geometries, in the framework of projective geometry, using the language of conics, quadrics and quadratic forms. Studying the three classical geometries in the setting of projective geometry is also a nineteenth century achievement; it is due to the visions of Cayley, Klein, Beltrami, Poincaré, and a few others.

Today, non-Euclidean geometry in this generalized sense is a very active research field and it seemed to us that editing a book containing self-contained surveys touching several aspects of this domain was desirable for researchers and those who want to learn this subject. This is the reason for which this book is published.

[1] The expression "non-Euclidean geometry," to denote spherical and hyperbolic geometry, was coined by Gauss.

[2] The first English edition of Menelaus' work just appeared in press: R. Rashed and A. Papadopoulos, *Menelaus' Spherics*: *early translation and al-Māhānī/al-Harawī's version,* Scientia Graeco-Arabica 21, Walter de Gruyter, Berlin, 2017.

[3] J. H. Lambert, "Theorie der Parallellinien," in *Die Theorie der Parallellinien von Euklid bis auf Gauss, eine Urkundensammlung zur Vorgeschichte der nichteuklidischen Geometrie* (P. Stäckel and F. Engel, eds.), B. G. Teubner, Leipzig 1895. French translation by A. Papadopoulos and G. Théret, *La théorie des lignes parallèles de Johann Heinrich Lambert,* Sciences dans l'Histoire, Blanchard, Paris, 2014. There is no English translation available of this work.

Geometry is one of these mathematical fields (probably the unique one to such a great extent) where the texts of the ancient great authors keep their full value, and going through them is not a matter of vain curiosity nor a matter of being interested in history, but it is the best way to understand present day mathematics. In this sense, the rapid growth of non-Euclidean geometry in the last few decades, after its revival by Thurston in the 1970s, is also a return to the sources of geometry. This is why the names of Menelaus, Pappus, Euler, Lambert, Lagrange, Lexell, Fuss, Schubert, Chasles, Study, Story, and several others are mentioned at several occasions in the various essays that constitute this volume. Going throughout these works also shows how slow is the process of development of geometry, despite the rapid (quantitative) growth of the literature in this field. The slow evolution process of mathematics is marked out by abrupt changes attached to names such as those we just mentioned.

Before describing in some detail the content of this volume, we would like to make a few remarks on two topics that are at the heart of the various surveys that constitute it, namely, the notion of area in non-Euclidean geometry, and the geometry of conics.

In spherical geometry, the area of a triangle is (up to a constant multiplicative factor) the excess of its angle sum with respect to two right angles. In other words, the area of a triangle with angles A, B, C is equal to $A + B + C - \pi$, up to a constant factor that does not depend on the choice of the triangle. In hyperbolic geometry, the area of a triangle is (again up to a constant factor) the deficiency of the angle sum with respect to two right angles, that is, $\pi - (A + B + C)$. The fact that in spherical (respectively hyperbolic) geometry the angle excess (respectively the angle deficiency) of an arbitrary triangle is positive, is probably the most important feature in that geometry. As a matter of fact, there exist classical proofs of the trigonometric formulae in spherical and hyperbolic geometries that are exclusively based on these properties. The oldest such proofs that we are aware of are due to L. Gérard,[4] a student of Poincaré. Since the trigonometric formulae contain in essence all the geometric information on a space, spherical and hyperbolic geometry are essentially the geometries where angle excess or, respectively, angle deficiency of all triangles are positive. Thus, it is not surprising that several chapters in the present volume concern the notion of area and its use in non-Euclidean geometry.

After area, one naturally considers volume in higher-dimensional spherical and Euclidean geometries, and here we enter into the realm of difficult problems. One might recall in this respect that Gauss, in a letter to Wolfgang Bolyai (the father of János Bolyai, the co-discoverer of hyperbolic geometry), dated March 6, 1832,[5] after he gave an outline of a proof of the area formula for a hyperbolic triangle as angle deficiency, asked his friend to suggest to his son to work on the determination of volumes

[4] L. Gérard, *Sur la géométrie non euclidienne,* Thèse № 768, Faculté des Sciences de l'Université de Paris, Gauthier-Villars, Paris, 1892.

[5] See Gauss's *Collected works*, Vol. 6, p. 221, and the article by P. Stäckel and F. Engel "Gauss, Die beiden Bolyai und die nichteuklidische Geometrie," *Math. Annalen* 2 (1897), 149–167.

of tetrahedra in three-dimensional hyperbolic space. It is conceivable that Gauss was unaware of the fact that this subject was extensively studied by Lobachevsky during the same period, but he probably knew from his own experience that obtaining formulae for volumes of tetrahedra in hyperbolic and spherical spaces is a difficult problem. It turns out indeed that there is no simple formula for the volume of a hyperbolic (or spherical) tetrahedron in terms of the dihedral angles that is comparable to the formula for the area of a triangle. The existing formulae (first obtained by Lobachevsky) give expressions of volume in terms of an integral function which bears the name Lobachevsky function and which is closely connected to the Euler dilogarithm function. These formulae involve the value of the Lobachevsky function at the dihedral angles of the tetrahedron.

Several computations of volumes of tetrahedra were conducted after Lobachevsky's work but the progress was slow. Some of the most important conjectures that are open in three-dimensional geometry of hyperbolic (or spherical) space concern volume. We mention incidentally that the computation of volumes of hyperbolic tetrahedra is the subject of Chapter 8 of Thurston's notes on three-dimensional geometry.[6]

Talking about volume, let us also mention that there are interesting formulae for volumes of Euclidean tetrahedra that are due to Euler. The latter, in a letter to his friend Christian Goldbach, dated November 14, 1750,[7] announced a formula for the volume of a simplex in terms of its side lengths, a three-dimensional analogue of Heron's formula for the area of a triangle.[8] Later on, he gave the proof of that formula, in a paper published in 1758.[9] In the same paper, he provided several other formulae for volumes of Euclidean simplices, including a formula in terms of side lengths, and another one in terms of lengths of the three edges abutting on a solid angle along with the three plane angles that these sides form.

Finally, let us note that computing area and volume, beyond those of polygons and polyhedra, is another subject that has a long history whose origin can be traced back to the work of Hippocrates of Chios (5th century B.C.) who studied areas of lunes (intersections of two discs) and other figures, and which was brought to a high degree of sophistication in the work of Archimedes (3rd century B.C.). The reader can imagine that computing areas and volumes without the use of the mechanical methods of modern calculus is always a challenging problem.

---

[6] W. P. Thurston, *Geometry and topology of three-manifolds*, Princeton Lectures Notes, Princeton, N.J., 1979.

[7] Letter № 149 in Euler's volume of his *Opera omnia* containing his correspondence with Goldbach, Series quarta A, Vol. IV, Birkhäuser, Basel, 2014.

[8] Heron's formula is contained in the *Codex costantinopolitanus Palatii Veteris,* ed. and transl. by E. M. Bruin, Leiden, 1964.

[9] L. Euler, Demonstratio nonnullarum insignium proprietatum, quibus solida hedris planis inclusa sunt praedita, *Novi Commentarii academiae scientiarum petropolitanae* 4 (1752), 140–160, in *Opera omnia,* Series 1, Vol. XXVI, 94–108.

Let us pass to our second topic, projective geometry and conics. We note right away that the study of conics dates back to Greek antiquity, where the most important treatise on the subject is the multi-volume work of Apollonius of Perga (3rd–2nd c. B.C.),[10] which remained until the seventeenth century one of the most fundamental and most read mathematical texts among mathematicians. A conic is the intersection of a plane with a right cone whose basis is circular. If we think of projective geometry as the study of properties of figures that are invariant by projections, it is in the study of conics that we find the root of this field: the cone vertex is the center of projection. The notion of cross-ratio as a projective invariant also dates back to Greek antiquity, it is used in Menelaus' *Spherics*.[2]

In his 1859 paper "A sixth memoir upon quantics,"[11] Cayley included the study of Euclidean and spherical geometries in the setting of projective geometry, where the ground space for each of these geometries is the interior of an appropriate conic in projective space, the conic itself becoming the "absolute" of the space. He noted that the distance function in such a geometry may be defined in terms of the logarithm of the cross-ratio. In the same paper, Cayley made his famous statement that "descriptive geometry is all geometry,"[12] an idea which was taken up by Klein later on, who, in his 1871 paper "Über die sogenannte Nicht-Euklidische Geometrie" (On the so-called non-Euclidean geometry),[13] included hyperbolic geometry as well in the picture, using different kinds of conics and quadrics in dimensions 2 and 3 respectively. In this work, Klein also gave formulae for the distance functions using the cross-ratio in the Euclidean and the two classical non-Euclidean geometries.

Among the various attempts to generalize hyperbolic geometry using the geometry of quadrics, we may mention a section in a paper of Poincaré which he wrote in 1887, called "Sur les hypothèses fondamentales de la géométrie" (On the fundamental hypotheses of geometry).[14] This paper is poorly known to mathematicians. The section which interests us here is concise, it contains ideas with no attempts for precise statements and proofs. We provide an English translation of it in the prologue of this volume, after the present introduction.

[10] The authoritative version is the one by R. Rashed, in 4 volumes: *Les Coniques,* Tome 1, Livre I, de Gruyter, Berlid 2008; Tome 2, Livre IV, ibid., 2009; Tome 3, Livre V, ibid., 2008; Tome 4, Livres VI et VII, ibid, 2009.

[11] A. Cayley, "A sixth memoir upon quantics," *Phil. Trans. R. Soc. Lond.* 149 (1859), 61–90. Reprinted in Vol. II of Cayley's *Collected mathematical papers.*

[12] "Descriptive geometry" is the name Cayley used for projective geometry.

[13] F. Klein, "Über die sogenannte Nicht-Euklidische Geometrie," Vorgelegt von A. Clebsch, *Nachricheten von der Kgl. Gesellschaft der Wissenschaften zu Göttingen,* № 17 (30 August 1871). French version, "Sur la géométrie dite non euclidienne," translated by J. Hoüel, *Bull. sci. math. et astr.* 2 (1871), 341–351. See also the commentary in N. A'Campo and A. Papadopoulos, "On Klein's *So-called non-Euclidean geometry,*" in *Sophus Lie and Felix Klein*: *the Erlangen program and its impact in mathematics and in physics* (L. Ji and A. Papadopoulos, eds.), EMS Publishing House, Zürich, 2015, 91–136.

[14] H. Poincaré, "Sur les hypothèses fondamentales de la géométrie." *Bull. Soc. Math. France* 15 (1887), 203–216.

Eduard Study, in a paper published in 1907[15] whose translation is contained in the present volume, included in the projective setting other geometries than the three classical geometries.

In the rest of this introduction, we will give a quick survey of the various essays that constitute the present volume. We have divided this collection of essays into three parts. The first part (Chapters 1–12) is the longest, and it is concerned with spherical and hyperbolic geometries. The second part (Chapters 13–16) deals with geometries defined in the setting of projective geometry. The third part (Chapters 17 and 18) concern two other geometries, namely, Hermitian geometry, that is, the geometry of complex projective spaces, and an axiomatic plane geometry which is termed "non-elliptic metric plane in which every segment has a midpoint."

The content of each chapter of this volume is now described in detail.

Part I. Spherical and hyperbolic geometries

Chapter 1, written by Norbert A'Campo and Athanase Papadopoulos, is a survey of classical material about area in spherical and hyperbolic geometry. A theorem of Albert Girard stating that the area of a spherical triangle is equal (up to a constant multiple) to its angle sum, as well as its analogue in hyperbolic geometry, are reviewed. A formula due to Euler for the area of a spherical triangle in terms of its side lengths, again with its analogue in hyperbolic geometry, are used in order to give an equality for the distance between the midpoints of two sides of a spherical (respectively hyperbolic) triangle in terms of the third side. These equalities are quantitative versions of the formula expressing the fact that the sphere (respectively hyperbolic plane) is positively (respectively negatively) curved in the sense of Busemann. The essay contains further results related to area in non-Euclidean geometry together with some historical comments.

Chapter 2, by Elena Frenkel and Weixu Su, is based on variational methods introduced by Euler in non-Euclidean geometry. The authors provide detailed proofs of hyperbolic analogues of spherical results obtained by Euler using these methods. This includes a derivation of the trigonometric formulae and an area formula for hyperbolic triangles in terms of their side lengths. Euler considered the use of variational methods in spherical geometry as an application of the techniques of the calculus of variations, a field of which he had laid the foundations.[16]

[15] E. Study, "Beiträge zur nichteuklidische Geometrie," I.–III., *Amer. J. Math.* 29 (1907), 101–167.

[16] Cf. L. Euler, "Principes de la trigonométrie sphérique tirés de la méthode des plus grands et des plus petits," *Mémoires de l'Académie royale des sciences et belles-lettres* 9 (1755), 223–257, in *Opera omnia,* Series 1, Vol. XXVI, 277–308.

In Chapter 3, Elena Frenkel and Vincent Alberge study the hyperbolic analogue of a problem in spherical geometry solved by Friedrich Theodor von Schubert, a young geometer who became a successor of Euler at the Saint Petersburg Academy of Sciences and who worked on spherical geometry. Schubert's results include the determination of the loci of the vertices of triangles satisfying some given conditions, in the spirit of problems solved by Euler and his other young collaborators Anders Johan Lexell and Paul Heinrich Fuss. In this chapter, the authors give a solution in the hyperbolic setting of a problem solved by Schubert on the sphere, namely, to find, for a triangle with prescribed base and whose vertex varies on a given hypercycle consisting of points equidistant to the line containing the base, the point(s) for which the area of this triangle is maximal or minimal. They provide two different solutions of this problem.

The reader may note that in the Euclidean case, if we fix two vertices of a triangle and if we look for a family of triangles having the same area, the locus of the third vertex is a straight line parallel to the base of the triangle (in fact, in the Euclidean plane, hypercycles are straight lines). The situations in the hyperbolic and spherical cases are different from the Euclidean case since in these cases, the area of a triangle is not determined by a side and the corresponding altitude. The desired locus in these cases is not a curve equidistant to the basis of the triangle. It is interesting to note in this respect that Herbert Busemann solved in 1947 the following related problem: "To characterize the geometries in which the following property, which is satisfied in Euclidean geometry, holds: *The area of a triangle ABC depends only on the length of BC and the distance from A to the segment BC*." Busemann found that the only geometries that satisfy this property are what he called the Minkowski planes (that is, the metric spaces underlying the 2-dimensional normed vector spaces) in which the perpendicularity relation is symmetric (for an appropriate notion of perpendicularity).[17]

Chapter 4, by Himalaya Senapati, concerns medians in non-Euclidean geometry. A result due to J. H. Lambert, from his *Theorie der Parallellinien* (Theory of parallel lines), written in 1766,[3] says that for any equilateral triangle ABC with medians AA', BB' and CC' intersecting at O, we have $OA' = \frac{1}{3}AA'$, $OA' > \frac{1}{3}AA'$ and $OA' < \frac{1}{3}AA'$ in Euclidean, spherical and hyperbolic geometry respectively. Senapati shows, using non-Euclidean trigonometry, that Lambert's inequalities hold for arbitrary triangles. He also presents a collinearity result in spherical geometry, as an application of Menelaus' theorem.

[17] Cf. Theorem 50.9 in H. Busemann, *The geometry of geodesics,* Academic Press, New York, 1955, and H. Busemann, "Two-dimensional geometries with elementary areas," *Bull. Amer. Math. Soc.* 53 (1947), 402–407, reprinted in H. Busemann, *Selected works* (A. Papadopoulos, ed.), Vol. 1, Springer Verlag, Cham, 2018, 379–384.

Chapter 5 is also due to Himalaya Senapati. It concerns the construction in spherical geometry of a triangle whose three vertices lie on a given circle and whose three sides produced pass through three given points. Such a construction, in the Euclidean case, is a theorem in Pappus' *Collection*,[18] in the special case where the three points are aligned. Euler generalized the problem to the case where the three points are not necessarily aligned and he gave a construction in this general case in the Euclidean as well as in the spherical setting.[19] In the same volume in which Euler's solution appeared, his young collaborator Nicolaus Fuss[20] published another proof of the same construction for the Euclidean case.[21] Lagrange found a new short proof for the Euclidean case, reducing the problem to finding roots of quadratic equations that can be solved using ruler and compass.[22] In Senapati's essay, Lagrange's solution is adapted to the non-Euclidean setting.

The notion of constructibility by ruler and compass is at the root of geometry. It underwent several transformations, from the Greeks to Galois, passing through Descartes, Gauss, and others. Senapati's construction on the sphere, in Chapter 5, based on the Euclidean Lagrange solution of the generalized problem of Pappus, is in the tradition of Descartes, that is, reducing the construction problem to solutions of quadratic equations.

We note incidentally that the first time Descartes introduced the xy coordinates (what we call the "Cartesian coordinates") is in Book I of his *Géométrie*, precisely in the solution of a construction problem of Pappus. We also recall that the title of this book is "Des problèmes qu'on peut construire sans y employer que des cercles et des lignes droites" that is, "On problems that can be constructed using only circles and straight lines."

Chapter 6 is again due to Himalaya Senapati. It is based on two propositions of Menelaus' *Spherics* that are comparison results for angles in a triangle cut by a geodesic arc joining the midpoints of two sides. The two propositions, in Menelaus'

---

[18] Pappus d'Alexandrie, *La collection mathématique,* Œuvre traduite pour la premiere fois du grec en français par P. Ver Eecke, Desclée de Brouwer, Paris and Bruges, 1933, Propositions 105, 107, 108, and 117 of Book VII.

[19] L. Euler, "Problematis cuiusdam Pappi Alexandrini constructio," *Acta Academiae scientarum imperialis petropolitanae*, Pars 1 (1780), 91–96. In *Opera omnia,* Series 1, Vol. XXVI, 237–242.

[20] Nicolaus Fuss (1755–1826) was initially Euler's secretary, and he became later his student, collaborator, and colleague at the Russian Academy of Sciences, and eventually the husband of his granddaughter Albertine.

[21] N. Fuss, "Solutio problematis geometrici Pappi Alexandrini," *Acta Academiae scientarum imperialis petropolitanae*, Pars 1 (1780), 97–104.

[22] J.-L. Lagrange, "Solution algébrique d'un problème de géométrie," in *Oeuvres de Lagrange*, Vol. IV, Gauthier-Villars, Paris 1868, 335–339. There is a confusion about the authorship of this work. It first appeared in a memoir under the name of J. de Castillon (Giovanni Francesco Mauro Melchiorre Salvemini, also called il Castiglione, after his birthplace, Castiglione del Valdarno), in which the latter declares that this proof is due to Lagrange, and after that, he provides his own proof, which is different from Lagrange's. The part of Castillon's memoir that corresponds to Lagrange's proof appeared eventually in Lagrange's *Œuvres* under his own name.

treatise, belong to a group of propositions in which the author proves for spherical triangles the property called today the Busemann property for positively curved metric spaces. Senapati gives an improved version of the angle comparison result in the case of spherical triangles as well as a version for hyperbolic triangles.

In Chapter 7, Dmitriy Slutskiy gives a proof of the trigonometric formulae of spherical and hyperbolic geometries using a kinematic method, following the work of Joseph Marie de Tilly (1837–1906). This approach is explained in a memoir by the latter, published in 1870 under the title *Études de mécanique abstraite* (Studies in abstract mechanics).[23] The basic tools used are two functions that de Tilly calls $\mathrm{eq}(r)$ and $\mathrm{circ}(r)$ respectively, the first one being the length of a curve at distance r from a geodesic of length one, and the second one being the length of a circle of radius r. The work is model-free.

Chapter 8, by Son Lam Ho, is concerned with the classical Gauss–Bonnet formula for surfaces of variable curvature. This is again a result on area. It says that the integral of the Gaussian curvature over a closed surface in the Euclidean space $\mathbb{R}^3$, with respect to the area form, is equal to 2π times the Euler characteristic of the surface. This formula is one of the first formulae that establish a relation between, on the one hand, topology (the Euler characteristic) and on the other hand, geometry (curvature and area). The author explains the relation between the Gauss–Bonnet formula and the formula for the area of a triangle in terms of angle excess.

Chapter 9, by Charalampos Charitos and Ioannis Papadoperakis, is based on a theorem of Euler on cartography. Euler's result says that there is no perfect map from an open subset of the sphere into the Euclidean plane. Here, a perfect map is a smooth map that preserves distances infinitesimally along the meridians and parallels as well as angles between these lines. The theorem is generally quoted with a different (and wrong) statement in the literature by historians of mathematics (and sometimes mathematicians who followed the historians' writings) who misunderstood the statement.[24] In the present essay, Euler's precise statement and a detailed proof of it are provided.

[23] J.-M. de Tilly, *Études de mécanique abstraite,* Mémoires couronnée et autres mémoires publiés par l'Académie Royale de Belgique, Vol. XXI, 1870.

[24] An instance of a wrong quote is contained in R. Ossermann's "Mathematical mapping from Mercator to the millenium," in *Mathematical adventures for students and amateurs* (D. F. Hayes and T. Shubin, eds.), The Mathematical Society of America, Washington (D.C.), 2004, 237–257, Spectrum. Theorem 1 there, attributed to Euler, states the following: *It is impossible to make an exact scale map of any part of a spherical surface*, an exact scale map meaning, in that paper, a map that preserves distances up to scale. The author refers to Euler's paper "De repraesentatione superficiei sphaericae super plano published" in *Acta Academiae scientarum imperialis petropolitanae,* Pars 1 (1777), 107–132 (*Opera omnia,* Series 1, Vol. XXVIII, 248–275). The theorem stated in Ossermann's paper is not what Euler proves. This result was known long before Euler, since Greek antiquity, and Euler's theorem is much more involved than that. The same error occurs in the paper "Curvature and the notion of space" by A. Knoebel, J. Lodder, R. Laubenbache, and D. Pengelley, in *Mathematical masterpieces,* Springer Verlag, New York, 2007, 159–227, and in many other papers.

Chapter 10, by Charalampos Charitos, is based again on certain projection maps from (subsets of) the sphere to the Euclidean plane that were considered by Euler. The study concerns more precisely the class of area-preserving projections of class C^2 that satisfy the further property that they send all meridians and parallels to perpendicular curves in the Euclidean plane. This class of maps was highlighted by Euler in his memoir "De repraesentatione superficiei sphaericae super plano" (On the representation of spherical surfaces onto the plane).[24] The distortion from conformality of these projections is compared with that of the so-called "Lambert cylindrical equal area projection," a map introduced by Lambert and rediscovered by Euler. The author shows that the distortion of the Lambert projection has remarkable extremal properties in the class considered.

Thus, Chapters 9 and 10 concern the search of maps between non-Euclidean and Euclidean geometries that are best in an appropriate sense. It is also interesting to see that by their study of such maps, Euler, Lambert and the geographers that preceded them are in fact predecessors of the modern theory of quasiconformal mappings. The interested reader may refer to the recent survey entitled "Quasiconformal mappings, from Ptolemy's geography to the work of Teichmüller."[25]

Chapter 11, by Nikolai Abrosimov and Alexander Mednykh, consists of two parts. The first one concerns area, and the second one volume, in spherical and hyperbolic geometries.

In the first part, the authors survey various classical formulae by Euler, Cagnoli, Lhuillier and others on the area of a triangle in spherical geometry and their counterparts in hyperbolic geometry. They also present non-Euclidean analogues of a Euclidean area formula due to Bretschneider and other recent formulae for the area of non-Euclidean triangles and quadrilaterals, together with non-Euclidean versions and generalizations of an identity of Ptolemy concerning cyclic quadrilaterals, and related identities due to Casey, and others.

In the second part, the authors survey classical formulae for volumes of various kinds of polyhedra due to Schläfli, Lobachevsky, Bolyai, Coxeter, and Vinberg. They discuss in particular the case of an orthoscheme, that is, a simplex in which the edges are mutually orthogonal. A three-dimensional orthoscheme has three right dihedral angles, the other dihedral angles being termed *esesential*. Schläfli provided a formula for the volume of a spherical orthoscheme in terms of its essential dihedral angles, using a function which is now called the Schläfli function. Lobachevsky and Bolyai obtained formulae for volumes of hyperbolic orthoschemes. A theory of volume of ideal hyperbolic tetrahedra originated in the work of Lobachevsky done in the second

[25] A. Papadopoulos, "Quasiconformal mappings, from Ptolemy's geography to the work of Teichmüller," in *Uniformization, Riemann–Hilbert correspondence, Calabi–Yau manifolds, and Picard–Fuchs equations* (L. Ji and S.-T. Yau, eds.), Advanced Lectures in Mathematics 42, Higher Education Press, Beijing, and International Press, Boston, 2018, pp. 237–315.

quarter of the nineteenth century. This theory was revived by Thurston and Milnor in the late 1970s. Vinberg considered in detail the case of tetrahedra having at least one vertex at infinity.

Together with these developments, the authors in Chapter 11 present several other relatively recent results on volume, including a result of Abrosimov on a problem of Seidel asking for the expression of the volume of an ideal hyperbolic tetrahedron in terms of the determinant and the permanent of its so-called “Gram matrix.” They also report on a formula, due to Derevin and Mednykh, expressing the volume of a compact hyperbolic tetrahedron in terms of its dihedral angles, and on a formula due to Sforza concerning the volume of a compact tetrahedron in hyperbolic or spherical 3-space. They also present formulae for volumes of spherical and hyperbolic octahedra with various kinds of symmetries.

Chapter 12 by Ivan Izmestiev is concerned with rigidity problems of bar-and-joint frameworks in Euclidean, spherical and hyperbolic geometries. A bar-and-joint framework is an object made of rigid bars connected at their ends by universal joints, that is, joints that allow the incident bars to rotate in any direction. The questions that are addressed in this chapter generalize rigidity questions for polyhedra. The author approaches them from two points of view, which he calls the static and the kinematic. In the scientific jargon, statics is the science of equilibrium of forces, and kinematics the science of motion. In the present setting, the static rigidity of a framework refers to the fact that every system of forces with zero sum and zero moment applied on it can be compensated by stresses in the bars, whereas its kinematic rigidity refers to the absence of deformations that keep the lengths of bars constant to the first order. The author shows that these two notions of rigidity of a framework are equivalent. More generally, he proves that a framework in a Euclidean, spherical, or hyperbolic space has the same number of static and kinematic degrees of freedom. Here, the number of static degrees of freedom is the dimension of the vector space of unresolvable loads, that is, of systems of forces applied to the nodes that cannot be compensated by stresses in the bars. In particular, a framework is statically rigid if the number of its static degrees of freedom is zero. The number of kinematic degrees of freedom is the dimension of the vector space of non-trivial infinitesimal isometric deformations, that is, deformations that are not induced by an isometry of the ambient space.

The subject of statics, in the sense used in this essay, finds its roots in the 19$^{\text{th}}$ century works of Poinsot, Möbius, Grassmann, and others. Questions of infinitesimal rigidity of smooth surfaces were addressed in the 20$^{\text{th}}$ century by H. Weyl, A. D. Alexandrov, and A. V. Pogorelov. Izmestiev in Chapter 12 provides a projective interpretation of statics which allows him to prove a projective invariance property of infinitesimal rigidity and to establish a correspondence between the infinitesimal motion of a Euclidean framework and its geodesic realization in the two geometries of constant nonzero curvature. He refers to the fact that the number of degrees of freedom of a Euclidean framework is a projective invariant as the Darboux–Sauer correspondence. He also reviews the so-called “Maxwell–Cremona” correspondence for a framework in Euclidean and in the two non-Euclidean geometries. This correspondence establishes an equivalence between the existence of a self-stress of a framework (a collection of stresses in its bars that resolves a zero load), a reciprocal diagram (a framework whose combinatorics is dual to the combinatorics of a given framework), and a so-called “polyhedral lift" (which, in the Euclidean case, is a vertical lift to 3-space which has the property that the images of the vertices of every face are coplanar).

Part II. Projective geometries

Chapters 13 and 14 inaugurate the second part of the volume. They consist of a translation (made by Annette A’Campo-Neuen) and a short commentary (by Annette A’Campo-Neuen and Athanase Papadopoulos) on a paper by Eduard Study, published in 1907, entitled “Beiträge zur Nicht-Euklidischen Geometrie. I.”[15] In this paper, Study introduces what he calls the *exterior plane hyperbolic geometry*, which turns out to be the geometry we call today the de Sitter geometry. This is a geometry of the complement of hyperbolic space, when this space is realized as the Cayley–Klein disc model sitting in the projective plane. The exterior space is also the space of lines in the hyperbolic plane. A distance function (which is not a metric in the sense we intend it today) is defined on pairs of points in this exterior space, and the way the distance between two points is defined depends on the position of these points, more precisely, on whether the line joining them intersects the unit circle, or is tangent to it, or is disjoint from it. Study gives a characterization of triangles for which the triangle inequality holds, and of triangles for which the reverse triangle inequality (called the time inequality) holds. He also discusses the cases where the distance between two points is equal to the length of the longest curve joining them.

It is interesting to see that the ideas expressed by Study in this paper are explored in modern research. They appear in the two essays that follow (Chapters 15 and 16).

Chapter 15 is also due to Ivan Izmestiev. It consists of an exposition of the theory of conics in spherical and hyperbolic spaces. The author presents this theory in full detail, including classifications of conics from the algebraic and analytic points of view, the theory of pencils, the characteristic properties of foci, axes, and centers, the bifocal properties of spherical conics, the various concepts of polarity and duality as well as the non-Euclidean versions of several results including Poncelet, Brianchon, Pascal and Chasles’ theorems, and Ivory’s lemma. The latter states that the diagonals of a quadrilateral formed by four confocal conics have equal lengths. De Sitter space is also included in the picture.

The topic of spherical conics is classical. Back in 1788, Nicolaus Fuss wrote a memoir on spherical conics, entitled "De proprietatibus quibusdam ellipses in superficie sphaerica descriptae" (On some properties of an ellipse traced on a spherical surface).[26] Michel Chasles made a systematic study of spherical conics in his 1860 article "Résumé d'une théorie des coniques sphériques homofocales et des surfaces du second ordre homofocales" (Summary of a theory of spherical homofocal conics and second-order homofocal surfaces).[27] William Story, in a paper on non-Euclidean geometry published in 1881,[28] computed areas of conics in the hyperbolic plane; his results are expressed in terms of elliptic integrals. In 1883, he wrote a paper on hyperbolic conics entitled "On non-Euclidean properties of conics."[29] Today, the subject of non-Euclidean conics is almost forgotten, and even, the study of Euclidean conics is no more part of the curricula. Izmestiev's essay, based on the articles by Chasles and Story, revives this subject, showing at the same time that Euclidean and non-Euclidean geometries (including de Sitter) can be approached via the study of (Euclidean) conics. From this point of view, a non-Euclidean conic is represented by a pair of quadratic forms on Euclidean space, the first one (assumed to be non-degenerate) representing the absolute of a non-Euclidean geometry, and the second one being the conic in the non-Euclidean space realized by the first conic.

In Chapter 16, François Fillastre and Andrea Seppi start from the fact, which we mentioned several times, that in dimension two, a model for each of the three classical geometries can be obtained as the interior of (a connected component of the complement of) a certain conic, the latter becoming the "absolute" of the space. The authors survey several two- and three-dimensional geometries that may be obtained, in the tradition of Klein, in the setting of projective geometry. These geometries include, besides the classical Euclidean, spherical and hyperbolic geometries, de Sitter and anti-de Sitter geometries, as well as geometries that the authors call co-Euclidean and co-Minkowski. The last two are respectively the geometry of the space of hyperplanes in Euclidean space and that of space-like hyperplanes in Minkowski geometry. Some of these geometries appear as dual to or as limits of other geometries. The developments of the theory presented in this essay use duality theory in Euclidean space associated to convex sets (polar duality). A convex complementary component of a quadric in $\mathbb{R}^3$ appears as a model of hyperbolic space whereas the other component appears as a model for de Sitter space and is also described as the space of hyper-

[26] N. Fuss, "De proprietatibus quibusdam ellipses in superficie sphaerica descriptae," *Nova acta Academiae scientiarum imperialis petropolitanae* 3 (1785), 90–99.

[27] M. Chasles, "Résumé d'une théorie des coniques sphériques homofocales et des surfaces du second ordre homofocales," *J. Math. Pures Appl.* (1860), 425–454.

[28] W. E. Story, "On the non-Euclidean trigonometry," *Amer. J. Math.* 4 (1881), 332–335.

[29] W. E. Story, "On non-Euclidean properties of conics," *Amer. J. Math.* 5 (1883), 358–382.

planes in hyperbolic space. The definition of anti-de Sitter space, as a Lorentzian manifold of curvature -1, uses bilinear forms of signature 2.

Questions regarding transitions between geometries are also addressed in Chapter 16, and in this setting, elliptic and de Sitter geometries limit to co-Euclidean geometry. Likewise, Minkowski and co-Minkowski geometries appear as limit geometries of classical ones. The authors also discuss connections and volume forms for their model spaces, leading to connections and volume forms for the degenerate co-Euclidean and co-Minkowski geometries. A map they call "infinitesimal Pogorelov map" is introduced, between the hyperbolic and Euclidean spaces, and between anti-de Sitter and Minkowski spaces. They then study embedded (especially convex) surfaces in the 3-dimensional model spaces, as well as geometric transitions of surfaces. They show that the notion of curvature transits between various geometries, under rescaling in co-Euclidean and co-Minkowski geometries.

Part III. Other geometries

In Chapter 17, Boumediene Et-Taoui provides a survey of Hermitian trigonometry, that is, trigonometry in complex projective space. In particular, he presents the two laws of sines and the law of cosines for a triangle in the complex projective plane. The subject of Hermitian trigonometry is classical. It was first studied by Blaschke and Terheggen in their paper "Trigonometria Hermitiana."[30] The exposition in Chapter 17 follows in part that of Ulrich Brehm in his article "The shape invariant of triangles and trigonometry in two point homogeneous spaces."[31] Et-Taoui defines at the same time the notion of polar triangle associated to a given triangle in complex projective space and he uses it to derive the second law of sines. He also establishes relations between invariants introduced by Brehm and others. He notes that the spherical geometry developed by Eduard Study is a special case of the Hermitian trigonometry developed in this chapter. In particular, the trigonometric formulae of spherical trigonometry can easily be deduced from the Hermitian trigonometric formulae. The essay also contains remarks on trigonometry in general symmetric spaces, a subject where almost all the basic questions are still open.

In Chapter 18, Victor Pambuccian presents a result on triangles of fixed area in a geometry he describes as the "Bachmann non-elliptic metric plane in which every pair of points has a midpoint." This geometry is defined in an axiomatic way in terms of

[30] W. Blaschke and H. Terheggen, "Trigonometria Hermitiana," *Rend. Sem. Mat. Univ. Roma Ser.* 4 (1939), 153–161.

[31] U. Brehm, "The shape invariant of triangles and trigonometry in two point homogeneous spaces," *Geom. Dedicata* 33 (1990), 59–76.

groups, involutions in these groups and axioms concerning the relations among these involutions. The expression "elliptic metric plane" refers to a geometry in which there are three line reflections whose composition is the identity. The result presented says that for any given pair of points A and C and for any point B varying in the plane in such a way that the area of the triangle ABC is constant, the locus of the midpoints of AB and CB consists of two lines symmetric with respect to AC. This result is in the trend of a theorem attributed to Lexell (with various proofs by Lexell, Euler, Legendre, Steiner, Lebesgue, and several others)[32] concerning the locus of the third vertex of a spherical triangle with two given vertices and fixed area.

This set of essays should give the reader a taste of the fundamental notions of non-Euclidean geometry and an idea of a variety of problems that are studied in this very rich branch of mathematics.

Vincent Alberge (New York),
Athanase Papadopoulos (Strasbourg and Beijing),
December 2018

[32] A. J. Lexell, "Solutio problematis geometrici ex doctrina sphaericorum," *Acta Academiae scientarum imperialis petropolitanae*, Pars 1 (1781), 112–126; L. Euler, "Variae speculationes super area triangulorum sphaericorum," *Nova acta Academiae scientarum imperialis petropolitanae* 10 (1792), 47–62, in *Opera omnia,* Series 1, Vol. XXIX, 253–266; A.-M. Legendre, *Éléments de géométrie* (11th ed.). Firmin Didot, Paris 1817; J. Steiner, "Sur le maximum et le minimum des figures dans le plan, sur la sphère et dans l'espace général," *J. Math. Pures Appl.* 6 (1841), 105–170; H. Lebesgue, "Démonstration du théorème de Lexell," *Nouvelles annales de mathématiques* 14 (1855), 24–26.