Prologue
An excerpt from "Sur les hypothèses fondamentales de la géométrie"[*]

Henri Poincaré

We already know three two-dimensional geometries:

1. Euclidean geometry, where the angle sum in a triangle is equal to two right angles;

2. Riemann's geometry, where this sum is greater than two right angles;

3. Lobachevsky's geometry, where it is smaller than two right angles.

These three geometries rely on the same fundamental hypotheses, if we except Euclid's *postulatum* which the first one admits and the two others reject. Furthermore, the principle according to which two points completely determine a line entails an exception in Riemann's geometry and entails no exception in the other two cases.

When we restrict ourselves to two dimensions, Riemann's geometry allows a very simple interpretation; it does not differ, as is well known, from spherical geometry, provided we decide to give the name *lines* to the great circles of the sphere.

I shall start by generalizing this interpretation in such a way that it can be extended to Lobachevsky's geometry.

Consider an arbitrary order-two surface.[1] We agree to call *lines* the diametral plane[2] sections of this surface, and *circles* the non-diametral plane sections.

[*] H. Poincaré "Sur les hypothèses fondamentales de la géométrie," *Bull. Soc. Math. France* 15 (1887) 203–216. Translation by A. Papadopoulos.

[1] [Translator's note] A degree-two algebraic surface, also called a quadric.

[2] [Translator's note] For a given quadric, the locus of midpoints of the system of chords that have a fixed direction is a plane, called a *diametral plane* of the quadric. In the case where the quadric has a center of symmetry, a diametral plane is a plane passing through this center of symmetry. There are more general points of symmetry for a quadric. In analogy with the case of the great circles and the small circles of a sphere, Poincaré calls a *line* the intersection of a quadric surface with a diametral plane, and a *circle* the intersection of a quadric with an arbitrary plane.

It remains to define what we mean by the angle made by two lines which intersect and by the length of a line segment.

Through a point on the surface, let two diametral plane sections pass (which we decided to call *lines*). Let us then consider the tangents to these two plane sections and the two rectilinear generatrices of the surface which pass through the given point. These four lines (in the usual sense of the word) have a certain anharmonic ratio.[3] The angle that we want to define will then be the logarithm of the anharmonic ratio if the two generatrices are real, that is, if the surface is a one-sheeted hyperboloid. In the other case, our angle will be the same logarithm divided by $\sqrt{-1}$.

Let us consider an arc of conic which is part of a diametral plane section (this is what we agreed to call a *line segment*). The two endpoints of the arc and the two points at infinity of the conic have, as any system of four points situated on a conic, a certain anharmonic ratio. Then we agree to call *length* of the given segment the logarithm of this ratio if the conic is a hyperbola and this same logarithm divided by $\sqrt{-1}$ if the conic is an ellipse.

There will be, between the angles and the lengths thus defined, a certain number of relations, and these will constitute a collection of theorems which are analogous to those of plane geometry.

This collection of theorems can then take the name *quadratic geometry*, since our starting point was the consideration of a quadric or a second order fundamental surface.

There are several quadratic geometries, because there are several kinds of second order surfaces.

If the fundamental surface is an ellipsoid, then the quadratic geometry does not differ from Riemann's geometry.[4]

If the fundamental surface is a two-sheeted hyperboloid, then the quadratic geometry does not differ from that of Lobachevsky.

If this surface is an elliptic paraboloid, then the quadratic geometry is reduced to that of Euclid; this is a limiting case of the preceding two.

It is clear that we have not exhausted the list of quadratic geometries; because we have not considered neither the one-sheeted hyperboloid nor its numerous degeneracies.

Thus, we can say that there are three main quadratic geometries, which correspond to the three types of second-order surfaces with center.

On the other hand, we have to add to them the geometries that correspond to the limiting cases and Euclidean geometry will be part of them.

[3] [Translator's note] The term *anharmonic ratio* means the cross ratio.

[4] [Translator's note] In the 19$^{\text{th}}$ century, the term *Riemann's geometry* was commonly used for the geometry of the sphere.

Why did the geometry of the one-sheeted hyperboloid, up to now, escape the theorists? This is because it implies the following propositions.

1. The distance between two points situated on the same rectilinear generator of the fundamental surface is zero.

2. There are two kinds of lines, the first one corresponding to the elliptic diametral sections and the other one to the hyperbolic diametral sections. It is impossible, using any real motion, to make a line of the first kind coincide with a line of the second.

3. It is impossible to make a line coincide with itself by a real rotation around one of its points, as it happens in Euclidean geometry when we rotate a line by 180° around one of its points.

All the geometers have implicitly assumed that these three propositions are false, and indeed these three propositions are so much contrary to the habits of our spirit that its was not possible to the founders of our geometries to make such a hypothesis and think of stating it.