Preface

This book, divided into two volumes, collects different cycles of lectures given at the IHP Trimester, "Geometry, Analysis and Dynamics on Sub-Riemannian Manifolds", held at the Institut Henri Poincaré in Paris, and the CIRM Summer School, "Sub-Riemannian Manifolds: From Geodesics to Hypoelliptic Diffusion", held at the Centre Internationale de Rencontres Mathématiques, in Luminy, during fall 2014.

Sub-Riemannian geometry is a generalization of Riemannian geometry, whose birth dates back to Carathéodory's 1909 seminal paper on the foundations of Carnot thermodynamics, followed by E. Cartan's 1928 address at the International Congress of Mathematicians in Bologna.

Sub-Riemannian geometry is characterized by nonholonomic constraints: distances are computed by minimizing the length of curves whose velocities belong to a given subspace of the tangent space. From the theoretical point of view, sub-Riemannian geometry is the geometry underlying the theory of hypoelliptic operators and degenerate diffusions on manifolds.

In the last twenty years, sub-Riemannian geometry has emerged as an independent research domain, with extremely rich motivations and ramifications in several parts of pure and applied mathematics. Let us mention geometric analysis, geometric measure theory, stochastic calculus, and evolution equations together with applications in mechanics and optimal control (motion planning, robotics, nonholonomic mechanics, quantum control) and another to image processing, biology, and vision.

Even if, nowadays, sub-Riemannian geometry is recognized as a transverse subject, researchers working in different communities are still using quite different language. The aim of these lectures is to collect reference material on sub-Riemannian structures for the use of both researchers and graduate students. Starting from basic definitions and extending up to the frontiers of research, this material reflects the point of view of authors with different backgrounds. The exchanges among the participants of the IHP Trimester and of the CIRM school are reflected here by several connections and interplays between the different chapters. This will hopefully reduce the existing gap in language between the different communities and favor the future development of the field.

Agrachev, Barilari, and Boscain give in their lecture notes an introductory text to sub-Riemannian geometry from the Hamiltonian point of view. The text revisits some basic notions of differential geometry and introduces the general framework. Classical results such as the Chow theorem and the existence of length-minimizers are presented in detail. The last part of the notes is devoted to the Hamiltonian description of normal and abnormal trajectories and to the proof that small pieces of normal trajectories are length-minimizers.

The lectures notes of Thalmaier focus on some probabilistic aspects related to sub-Riemannian geometry. The main intention is to give an introduction to hypoelliptic and subelliptic diffusions. The notes are written from a geometric point of view trying to minimize the weight of “probabilistic baggage” necessary to follow the arguments. They discuss in particular the following topics: stochastic flows to second-order differential operators; smoothness of transition probabilities under Hörmander’s brackets condition; control theory and Stroock–Varadhan’s support theorems; Malliavin calculus; Hörmander’s theorem.

The contribution of Friz and Gassiat presents, in a self-contained way, the foundations of geometric rough paths theory, starting from simple examples and key ideas up to rough integration. They also present some new results showing how classes of Markov processes provide natural examples of (random) rough paths. The authors accomplish keeping the connection to the language of sub-Riemannian geometry and nilpotent Lie groups.

The lecture notes by Ambrosio and Ghezzi discuss three different definitions of Sobolev spaces in the context of metric measure spaces. These three definitions, called W-, H-, and BL-spaces, are conceptually already quite different in the Euclidean case. Indeed, the three constructions rely on different objects: W-spaces are based on coordinate vector fields and H-spaces exploit approximations by smooth functions, whereas BL-spaces take account of the behavior of a function along special curves, and are characterized by a pointwise definition. The goal of these lectures is to show the equivalence of these three spaces under almost no assumption on the metric measure structure. The last section analyzes different approaches to defining BV functions in the framework of metric measure spaces, following two among the three main ideas developed for Sobolev functions: approximation by smooth functions and Beppo Levi’s point of view.

In his lecture notes, Zhitomirskii considers several classification problems for singularities of vector distributions. The notes are introduced by a general discussion on the role and scope of normal forms in singularity theory and differential geometry. The author presents normal forms for Riemannian metrics and conformal structures and explains, in terms of these normal forms, the classical invariants. The author then presents several results concerning the classification of generic low-dimension vector and affine distributions, as well as special classes of distributions such as Goursat ones.

Davide Barilari

Ugo Boscain

Mario Sigalotti