A casual preface

In einem unbekannten Land
vor gar nicht allzu langer Zeit
war eine Biene sehr bekannt
von der sprach alles weit und breit ...

To Maya

About ten years after Manin's lecture notes "Lectures on zeta functions and motives (according to Deninger and Kurokawa)" (dated 1995), and fifty years after Tits's influential paper "Sur les analogues algébriques des groupes semi-simples complexes" (1957), in which buildings over a "field with one element" $\mathbb{F}_1$ are described in order to see symmetric groups as Chevalley groups over this "field," the first papers got published in which scheme theories over the non-existing object $\mathbb{F}_1$ were developed. One fundamental paper is Deitmar's paper "Schemes over $\mathbb{F}_1$" in 2005 (inspired by Kato's *log schemes*); a year before, Soulé already published his $\mathbb{F}_1$-approach to varieties in "Les variétés sur le corps à un élément."

Other researchers such as Borger, Connes, Consani, Kurokawa, Lorscheid, Manin, Marcolli, Toën, and Vaquié contributed further to this rapidly emerging theory, and this stream of thoughts eventually culminated in the very recent mini-symposium "Absolute Arithmetic and $\mathbb{F}_1$-Geometry" at the 6th *European Congress of Mathematics* (Kraków, Poland) in 2012, organized by myself. The goal of the mini-symposium itself was to present the state of the art of this mysterious theory; speakers were Lieven Le Bruyn, Oliver Lorscheid, Yuri I. Manin, and myself as an extra.

Soon after, the idea grew to assemble the talks into a proceedings volume, and later Yuri Manin convinced me to see it bigger, and to aim rather for a proper monograph with chapters by various authors, so as to provide the first book on the subject of the mythical beast $\mathbb{F}_1$. And this volume is the outcome.

The book. The book is divided into four main parts:

(1) *Combinatorial Theory*—which contains one chapter (by myself);

(2) *Homological Algebra*—also containing one chapter (by Deitmar);

(3) *Algebraic Geometry*—with chapters by Borger, Le Bruyn, Lorscheid, Manin & Marcolli and myself (I refer to the table of contents for the precise titles); and

(4) *Absolute Arithmetic*—containing one chapter (by myself).

The first chapter should be seen as a combinatorial introduction on the one hand, and as a description of various combinatorial and incidence geometrical aspects of the theory on the other.

Deitmar's chapter paves a solid base for Homological Algebra of "belian categories" (certain non-additive categories like categories of modules of $\mathbb{F}_1$-algebras or $\mathbb{F}_1$-module sheaves in various $\mathbb{F}_1$-theories).

In Borger's chapter, the author extends the big and p-typical Witt vector functors from commutative rings to commutative semirings (and explains its connections with $\mathbb{F}_1$-theory).

Le Bruyn explores the origins of a new topology on the roots of unity μ_∞ introduced and studied by Kazuo Habiro in order to unify invariants of 3-dimensional homology spheres. He also seeks a meaning for the object $\mathtt{Spec}(\mathbb{Z})$ over $\mathbb{F}_1$.

Lorscheid reviews the development of $\mathbb{F}_1$-geometry from the first mentioning by Jacques Tits in 1956 until the present day. After that he explains his theory of *blueprints* in much depth (describing various connections with other scheme theories over $\mathbb{F}_1$).

Manin and Marcolli answer a question raised in the recent paper "Cyclotomy and analytic geometry over $\mathbb{F}_1$" by Manin, by showing that the genus zero moduli operad $\{\overline{M}_{0,n+1}\}$ can be endowed with natural descent data that allow it to be considered as the lift to $\mathtt{Spec}(\mathbb{Z})$ of an operad over $\mathbb{F}_1$. (They also describe a blueprint structure on $\{\overline{M}_{0,n}\}$.)

In my second chapter I first review Deitmar's theory of monoidal schemes; it is then explained how one can combinatorially study such schemes through a generalization of graph theory. In a more general setting I introduce "Υ-schemes," after which I study Grothendieck's motives in some detail in order to pass to "absolute motives" and "absolute zeta functions" (after Manin). In a last part of the chapter, I describe a marvelous connection between certain group actions on projective spaces and $\mathbb{F}_1$-theory.

Finally, I mention some aspects of "Absolute Arithmetic" in my last chapter, which may be considered as an appendix to the first three parts of the book.

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Famous last words As for those readers who want to know what paintings of Velázquez and Bacon are doing in this monograph—just think of the Weyl functor.

Koen Thas
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