Preface

This book, divided into two volumes, collects different cycles of lectures given at the IHP Trimester "Geometry, Analysis and Dynamics on Sub-Riemannian Manifolds", held at Institut Henri Poincaré in Paris, and the CIRM Summer School "Sub-Riemannian Manifolds: From Geodesics to Hypoelliptic Diffusion", held at Centre Internationale de Rencontres Mathématiques, in Luminy, during fall 2014.

Sub-Riemannian geometry is a generalization of Riemannian geometry, whose birth dates back to Carathéodory's 1909 seminal paper on the foundations of Carnot thermodynamics, followed by E. Cartan's 1928 address at the International Congress of Mathematicians in Bologna.

Sub-Riemannian geometry is characterized by non-holonomic constraints: distances are computed by minimizing the length of curves whose velocities belong to a given subspace of the tangent space. From the theoretical point of view, sub-Riemannian geometry is the geometry underlying the theory of hypoelliptic operators and degenerate diffusions on manifolds.

In the last twenty years, sub-Riemannian geometry has emerged as an independent research domain, with extremely rich motivations and ramifications in several parts of pure and applied mathematics. Let us mention geometric analysis, geometric measure theory, stochastic calculus and evolution equations together with applications in mechanics and optimal control (motion planning, robotics, nonholonomic mechanics, quantum control) and another to image processing, biology and vision.

Even if, nowadays, sub-Riemannian geometry is recognized as a transverse subject, researchers working in different communities are still using quite different language. The aim of these lectures is to collect reference material on sub-Riemannian structures for the use of both researchers and graduate students. Starting from basic definitions and extending up to the frontiers of research, this material reflects the point of view of authors with different backgrounds. The exchanges among the participants of the IHP Trimester and of the CIRM school are reflected here by several connections and interplays between the different chapters. This will hopefully reduce the existing gap in language between the different communities and favour the future development of the field.

The notes of Francesco Serra Cassano give an extensive presentation of geometric measure theory in Carnot groups. The first part of the notes discusses differential calculus for maps between Carnot groups in relation with the underlying metric structure. The text then focuses on differential calculus within Carnot groups and uses it to investigate intrinsic regular and Lipschitz surfaces in Carnot groups and their relation with rectifiability. The final section deals with sets of finite perimeter and with the related notions of reduced and minimal boundary. An application to minimal graphs in Heisenberg groups is developed.

The lecture notes by Nicola Garofalo are a quite comprehensive compendium of results in geometric analysis. In the first part, starting from basic examples and definitions of sub-Riemannian manifolds, length-spaces and Carnot groups, he discusses, in the sub-Riemannian context, Sobolev spaces, BV functions and Sobolev embedding theorems, passing through isoperimetric inequalities. In the second part he discusses classical results in geometric analysis in Riemannian manifolds and the now classical contributions by Folland–Stein, Rothschild–Stein and Nagel–Stein–Wainger in the sub-Riemannian context. Besides giving estimates for the fundamental solution of the heat equation, the goal is to discuss Li–Yau inequalities and curvature dimensional inequalities in the sub-Riemannian case.

The lecture notes by Fabrice Baudoin study hypoelliptic diffusion operators from the viewpoint of geometric analysis. The main focus is on sub-Riemannian Laplacians that arise as horizontal Laplacians of a Riemannian foliation. For this kind of operator an extensive theory is developed, with special attention to subelliptic Weitzenböck identities and different applications, from Li–Yau inequalities to spectral gap inequalities and the Bonnet–Myers theorem. The last section is devoted to the analysis of some Kolmogorov-type hypoelliptic diffusion operators and hypocoercive estimates.

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