# Introduction

The project of editing this Handbook arose from the observation that Hilbert geometry is today a very active field of research, and that no comprehensive reference exists for it, except for results which are spread in various papers and a few classical (and very inspiring) pages in books of Busemann. We hope that this Handbook will serve as an introduction and a reference for both beginners and experts in the field.

Hilbert geometry is a natural geometry defined in an arbitrary convex subset of real affine space. The notion of convex set is certainly one of the most basic notions in mathematics, and convexity is a rich theory, offering a large supply of refined concepts and deep results. Besides being interesting in themselves, convex sets are ubiquitous; they are used in a number of areas of pure and applied mathematics, such as number theory, mathematical analysis, geometry, dynamical systems and optimization.

In 1894 Hilbert discovered how to associate a length to each segment in a convex set by way of an elementary geometric construction and using the cross ratio. In fact, Hilbert defined a canonical metric in the relative interior of an arbitrary convex set. Hilbert geometry is the geometric study of this canonical metric. The special case where the convex set is a ball, or more generally an ellipsoid, gives the Beltrami–Klein model of hyperbolic geometry. In this sense, Hilbert geometry is a generalization of hyperbolic geometry. Hilbert geometry gives new insights into classical questions from convexity theory, and it also provides a rich class of examples of geometries that can be studied from the point of view of metric geometry or differential geometry (in particular Finsler geometry).

Let us recall Hilbert's construction. The line joining two distinct points x and y in a bounded convex domain intersects the boundary of that domain in two other points p and q. Assuming that y lies between x and p, the *Hilbert distance* from x to y is the logarithm of the cross ratio of these four points:
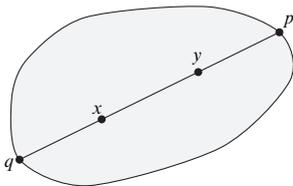
$$d(x, y) = \frac{1}{2} \log \left(\frac{|x - p|}{|y - p|} \cdot \frac{|y - q|}{|x - q|} \right).$$

This distance was defined in a letter to Klein written by Hilbert in 1894. It is a distance in the usual sense, and the relative interior of the convex set is a complete metric space for this distance. The Hilbert metric is invariant under projective transformations and depends in a monotonic way on the domain: a larger domain induces a smaller Hilbert distance. The Hilbert metric is projective in the sense that the straight

lines are geodesics. In other words,

$$d(x, y) = d(x, z) + d(z, y)$$

whenever $z \in [x, y]$. Furthermore, if the convex domain is strictly convex, then the affine segment is the unique geodesic joining two points.

The fourth Hilbert problem asks for a description of all projective metrics in a convex region, that is, metrics for which the straight lines are geodesics. At the beginning of the twentieth century, Hamel, who was a student of Hilbert, worked on this problem; he discovered new examples of projective metrics and found some deep results using differential calculus and the calculus of variations. The subject of Finsler geometry gradually emerged as an independent topic, and at the end of the 1920s, Funk and Berwald gave a differential geometric characterization of Hilbert metrics among all Finsler metrics on a domain with a smooth and strongly convex boundary. All these facts and several others which we describe below are reported on in this Handbook.

The deepest and most thorough studies in Hilbert geometry during the twentieth century are due to Busemann and his students and collaborators. During the period from the 1940s to the 1990s this school investigated Hilbert geometry from the viewpoint of metric geometry. A variety of questions regarding these metrics were studied, concerning their geodesics, their convexity theory (convexity of the distance function, of the spheres, etc.), their curvature, area, asymptotic geometry, limit cycles (horocycles) and limit spheres (horospheres), and several other features. For instance, these authors gave several characterizations of the ellipsoid in terms of its Hilbert geometry. They noticed (like Hilbert did before them) that the special case of the simplex is particularly interesting and they studied it in detail. They worked on the metrical properties of the Hilbert metric as well as on the axiomatic theory, making relations with the axioms and the basic notions of Euclidean and of non-Euclidean geometries, in particular the theory of parallels. They established relations between Hilbert geometry and other fields, including the foundations of mathematics, the calculus of variations, convex geometry, Minkowski geometry, geometric group theory and projective geometry. They also developed the basics of the closely related Funk metric.

Busemann formulated and initiated the study of several problems of which he (and his collaborators) gave only partial solutions, and a large amount of the research on Hilbert geometry that was done after him is directly or indirectly inspired by his work.

Let us briefly mention two further important directions in which the subject developed in the last century. In the late 1950s, Birkhoff found a new proof of the classical Perron–Frobenius theorem on eigenvectors of non-negative matrices based on the Hilbert metric in the positive cone. This new proof brought a new point of view on the subject and initiated a rich generalization of Perron–Frobenius theory. During the same period, Benzécri initiated the theory of divisible convex domains, that is, convex domains admitting a discrete cocompact group of projective transformations. The quotient manifold or orbifold naturally carries a Finsler structure whose universal cover is a Hilbert geometry. The reader will find more information on twentieth

century developments in Chapters 3, 10 and 15 of this Handbook. During the last fifteen years, the subject grew rapidly and a number of these recent developments are discussed in the other chapters.

We now describe the content of the book. The various chapters are written by different authors and are meant to be read independently from each other. Each chapter has its own flavor, due to the variety of tastes and viewpoints of the authors. Although we tried to merge the chapters into a coherent whole, we did not unify the different notation systems, nor did we try to avoid repetitions, hopefully to the benefit of the reader.

The book is divided into four parts.

Part I contains surveys on Minkowski, Funk and Hilbert geometries and on the relations between them.

In Chapter 1, A. Papadopoulos and M. Troyanov treat *weak Minkowski spaces*. A *weak metric* on a set is a non-negative distance function δ that satisfies the triangle inequality, but is allowed to be non-symmetric (we may have $d(x, y) \neq d(y, x)$) or degenerate (we may have $d(x, y) = 0$ for some $x \neq y$). A weak Minkowski metric on a real vector space is a weak metric that is translation-invariant and projective. The authors define the fundamental concept of weak Minkowski space and they give several examples and counterexamples. The basic results of the theory are stated and proved. Minkowski geometry shares several properties with Hilbert geometry, one of them being that the Euclidean geodesics are geodesics for that geometry. An important observation is that the infinitesimal – or tangential – geometry of a Hilbert or a Funk geometry is of Minkowski type. One of the main results of this chapter is the following: *A continuous weak metric δ on $\mathbb{R}^n$ is a weak Minkowski metric if and only if it satisfies the midpoint property, that is, $\delta(p, q) = 2\delta(p, m) = 2\delta(m, q)$ for any points p, q where m is the affine midpoint.* Other characterizations of weak Minkowski distances are given, providing various important aspects of this geometry. The relations with Busemann's G-spaces and Desarguesian spaces and comparisons with the Funk and the Hilbert metrics are also highlighted.

Chapter 2, *From Funk to Hilbert geometry*, by the same authors, is devoted to the study of the distance in a convex domain introduced by P. Funk in 1929. Using the notation of the figure on page 1, the Funk distance is defined by

$$F_\Omega(x, y) = \log\left(\frac{|x - p|}{|y - p|}\right).$$

Observe that this distance is a non-symmetric version of the Hilbert distance. Many properties of the Hilbert distance can be obtained as consequences of similar properties of the Funk distance. In this chapter, metric balls, the topology, convexity and orthogonality properties in Funk geometry are studied. A full description of Funk geodesics is also given. In the case of smooth curves, the property can be described as follows: *A smooth curve $\gamma(t)$ in a convex domain Ω is geodesic for the Funk metric if and only if there is a face D in $\partial\Omega$ such that the velocity vector $\dot{\gamma}(t)$ points toward D for all t.*

Chapter 3, by M. Troyanov, concerns the Funk and the Hilbert metrics from the point of view of Finsler geometry. This approach dates back to works done at the end of the 1920s, by Funk and by Berwald, who gave a characterization of Hilbert geometry from the Finslerian viewpoint. Funk and Berwald proved the following theorem: *A smooth Finsler metric defined on a convex bounded domain Ω of $\mathbb{R}^n$ is the Hilbert metric of that domain if and only if this geometry is complete (in an appropriate sense), if its geodesics are straight lines and if its flag curvature is equal to -1.* The author explains these notions in detail and he gives a complete proof of this result. At the same time, the chapter constitutes an introduction to the Finsler nature of the Funk and Hilbert metrics, where the Funk and the Hilbert Finsler structures appear respectively as the tautological and the symmetric tautological Finsler structures on Ω.

Chapter 4, *On the Hilbert geometry of convex polytopes*, by C. Vernicos, concerns the Hilbert geometry of an open set $\Omega \subset \mathbb{R}^n$ which is a polytope. A bounded convex domain is a polytope if and only if its Hilbert metric is bi-Lipschitz equivalent to a Euclidean space. An equivalent condition is that the domain is isometrically embeddable in a finite-dimensional normed vector space. Another characterization states that the volume growth is polynomial of order equal to the dimension of the convex domain. The author discusses several other aspects of the Hilbert geometry of polytopes.

The main goal of Chapter 5, by C. Walsh, is to give an explicit description of the horofunction boundary of a Hilbert geometry. This notion is based on ideas that go back to Busemann but which were formally introduced by Gromov. The results in this chapter are mainly due to Walsh. Walsh gives a sketch of how this boundary may be used to study the isometry group of these geometries. The main result in the chapter is that the group of isometries of a bounded convex polyhedron which is not a simplex coincides with the group of projective transformations leaving the given domain invariant.

Let us note that the horofunction boundaries of several other spaces have been described during the last few years (mostly by Walsh), in particular, Minkowski spaces and Teichmüller spaces equipped with the Thurston and with the Teichmüller metrics. These descriptions have also been applied to the characterization of the isometry groups of the corresponding metrics spaces.

Chapter 6, by R. Guo, gives a number of characterizations of hyperbolic geometry (or, equivalently, the Hilbert geometry of an ellipsoid) among Hilbert geometries. All these geometric characterizations are formulated in simple geometric terms.

Part II concerns the dynamical aspects of Hilbert geometry. It consists of four chapters: Chapters 7 to 10.

Chapter 7, by M. Crampon, concerns the geodesic flow of a Hilbert geometry. The study is based on a comparison of this flow with the geodesic flow of a negatively curved Finsler or Riemannian manifold. The main interest is in Hilbert geometries that have some hyperbolicity properties. Such geometries correspond to convex sets with C^1 boundary. In this case, stable and unstable manifolds exist, and there is a relation between the asymptotic behaviour of these manifolds along an orbit of the flow and the shape of the boundary at the endpoint of that orbit. The author then

studies the particular case of the geodesic flow associated to a compact quotient of a strictly convex Hilbert geometry, and he shows that such a flow satisfies an Anosov property. This property implies some regularity properties at the boundary of the convex set. The author then describes the ergodic properties of the geodesic flow, and he also surveys several notions of entropy that are associated to Hilbert geometry.

In Chapter 8, L. Marquis studies Hilbert geometry in the general setting of projective geometry. More precisely, the author surveys the various groups of projective transformations that appear in Hilbert geometry. In the first part of the chapter, he describes the projective automorphism group of a convex set in terms of matrices and then from a dynamical point of view. He shows the existence of convex sets with large groups of symmetries. He exhibits relations with several areas in mathematics, in particular with the theory of spherical representations of semi-simple Lie groups and with Schottky groups. He then explains how Hilbert geometry involves geometric group theory in various contexts. For instance, the Gromov hyperbolicity of a so-called *divisible* Hilbert metric (that is, one that admits a compact quotient action by a discrete group of isometries) is equivalent to a smoothness property of the boundary of the convex set. Note that the fact that the convex set is divisible means in some sense that it has a large group of symmetries. The other aspects of Hilbert geometry in which group theory is involved include differential geometry, convex affine geometry, real algebraic group theory, hyperbolic geometry, the theories of moduli spaces, of symmetric spaces, of Hadamard manifolds, and the theory of geometric structures on manifolds.

In Chapter 9, A. Karlsson considers the dynamical aspect of the theory of non-expansive (or Lipschitz) maps in Hilbert geometry. He makes relations with works of Birkhoff and Samelson done in the 1950s on Perron–Frobenius theory, and with a more recent work by Nussbaum and Karlsson–Noskov. He explains in particular how the theory of Busemann functions, horofunctions and horospheres in Hilbert geometries appear in the study of nonexpansive maps of these spaces and in their asymptotic theory. This is another example of the fact that a theory which is quite developed in the setting of spaces of negative curvature can be generalized and used in an efficient way in Hilbert geometry, which is not negatively curved (except in the case where the convex set is an ellipsoid).

In Chapter 10, B. Lemmens and R. Nussbaum give a thorough survey of the development of the ideas of Birkhoff and Samelson on the applications of Hilbert geometry to the contraction mapping principle and to the analysis of non-linear mappings on cones and in particular to the so-called non-linear Perron–Frobenius theory. The setting is infinite-dimensional. The authors also show how this theory leads to the result that the Hilbert metric of an n-simplex is isometric to a Minkowski space whose unit ball is a polytope having $n(n+1)$ facets.

Part III contains extensions and generalizations of Hilbert and Funk geometries to various contexts. It consists of three chapters: Chapters 11 to 13.

Chapter 11, written by I. Kim and A. Papadopoulos, concerns the projective geometry setting of Hilbert geometry. A Hilbert metric is defined on any convex subset

of projective space and descends to a metric on convex projective manifolds, which are quotients of convex sets by discrete groups of projective transformations. Thus, it is natural in this Handbook to have a chapter on convex projective manifolds and to study the relations between the Hilbert metric and the other properties of such manifolds. There are several parametrizations of the space of convex projective structures on surfaces; a classical one is due to Goldman and is an analogue of the Fenchel–Nielsen parametrization of hyperbolic structures. A more recent parametrization was developed by Labourie and Loftin in terms of hyperbolic structures on a Riemann surface together with cubic differentials, making use of the Cheng–Yau classification of complete hyperbolic affine spheres. Chapter 11 contains an introduction to these parametrizations and to some related matters. Teichmüller spaces appear naturally in this setting as they are important subspaces of the deformation spaces of convex projective structures of surfaces. The authors also discuss higher-dimensional analogues, covering in particular the work of Johnson–Millson and of Benoist and Kapovich on deformations of convex projective structures on higher-dimensional manifolds. Relations with geodesic currents and topological entropy are also treated. The Hilbert metrics which appear via their length spectra in the parametrization of the deformation spaces of convex projective structures also show up in the compactifications of these spaces. Some of the subjects mentioned are only touched on; this chapter is intended to open up new perspectives.

In Chapter 12, S. Yamada, K. Ohshika and H. Miyachi report on a new weak metric which Yamada defined recently on Teichmüller space and which he calls the *Weil–Petersson Funk metric*. This metric shares several properties of the classical Funk metric. It is defined using a similar variational formula, involving projections on hyperplanes in an ambient space that play the role of support hyperplanes, namely, they are the codimension-one strata of the Weil–Petersson completion of the space. The ambient Euclidean space of the Funk metric is replaced here by a complex which Yamada introduced in a previous work, and which he calls the Teichmüller–Coxeter complex. It is interesting that the Euclidean setting of the classical Funk metric can be adapted to a much more complex situation.

In Chapter 13, A. Papadopoulos and S. Yamada survey analogues of the Funk and Hilbert geometries on convex sets in hyperbolic and in spherical geometries. These theories are developed in a way parallel with the classical Funk theory. The existence of a Funk geometry in these non-linear spaces is based on some non-Euclidean trigonometric formulae, and the fact that the analogies can be carried over is somehow surprising because the study of the classical Funk metric on convex subsets of Euclidean space involves a lot of similarity properties and the use of parallels, which do not exist in the non-Euclidean settings. The geodesics of the non-Euclidean Funk and Hilbert metrics are studied, and variational definitions are given of these metrics. These metrics are shown to be Finsler. The Hilbert metric in each of the constant curvature convex sets is also a symmetrization of the Funk metric. The Hilbert metric of a convex subset in a space of constant curvature can also be defined using a notion

of a cross ratio which is adapted to that space. A relation is made with a generalized form of Hilbert's Problem IV.

Part IV consists of a two chapters which have a historical character. Chapter 14, written by M. Troyanov, contains a brief description of the historical origin of Hilbert geometry. The author presents a summary with comments of a letter of Hilbert to Klein in which Hilbert announces the discovery of that metric. Chapter 15, written by A. Papadopoulos, is a report on Hilbert's Fourth Problem, one of the famous twenty-three problems that Hilbert presented at the second ICM held in Paris in 1900. The problem asks for a characterization and a study of metrics on subsets of projective space for which the projective lines are geodesics. The Hilbert metric is one of the finest examples of metrics that satisfy the requirements of this problem. The author also reports on the relations between this problem and works done before Hilbert on metrics satisfying this requirements, in particular, by Darboux in the setting of the calculus of variations and by Beltrami in the setting of differential geometry.

Making historical comments and giving information on the origin of a problem bring into perspective the motivations behind the ideas. The comments that we include also make relations between the theory that is surveyed here and other mathematical subjects. We tried to pay tribute to the founders of the theory as we feel that historical comments usually make a theory more attractive.

Several chapters contain questions, conjectures and open problems, and the book also contains a special section on open problems proposed by various authors.