Introduction

Triangulated categories are subjects of newly developing research areas which attract young people from various fields of mathematics including abstract algebra, algebraic geometry, representation theory and theoretical physics. The derived category of complexes of sheaves on a space is a triangulated category appearing in algebraic geometry. An important observation is that the derived category has more hidden symmetries than the abelian category of sheaves. Namely there are non-isomorphic or even non-birational algebraic varieties which have equivalent derived categories. One might say that the derived categories provide a new concept of spaces. This aspect provides a new insight in birational geometry. The homological mirror symmetry conjecture of Kontsevich is another source of inspirations.

The concept of a triangulated category such as the derived category of sheaves on an algebraic variety was invented by Grothendieck and Verdier in the 1960s as a tool to express important results in algebraic geometry such as the duality theorem. The study of derived categories of individual algebraic varieties started in Russia and Japan simultaneously and independently in the 1970s. Beilinson and independently Gelfand and Gelfand proved that the derived categories of projective spaces have very nice explicit structures. Kapranov extended this result to more varieties of negative Kodaira dimension. On the other hand, Mukai studied moduli spaces of sheaves on abelian varieties, and discovered so-called Fourier–Mukai transforms. He proved that a non-isomorphic pair of varieties may have equivalent derived categories, as in the case of an abelian variety and its dual variety. He also proved similar results for K3 surfaces. The Fourier–Mukai transform is a new method for analyzing sheaves on varieties by changing the slices of the derived category given by its t-structures. In the 1990s, Bondal and Orlov proved a reconstruction theorem saying that a Fano manifold or canonically polarized manifold is recovered from its derived category. Moreover they proved that some standard birational transformations such as simple blowing ups along smooth centers correspond to semi-orthogonal decompositions of derived categories. This result suggested that there is a close relationship between the theory of derived categories and the minimal model theory in the birational geometry, in a way that varieties related by a flop have equivalent derived categories, and those related by a divisorial contraction or a flip are described by a semi-orthogonal decomposition of derived categories. Bridgeland, King and Reid proved a generalized McKay correspondence for three dimensional Gorenstein quotient singularities using the language of derived categories. They found a new application of the derived categories in showing that even the existence of crepant resolutions of such singularities can be proved.

This book contains articles concerning more recent development of this rapidly developing field. These are mostly survey articles. The contribution by Bernardara and

Bolognesi treats the conjectural correspondence by Orlov between the semi-orthogonal decompositions of the derived categories and the direct sum decompositions of the motives, which is reflected to the intermediate jacobians and the rationality question. Canonaco and Stellari review results on the existence and the uniqueness of the kernels for an equivalence functor between derived categories. Cautis surveys a new technique for proving a conjecture which says that the varieties connected by a flop have equivalent derived categories. The case of stratified Atiyah or Mukai flops is confirmed by using the representation theory of categorical Lie group actions, revealing an unexpected relation to the knot homology groups. Ishii and Ueda give a nice concrete description for the derived category of a Fermat variety by using an exceptional collection on its quotient stack. Kaledin proves a simple expression for the homology of a connected spectrum in terms of the associated special Γ-space. Keller's contribution is an excellent survey on the relationship between the cluster algebras and the derived categories by an authority of this field. Mori surveys non-commutative geometry from the view point of the derived categories, where non-commutative schemes and non-commutative algebras appear as incarnations of derived categories carrying the new concept of spaces. Polishchuk's paper treats the generalization of semi-homogeneous vector bundles, which appeared in the context of the original Fourier–Mukai transform for abelian varieties, to the Lagrangian invariant objects. Popa reviews a natural but unexpected relationship between the generic vanishing theorems in birational geometry and the perverse coherent sheaves in the context of derived categories. The short and elegant paper by Schnell proves that the fundamental group is not an invariant of the derived category though the Hodge numbers are conjecturally derived invariants. Toda surveys Donaldson–Thomas theory on the counting problem in terms of derived categories, explaining relationships to the stability conditions and Gromov–Witten invariants, and the invariance under flops. Van den Bergh treats a non-commutative analogue of a classical result that the quadrics and projective line bundles are preserved under deformations.

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