Introduction

The papers collected in this volume were all written in celebration of Bodil Branner's 60 year birthday. Most of them were presented at the 'Bodil Fest', a symposium on holomorphic dynamics held in June 2003 at the 'Søminestationen' in Holbæk, Denmark.

JOHN MILNOR gives an exhaustive survey of the so called Lattès maps, their history, their properties and significance within holomorphic dynamics in general and within Thurston theory in particular.

CARSTEN LUNDE PETERSEN and TAN LEI survey the Branner-Hubbard motion and illustrate its power by old and new examples.

MICHAEL LYUBICH and ARTHUR AVILA study the Hausdorff dimension of the Julia sets of the sequence of infinitely renormalizeable real quadratic polynomials with p-periodic combinatorics closest to the Chebychev polynomial, $p \to \infty$, using their Poincaré Series technicque.

ARNAUD CHERITAT surveys his joint work with XAVIER BUFF on quadratic Siegel disks with prescribed boundary regularity.

ROBERT DEVANEY, DANIEL M. LOOK, MONICA MORENO ROCHA, PRADIPTA SEAL, STEFAN SIEGMUND, and DAVID UMINSKY portray the family of quartic rational maps $z \mapsto z^2 + \lambda/z^2$ which exhibits many interesting properties, and they pose several questions about this family.

PASCALE ROESCH subsequently answers one of these questions affirmatively.

TOMOKI KAWAHIRA studies small perturbations of geometrically finite maps into other geometrically finite maps which are (semi)-conjugate on the Julia set to the original map.

WOLF JUNG presents his interesting thesis work on self-homeomorphisms of the Mandelbrot set. He shows among other things that the group of such homeomorphisms has the cardinality of $\mathbb{R}$.

NURIA FAGELLA and CHRISTIAN HENRIKSEN study the natural complexifications of the so-called standard maps and Arnold disks in the corresponding complexifications in parameterspace, of the irrational Arnold tongues for rotation numbers yielding Herman rings.

TAN LEI surveys and extends the results of the unpublished thesis of PIA WILLUMSEN, who was a Ph.D. student of Bodil Branner.

ADRIEN DOUADY, in the final paper, describes Branner-Hubbard motions of compact sets in the plane, poses several convergence conjectures and proves new results on semi-hyperbolic parameters in the Mandelbrot set.

Bodil Branner is a graduate of Aarhus University. In 1967 she submitted her master thesis in the area of algebraic topology. Since 1969 she has worked, first as an assistant

professor, later as associate professor of mathematics at the Department of Mathematics, Technical University of Denmark, which is situated in Lyngby, near Copenhagen.

In the summer of 1983 Branner had the fortune to be introduced to holomorphic dynamics when she met Adrien Douady and John H. Hubbard.

All three participated in the *Chaos* workshop at the Niels Bohr Institute at the University of Copenhagen, organized by Predrag Cvitanovic. The workshop poster displayed the Douady Rabbit. Branner had learned from Cvitanovic that Hubbard had generalized kneading sequences – used to classify one-dimensional real uni-modal maps such as real quadratic polynomials – to the complex setting, the Hubbard trees. At the time Branner worked on iterations of real cubic polynomials. This was initiated by Peter Leth Christiansen who in 1980 suggested, as a topic for a student's master thesis, the Nature paper of 1976 by Robert May on *Complicated Behavior of Simple Dynamical Systems*, dealing with the logistic family of real quadratic polynomials. The master student, Henrik Skjolding, made under supervision by Branner a careful numerical study of monic cubic polynomials, and afterwards Branner continued to iterate polynomials.

At the *Chaos* meeting, Branner asked Hubbard to tell her about the Hubbard trees. This became the starting point of a very fruitful collaboration. Hubbard convinced her easily that cubic polynomials were better treated when studied over the complex field. Moreover, if she wanted to shift to holomorphic dynamics there would be a great opportunity later that summer, when William Thurston would be lecturing on his groundbreaking topological characterization of rational functions at an NSF summer conference in Duluth.

Less than two months later Branner flew to the United States for the very first time and was introduced to an inspiring group of mathematicians working in dynamical systems, complex analysis, topology and differential geometry. After the conference, Branner continued with Hubbard to Cornell University. Already during the first week they became convinced that there was a wonderful structure in the cubic parameter space to be unfolded. Computer pictures generated by Homer Smith on the supercomputer at Cornell University supported this belief. As a result, Branner was invited as visiting professor to Cornell University, starting one year later.

Before then, in the summer of 1984, Douady, Hubbard and Branner spent some time at the Mittag-Leffler Institute in Sweden. Douady and Hubbard were most of the time working on the understanding of, and the filling in of details in, Thurston's proof

from the previous summer (where Douady had not been present), with Branner assigned 'guinea pig', taking notes while they were lecturing to each other, using the blackboard. They completed a joint preprint on Thurston's theorem. Several years later the manuscript was published in Acta Mathematica. Thurstons theorem is a milestone in holomorphic dynamics. It asserts the equivalence of any un-obstructed post-critically finite branched self-cover of the sphere with hyperbolic orbifold to a unique rational map up to Möbius equivalence. In this theory the so-called Lattès maps play a special role (see also Milnor's paper in this volume).

During the next one and a half year, while Branner was a visiting professor at Cornell University, Part I of [BH] on *The iteration of cubic polynomials* was finished. It describes the global topology of the parameter space $\mathbb{C}^2$ of monic, centered cubic polynomials of the form $P_{a,b}(z) = z^3 - a^2 z + b$. Several decompositions of the parameter space are considered. The first splitting is to separate the connectedness locus, where both critical points have bounded orbit and the Julia set therefore is connected, from the escape locus, where at least one critical point escapes to infinity. The second splitting is to foliate the escape locus into different hyper-surfaces, each one corresponding to a fixed maximal escape rate of the critical points. A particular way of constructing Teichmüller almost complex structures, invariant under $P_{a,b}$, were introduced as *wringing and stretching* the complex structure. This technique is now referred to as Branner-Hubbard motion (see also the three papers by Petersen & Tan, Tan and Douady in this volume). The wring and stretch operation is continuous on the cubic escape locus (it is not continuous on the entire cubic locus, see Tan in this volume). It follows that each

hyper-surface of fixed escape rate is homeomorphic to the three-dimensional sphere, and that the connectedness locus is cell-like: an infinite intersection of a nested sequence of closed topological three-dimensional balls. The third splitting is within each hyper-surface. Measuring the argument of the faster (first) escaping critical value (a choice of faster critical value if both escape at a common rate) one obtains a fibration over the circle with fibers a trefoil clover leaf. The final splitting in each clover leaf is governed by the behavior of the second critical point, which may escape or not. The structure of the set of parameters for which the second critical point does not escape can be described combinatorially. It includes infinitely many copies of the Mandelbrot set each with its own combinatorics. These stem from quadratic-like restrictions of (iterates of) cubic polynomials. In fact, the starting point of it all had been the understanding of this combinatorial structure.

In January 1986 Branner returned to Denmark. In Spring she learned from Douady that Yoccoz had observed that the combinatorial structure in a particular clover leaf over the zero-argument was similar to the combinatorial structure in the $\frac{1}{2}$-limb of the Mandelbrot set. How could this be justified? The comparison of quadratic polynomials with connected Julia set to cubic polynomials with disconnected Julia set was not obvious. However, moving along a stretching ray through such a cubic polynomial towards the connectedness locus the escape rate of the critical point decreases and in the limit the critical point is mapped onto the fixed point, which is the landing point of the zero-ray. In the cubic one-parameter family of polynomials with one critical point being pre-fixed, there is a limb corresponding to the fixed point being the landing point of the zero-ray. The correspondence between the post-critically finite quadratic and cubic polynomials can be understood in terms of Hubbard trees. It is much harder to establish a homeomorphism between the two relevant limbs.

Douady was at the time trying to embed the $\frac{1}{2}$-limb of the Mandelbrot set into the $\frac{1}{3}$-limb. He was motivated by the Corollary that would follow: The main vein in the $\frac{1}{3}$-limb is a topological arc, being the image of the line segment of the real axis included in the $\frac{1}{2}$-limb. It turned out that it was easier to work out the surgery technique that was needed to obtain the homeomorphisms between the relevant sets of quadratic and cubic polynomials than the one between the different sets of quadratic polynomials. Therefore, the surgery technique was developed in that order, and resulted in a joint paper [BD] *Surgery on Complex Polynomials* in Proceedings of Symposium on Dynamical Systems, Mexico 1986, with Theorem A relating the quadratic polynomials via a homeomorphism, and Theorem B relating the quadratic and cubic polynomials via a homeomorphism.

Part II of [BH] describing patterns and para-patterns was finished in the spring of 1988 during the special semester on dynamical systems at the Max-Planck-Institute für Mathematik in Bonn. The main inventions were on one hand the tableaus, combinatorial schemes, which catch enough information in the dynamical plane in order to estimate the modulus of annuli between consecutive critical levels of Green's function; and on the other hand the use of Grötzsch' inequality on moduli of annuli together with the result that for an open, bounded annulus of infinite modulus the bounded component of the complement is just one point. Hence, if for an infinite sequence of disjoint open annuli A_n, embedded in an open bounded annulus A, the infinite series of moduli of the A_n is divergent, then the annulus A has infinite modulus and surrounds exactly one point. This method of proving components to be point components has been called the divergence method. An infinite tree of patterns captures the structure in the dynamical planes. Out of these one builds para-patterns, which correspond to the structure in the clover leaves. The ends are either point components, as proved by the divergence method, or copies of the Mandelbrot set, corresponding to quadratic-like families.

The divergence method was later extended by Yoccoz to be applied to prove local connectivity of Julia sets of non-renormalizeable quadratic polynomials $z^2 + c$ with c in the Mandelbrot set and both fixed points repelling. He also proved local connectivity of the Mandelbrot set at the corresponding c-values. The complications in the quadratic setting is much more profound than in the cubic case, in particular the estimates in the parameter plane. Yoccoz called the division in the dynamical planes puzzles (instead of patterns) and the one in the parameter space para-puzzles (instead of para-patterns).

In the summer of 1993 Branner and one of the editors (PGH) organized a NATO advanced study institute in Hillerød, entitled *Real and Complex Dynamics*. For two weeks more than 100 participants, of which about two thirds were Ph.D.-students and Post. Docs., stayed together in ideal surroundings listening to lectures of 15 main speakers, combined with numerous talks by other participants and lots of informal discussions.

Much collaboration grew out of this summer school, and these young mathematicians are now leading the new developments. Branner began to work together with Nuria Fagella, further developing the surgery technique with the aim to prove certain symmetries in the Mandelbrot set, through comparisons with higher degree polynomials. Their first joint paper *Homeomorphisms between limbs of the Mandelbrot set* was finished in 1995 during the special semester on Conformal Dynamics at MSRI in Berkeley, arranged by Curt McMullen (see also the paper by Jung in this volume.)

10 years after the Hillerød meeting, in 2003, a large number of eminent researchers, colleagues young and old, gathered at the Søminestationen, a former Danish navy training site, now a peaceful conference centre picturesquely located by one of the many quiet fjords of the Danish coast. It was mid-Summer in Scandinavia. The symposium became a celebration of an exciting active area of mathematics, of warm and long lasting international friendships, and, not least, of the wonderful life and inspiring scholarship of Bodil Branner.

PGH & CLP