Preface

In differential geometry one studies local and global properties of smooth manifolds M equipped with a metric tensor g – that is, a smooth field of symmetric bilinear forms of fixed signature (p, q) on the tangent spaces of M – which encodes the geometry. If the metric tensor g is positive definite, the pair (M, g) is called a *Riemannian* manifold. If g is indefinite, (M, g) is referred to as a *pseudo-Riemannian* manifold. The difference in the signature of the metric g has essential consequences for the geometric structures as well as for the methods of their investigation.

In *Riemannian* geometry, important progress has been made over the past thirty years in understanding the relations between the local and global structure of Riemannian manifolds. Many classification results for different classes of Riemannian manifolds were obtained: manifolds with additional geometric structures, manifolds satisfying curvature conditions, symmetric and homogeneous Riemannian spaces, etc. Similar results for *pseudo-Riemannian manifolds* are rare, and many problems are still open. For a long time, the main source of problems in pseudo-Riemannian geometry was general relativity, which deals with 4-dimensional Lorentzian manifolds (space-times) where the signature of the metric is $(1, 3)$. However, the developments in theoretical physics (supergravity, string theory) require a deeper understanding of the geometric structure of higher dimensional manifolds with indefinite metrics of Lorentzian and other signatures. Moreover, pseudo-Riemannian metrics naturally appear in different geometric problems, e.g. in CR geometry or on moduli spaces of geometric structures. Sometimes, one can use a special Ansatz or "Wick-rotations" to transform problems of pseudo-Riemannian geometry into questions of Riemannian geometry. But in many aspects, pseudo-Riemannian and Riemannian geometry differ *essentially*, and many specific, highly nontrivial and interesting new questions appear in the pseudo-Riemannian setting. There has been substantial progress over the past few years in solving some of these problems.

In order to stimulate cooperation between different groups of researchers working in this field, we organized a scientific programme *Geometry of Pseudo-Riemannian Manifolds with Applications in Physics*, which was held at the Erwin Schrödinger International Institute for Mathematical Physics (ESI) in Vienna between September and December of 2005. In the course of this programme, the idea of this volume was born. It aims to introduce a broader circle of mathematicians and physicists to recent developments of pseudo-Riemannian geometry, in particular to those developments which were discussed during the ESI Special Research Semester.

We now briefly sketch the contents of this book. A basic problem of differential geometry, which is completely solved for Riemannian manifolds, but becomes quite complicated in the pseudo-Riemannian setting, is to determine all possible holonomy groups of pseudo-Riemannian manifolds. Contrary to the Riemannian case, the holonomy representation of an indefinite metric must not decompose into irreducible representations. In the indefinite case, additional holonomy representations occur, which have an isotropic holonomy invariant subspace without an invariant complement. Such representations are difficult to classify.

The first two contributions to this book describe recent progress concerning this question. The article of Ines Kath and Martin Olbrich deals with pseudo-Riemannian *symmetric* spaces. The classification of symmetric spaces with completely reducible holonomy representation, i.e., any invariant subspace has an invariant complement, was established long ago. In this case the transvection group is semi-simple. For symmetric spaces with non-reductive holonomy representation, the transvection group is more complicated. It has a proper Levi decomposition equipped with a biinvariant metric. Kath and Olbrich give a survey of new approaches to the classification of this type of pseudo-Riemannian symmetric spaces and explain applications to the classification of pseudo-hermitian, quaternionic Kähler and hyper-Kähler symmetric spaces. Furthermore, they describe the complete classification of Lorentzian symmetric spaces and of symmetric spaces with metrics of index 2.

The article of Anton Galaev and Thomas Leistner is about the classification of holonomy groups of *Lorentzian* manifolds, which was completed only recently. They describe the list of all possible Lorentzian holonomy groups, outline the proof of this result, and explain a method to construct local metrics for all possible holonomy groups. Furthermore, they give a brief outlook on the classification problem for metrics with higher signature.

Besides Lorentzian manifolds, which are basic for general relativity, pseudo-Riemannian manifolds of split signature (m, m) are of special interest. The article of Andrew Dancer and Andrew Swann, and that of Maciej Dunajski and Simon West, deal with this class of pseudo-Riemannian manifolds.

Dancer and Swann give a survey of so-called hypersymplectic manifolds, which were introduced by Hitchin as a cousin of hyper-Kähler manifolds. They are based on the algebra of split quaternions rather than the usual quaternions. Hypersymplectic manifolds are also Ricci-flat and Kähler, but of split signature $(2n, 2n)$. The article describes construction methods for hypersymplectic manifolds using ideas from symplectic and toric geometry.

Dunajski and West cover the special case of dimension 4. More generally, they survey the geometry of 4-dimensional anti-selfdual manifolds of signature $(2, 2)$, local and global construction methods for these, and relations to integrable systems.

The article of Brendan Guilfoyle and Wilhelm Klingenberg links special geometry in signature $(2, 2)$ to classical surface theory and geometric optics. Here a neutral Kähler metric appears on the space $\mathbb{L}$ of oriented lines in Euclidean 3-space. The fundamental objects in geometric optics are 2-parameter families of oriented lines, hence immersed surfaces in the pseudo-Kähler space $\mathbb{L}$.

These surveys of the geometry of manifolds with split signature are followed by a group of articles dealing with special properties of conformal transformations in the pseudo-Riemannian setting. Some of the main differences between Riemannian and pseudo-Riemannian conformal geometry are already manifested in the flat model of conformal Lorentzian geometry – the conformal compactification of Minkowski space, or equivalently, the ideal boundary of anti-de Sitter space – often referred to as the Einstein Universe in the physics literature.

The article by Thierry Barbot, Virginie Charette, Todd Drumm, William M. Goldman, and Karin Melnick gives a detailed introduction to the geometric and causal structure of the Einstein Universe. In particular, the article analyzes the 3-dimensional case, where the group of conformal transformations is locally isomorphic to the group of linear symplectomorphisms of $\mathbb{R}^4$. The authors explain the dynamics of actions of discrete subgroups of the symplectic (conformal) group on the 3-dimensional Einstein Universe. They also describe the actions of discrete subgroups of Lorentzian transformations, which act freely and properly on the 3-dimensional Minkowski space, and describe the construction of complete flat Lorentzian manifolds.

The article of Charles Frances considers essential conformal structures. A group G of conformal transformations of a manifold (M, g) is called essential if no metric in the conformal class of g is preserved by G. It is well known that the conformal group of a Riemannian manifold (M, g) of dimension $n \geq 3$ is essential if and only if (M, g) is conformally diffeomorphic to $\mathbb{S}^n$ or $\mathbb{R}^n$ with the canonical flat metric. In the pseudo-Riemannian case the situation is quite different and much more complicated, as Frances' contribution illustrates. Using the special dynamics of discrete subgroups of the conformal group acting on the Einstein Universe, he constructs a large class of conformally flat, compact Lorentzian manifolds with non-equivalent essential conformal structures.

Wolfgang Kühnel and Hans-Bert Rademacher study other aspects of pseudo-Riemannian conformal geometry. In their article they elucidate pseudo-Riemannian manifolds with essential infinitesimal conformal transformations (conformal Killing fields), in particular gradient fields. Furthermore, they study manifolds which are conformally equivalent to Einstein spaces, describe the conformal group of plane wave metrics which occur as Penrose limits of arbitrary space-times, and finally, discuss manifolds with conformal Killing spinors, which induce special kinds of infinitesimal conformal transformations.

The article of Ettore Minguzzi and Miguel Sanchez, and that of Anna Maria Candela and Miguel Sanchez, give an introduction to causality theory of Lorentzian manifolds and properties of geodesics in pseudo-Riemannian geometry. The former article describes the different causality notions for space-times and their relations to each other, from non-totally vicious to global hyperbolic space-times. In particular, recent results on the existence of smooth Cauchy surfaces and smooth time-functions of globally hyperbolic manifolds are discussed. One of the main differences between Riemannian and pseudo-Riemannian manifolds is in the behavior of geodesics. Whereas compact Riemannian manifolds are always geodesically complete, and geodesically complete Riemannian manifolds are always geodesically connected, both properties fail for indefinite metrics.

In the article of Candela and Sanchez these differences are illustrated by means of special space-times. Moreover, conditions and properties for geodesically complete manifolds and for incompleteness (Singularity Theorems) are discussed. Furthermore, the authors give an introduction to and survey of variational approaches for studying geodesic connectedness for stationary and orthogonal splitting space-times.

The volume concludes with three articles which describe applications of methods and results from pseudo-Riemanian geometry to mathematical physics.

Jose Figueroa-O'Farrill shows how Lorentzian symmetric spaces arise as supersymmetric supergravity backgrounds. His article is devoted to the local classification of supergravity theories in dimension 11, 10, 6 and 5. Lorentzian symmetric spaces appear as relevant geometries for the so-called maximal supersymmetric backgrounds, and play a role for the determination of parallelisable supergravity backgrounds in type II supergravity.

Applications of methods from the geometry of split signature metrics can be seen in the article of Frederik Witt. He studies the geometry of type II supergravity compactifications in terms of an oriented vector bundle E endowed with a bundle metric of split signature, which is associated with a so-called generalized G-structure, introduced by Hitchin. In particular, integrable generalized G-structures are considered.

Finally, the article of Gaetano Vilasi is about Einstein metrics with 2-dimensional Killing leaves and their physical interpretations. Here, solutions of the vacuum Einstein field equations for metrics with a non-abelian 2-dimensional Lie algebra of Killing fields are explicitly described.

Although pseudo-Riemannian geometry has experienced a rapid development in recent years and essential results were obtained, many fundamental questions are still open. We hope that this volume will stimulate interest for studying and solving geometric problems of pseudo-Riemannian geometry, which arise naturally in differential geometry and mathematical physics.

We would like to thank the Erwin Schrödinger Institute in Vienna for the opportunity to organize the special programme *Geometry of Pseudo-Riemannian Manifolds with Applications in Physics* at ESI, for the financial and the organizational support, and for the stimulating scientific atmosphere, without which this research programme would not have been so successful.

Moreover, we would like to thank Manfred Karbe and Irene Zimmermann from the EMS Publishing House for pleasant cooperation during the preparation of the volume.

Berlin, May 2008 Dmitri Alekseevski and Helga Baum