Introduction

Since its inception 50 years ago, K-theory has been a tool for understanding a wide-ranging family of mathematical structures and their invariants: topological spaces, rings, algebraic varieties and operator algebras are the dominant examples. The invariants range from characteristic classes in cohomology, through determinants of matrices, to Chow groups of varieties, as well as to traces and indices of elliptic operators. Thus K-theory is notable for its connections with other branches of mathematics.

Noncommutative geometry, on the other hand, develops tools which allow one to think of noncommutative algebras in the same footing as commutative ones; as algebras of functions on (noncommutative) spaces. The algebras in question come from problems in various areas of mathematics and mathematical physics; typical examples include algebras of pseudodifferential operators, group algebras, and other algebras arising from quantum field theory.

To study noncommutative geometric problems one considers invariants of the relevant noncommutative algebras. These invariants include algebraic and topological K-theory, and also cyclic homology, discovered independently by Alain Connes and Boris Tsygan, which can be regarded both as a noncommutative version of de Rham cohomology, and as an additive version of K-theory. There are primary and secondary Chern characters which pass from K-theory to cyclic homology. These characters are relevant both to noncommutative and commutative problems, and have applications ranging from index theorems to the detection of singularities of commutative algebraic varieties.

The contributions to this volume represent this range of connections between K-theory, noncommmutative geometry, and other branches of mathematics.

An important connection between K-theory, topology, geometric group theory and noncommutative geometry is given by the isomorphism conjectures, such as those due to Baum–Connes, Bost–Connes and Farrell–Jones, which predict certain topological formulas for various kinds of K-theory of a crossed product, in both the C^*-algebraic and the purely algebraic contexts. These problems have received a lot of attention over the past twenty years. Not surprisingly, then, three of the articles in this volume discuss these problems. The first of these, by R. Meyer, is a survey on bivariant Kasparov theory and E-theory. Both bivariant K-theories are approached via their universal properties and equipped with extra structure such as a tensor product and a triangulated category structure. The construction of the Baum–Connes assembly map via localisation of categories is reviewed and the relation with the purely topological construction by Davis and Lück is explained. In the second article, A. Bartels, S. Echterhoff and W. Lück investigate when Isomorphism Conjectures, such as the ones due to Baum–Connes, Bost and Farrell–Jones, are stable under colimits of groups over directed sets (with not necessarily injective structure maps). They show in particular that both the K-theoretic Farrell–Jones Conjecture and the Bost Conjecture with coefficients hold for those groups for which Higson, Lafforgue and Skandalis have disproved the Baum–Connes Conjecture with coefficients. H. Emerson and R. Meyer study in their contribution an equivariant co-assembly map that is dual to the usual Baum–Connes assembly map and closely related to coarse geometry, equivariant Kasparov theory, and the existence of dual Dirac morphisms. As applications, they prove the existence of dual Dirac morphisms for groups with suitable compactifications, that is, satisfying the Carlsson–Pedersen condition, and study a K-theoretic counterpart to the proper Lipschitz cohomology of Connes, Gromov and Moscovici.

Many applications of K-theory to geometric topology come from the K-theory of Waldhausen categories. The article by F. Muro and A. Tonks is about the K-theory of a Waldhausen category. They give a simple representation of all elements in K_1 of such a category and prove relations between these representatives which hold in K_1.

Another connection between K-theory, topology and analysis comes from twisted K-theory, which has received renewed attention in the last decade, in the light of new developments inspired by Mathematical Physics. The next article is a survey on this subject, written by M. Karoubi, one of its founders. The author also proves some new results in the subject: a Thom isomorphism, explicit computations in the equivariant case and new cohomology operations.

As mentioned above, one of the most interesting points of contact between K-theory and noncommutative geometry comes through cyclic homology. C. Voigt's contribution is about this theory. He defines equivariant periodic cyclic homology for bornological quantum groups. Generalizing corresponding results from the group case, he shows that the theory is homotopy invariant, stable, and satisfies excision in both variables. Along the way he proves Radford's formula for the antipode of a bornological quantum group. Moreover he discusses anti-Yetter–Drinfeld modules and establishes an analogue of the Takesaki–Takai duality theorem in the setting of bornological quantum groups.

A central theme in noncommutative geometry is index theory. A basic construction in this area consists of associating a C^*-algebra to a geometric problem, and then compute an index using K-theory. P. Carrillo Rouse's article deals with the problem of index theory on singular spaces, and in particular with the construction of algebras and indices between the enveloping C^*-algebra and the convolution algebra of compactly supported functions.

The article by J. Cuntz is also about C^*-algebras. It presents a C^*-algebra which is naturally associated to the $ax + b$-semigroup over $\mathbb{N}$. It is simple and purely infinite and can be obtained from the algebra considered by Bost and Connes by adding one unitary generator which corresponds to addition. Its stabilization can be described as a crossed product of the algebra of continuous functions, vanishing at infinity, on the space of finite adeles for $\mathbb{Q}$ by the natural action of the $ax + b$-group over $\mathbb{Q}$.

W. Werner applies C^*-algebraic methods in infinite dimensional differential geometry in his contribution. It deals with a special class of 'non-compact' hermitian infinite dimensional symmetric spaces, generically denoted by U. The author calculates their invariant connection very explicitly and uses the concept of a Hilbert C*-manifold so that the Banach manifold in question is of the form $\operatorname{Aut} U/H$, where $\operatorname{Aut} U$ is the automorphism group of the Hilbert C*-manifold. Using results previously obtained with D. Blecher, he characterizes causal structure on U that comes from interpreting the elements of U as bounded Hilbert space operators.

The subject of the next article, by U. Bunke, T. Schick, M. Spitzweck and A. Thom, connected with group theory, topology, as well as to mathematical physics and noncommutative geometry, is Pontrjagin duality. The authors extend Pontrjagin duality from topological abelian groups to certain locally compact group stacks. To this end they develop a sheaf theory on the big site of topological spaces $\mathbf{S}$ in order to prove that the sheaves $\underline{\mathsf{Ext}}^i_{\mathsf{ShAb}\mathbf{S}}(\underline{G}, \underline{\mathbb{T}})$, $i = 1, 2$, vanish, where $\underline{G}$ is the sheaf represented by a locally compact abelian group and $\mathbb{T}$ is the circle. As an application of the theory they interpret topological T-duality of principal $\mathbb{T}^n$-bundles in terms of Pontrjagin duality of abelian group stacks.

An important source of examples and problems in noncommutative geometry comes from deformations of commutative structures. In this spirit, the article by P. Bressler, A. Gorokhovsky, R. Nest and B. Tsygan investigates the 2-groupoid of deformations of a gerbe on a C^∞ manifold. They identify the latter with the Deligne 2-groupoid of a corresponding twist of the differential graded Lie algebra of local Hochschild cochains on C^∞ functions.

The next article, by G. Garkusha and M. Prest, provides a bridge between the noncommutative and the commutative worlds. A common approach in noncommutative geometry is to study abelian or triangulated categories and to think of them as the replacement of an underlying scheme. This idea goes back to work of Grothendieck and Gabriel and continues to be of interest. The approach is justified by the fact that a (commutative noetherian) scheme can be reconstructed from the abelian category of quasi-coherent sheaves (Gabriel) or from the category of perfect complexes (Balmer). A natural question to ask, is whether the hypothesis on the scheme to be noetherian is really necessary. This is precisely the theme of recent work of Garkusha and Prest. They show how to deal with affine schemes, without imposing any finiteness assumptions. More specifically, the article in this volume discusses for module categories over commutative rings the classification of torsion classes of finite type. For instance, the prime ideal spectrum can be reconstructed from this classification.

The final two contributions are about K-theory and (commutative) algebraic geometry.

T. Geisser's article is about Parshin's conjecture. The latter states that $K_i(X)_{\mathbb{Q}} = 0$ for $i > 0$ and X smooth and projective over a finite field $\mathbb{F}_q$. The purpose of this article is to break up Parshin's conjecture into several independent statements, in the hope that each of them is easier to attack individually. If $CH_n(X, i)$ is Bloch's higher Chow group of cycles of relative dimension n, then in view of $K_i(X)_{\mathbb{Q}} \cong \bigoplus_n CH_n(X, i)_{\mathbb{Q}}$, Parshin's conjecture is equivalent to Conjecture $P(n)$ for all n, stating that $CH_n(X, i)_{\mathbb{Q}} = 0$ for $i > 0$, and all smooth and projective X. Assuming resolution of singularities, the author shows that Conjecture $P(n)$ is equivalent to the conjunction of three conjectures $A(n)$, $B(n)$ and $C(n)$, and gives several equivalent versions of these conjectures.

Finally, C. Weibel's article gives an axiomatic framework for proving that the norm residue map is an isomorphism (i.e., for settling the motivic Bloch–Kato conjecture).

This framework is a part of the Voevodsky–Rost program.

In conclusion, this volume presents a good sample of the wide range of aspects of current research in K-theory, noncommutative geometry and their applications.

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