Foreword

The CIRM meeting revolved around some of the interactions of four very active topics with origin in pure mathematics and in physics:

- Renormalization in Quantum Field Theory
- Differential Galois theory
- Noncommutative geometry
- Motives and Galois theory

The articles that form this volume all illustrate the richness of these interactions. The recent years have witnessed the emergence of a much better mathematical understanding of perturbative renormalization both in its Galois aspects related to the ambiguity inherent in the renormalization group and the role of the Birkhoff decomposition, as well as in the deep arithmetic nature of the numbers that appear as residues of Feynman graphs in the renormalization process. The Birkhoff decomposition plays a crucial role, together with the Hopf algebra of Feynman graphs, in the conceptual understanding of the perturbative renormalization process. This result of our collaboration with D. Kreimer is recalled, and largely extended to regularization processes more general than dimensional regularization, in the article of Kurusch Ebrahimi-Fard and Dominique Manchon.

There is a striking similarity between the ambiguity group occurring in physics, and its underpinning as the "cosmic Galois group" in joint work with M. Marcolli, and the ambiguities that occur in the resummation processes of divergent series (Stokes phenomenon, resurgence, etc.). Frederic Menous has developed a mechanism of Birkhoff decomposition to treat conjugacy problems that arise in the study of solutions of differential equations (analytic or formal). For some equations, the obstacles in the formal conjugacy are reflected in the fact that the associated characters of a Hopf algebra (here a shuffle Hopf algebra) appear to be ill-defined. The analogy with the need for a renormalization scheme (dimensional regularization, Birkhoff decomposition) in Quantum Field Theory becomes obvious for such equations and delivers a wide range of toy models.

The article of David Sauzin is a careful account of the mould calculus of Écalle, a powerful combinatorial tool which yields surprisingly explicit formulas for the series attached to an analytic germ of singular vector field or of map. His article for this volume illustrates the method for the case of the saddle-node of a two-dimensional vector field.

The work of Kreimer and Broadhurst opened up a new area of interactions between the highly involved computations of Quantum Field Theory, where numbers such as $\zeta(3)$ naturally occur as ingredients of computations of physical quantities, and deep aspects of arithmetic, such as the study of periods, in the theory of motives.

Stefan Weinzierl reviews in his contribution the connections between Feynman integrals and multiple polylogarithms. He gives a thorough introduction to loop integrals, the Mellin–Barnes transformation, shuffle algebras and multiple polylogarithms. Finally, he discusses how certain Feynman integrals evaluate to multiple polylogarithms.

In his article, Michael Hoffman copes with various Hopf algebras which have their origin in renormalization; he shows in particular how combinatorial Dyson–Schwinger equations can profitably be solved by lifting them from a commutative algebra to a non-commutative one.

The aim of the enticing text of Yves André is to indicate what Grothendieck's theory of motives has to say, at least conjecturally, on the question of an analogue of Galois theory for transcendental numbers, and to promote the idea that periods should have well-defined conjugates and a Galois group that permutes them transitively.

The article of Caterina Consani starts by an enlightening historical survey of the theory of pure motives in algebraic geometry and then reviews some of the recent developments of this theory in noncommutative geometry, namely the new theory of endomotives, which was developed in joint work with C. Consani and M. Marcolli. It shows how a natural extension of the simplest category of motives, namely the Artin motives, to the larger category of algebraic endomotives quickly yields, by using cyclic cohomology and the thermodynamics of noncommutative spaces, the spectral realization of the zeros of the Riemann zeta function.

Finally, the article of Vincent Rivasseau and Fabien Vignes-Tourneret is a review of the fast growing subject of renormalization of Quantum Field Theory on noncommutative spaces. The Grosse–Wulkenhaar breakthrough opened up a new era by realizing that the right propagator in noncommutative field theory is not the ordinary commutative propagator, but it has to be modified to obey Langmann–Szabo duality. Grosse and Wulkenhaar were able to compute the corresponding propagator in the so-called "matrix base" which transforms the Moyal product into a matrix product, and to use this representation to prove perturbative renormalisability. V. Rivasseau, in joint work with R. Gurau, J. Magnen and F. Vignes-Tourneret, using direct space methods, provided recently a new proof that the Grosse–Wulkenhaar scalar Φ^4-theory on the Moyal space $\mathbb{R}^4$ is renormalisable to all orders in perturbation theory. Their review is remarkably clear and accessible.

In all of the mentioned subjects there is now a flurry of activity and the present book is a perfect introduction to these very lively topics of research, relevant both in physics and pure mathematics.

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