Introduction

This volume is a collection of expository articles on mathematical topics of current interest that have been partly stimulated or influenced by interactions with physical theory. Some of these articles are based on lectures given at various venues over the last few years, revised and written especially for this volume. They cover a diverse range of mathematical topics stemming from different parts of physics.

The major underlying theme of the volume involves the interactions between physics and number theory. This theme is manifested in two ways, first through the study of Feynman integrals and renormalisation theory, and second, through the application of methods from quantum statistical mechanics. In the former, the work of Bogoliubov–Parasuik–Hepp–Zimmermann (BPHZ) on renormalisation theory gave a method for step by step control of divergences and of their regularisation in Feynman's approach to perturbative quantum field theory. Already, in the evaluation of Feynman integrals, the occurrence of multizeta values hinted at deeper mathematical connections. This deeper underlying structure was found by Alain Connes and Dirk Kreimer in the form of associating to each renormalisable quantum field theory a Hopf algebra that provided a systematic understanding of the BPHZ procedure in terms of a Birkhoff factorisation in a Lie group associated to the Hopf algebra. A contemporary view of this fundamental work is provided here by Christoph Bergbauer. The latter strand, the relationship between statistical mechanics and number theory, began much earlier through the work of Jean-Benoît Bost and Alain Connes. Bost–Connes introduced a quantum statistical mechanical dynamical system that captures information on the primes and on the Riemann zeta function. They determined the equilibrium states of their model (the so-called Kubo–Martin–Schwinger (KMS) states) and its phase transitions. Subsequent extensions of this basic idea to number theoretic questions resulted in the theory of 'endomotives'. Motives also arise in the study of Feynman integrals and hence we have provided here an introduction to these ideas in the Ramdorai–Plazas–Marcolli article.

We now give a brief overview of the contents of this volume.

Bergbauer's article discusses Feynman integrals, regularization and renormalization following the algebraic approach to the Feynman rules developed by Bloch, Connes, Esnault, Kreimer, and others. It reviews several renormalization methods found in the literature from a single point of view using resolution of singularities, and includes a discussion of the motivic nature of Feynman integrals.

Motives are explained in much greater detail in the article of Sujatha Ramdorai and Jorge Plazas. The construction of the category of pure motives is explained here starting from the category of smooth projective varieties. They also survey the theory of endomotives developed by D. C. Cisinski and G. Tabuada, which links the theory of motives to the quantum statistical mechanical techniques that connect number theory and noncommutative geometry. The appendix to this article, contributed by Matilde Marcolli, elaborates these latter ideas providing a useful introduction to, and summary of, the role of KMS states. Also described is the view of motives that arises from noncommutative geometry along with a detailed account of the interweaving of number theory with statistical mechanics, noncommutative geometry and endomotives.

Hopf algebras associated to rooted trees are known to systemetise the combinatorics of Feynman graphs through the work of Connes–Kreimer. Dominique Manchon describes another algebraic object associated to rooted trees, namely pre-Lie algebras. His article reviews the basic theory of pre-Lie algebras and also describes how they arise from operads. Their relation to other algebraic structures and their application to numerical analysis are described as well.

Multiple zeta values arise from the evaluation of Feynman integrals. Sylvie Paycha's contribution discusses generalisations of renormalised multiple zeta values at nonpositive integers. As with some of the other articles, the exposition is partly inspired by renormalised Feynman integrals in physics using pseudodifferential symbols.

Zeta residues and pseudodifferential analysis, both of which play a role in earlier articles, also arise in index theory. In turn, index theory is well known to play a role in gauge field theories via the study of anomalies. These are described mathematically by the families index theorem. Under the influence of Alain Connes and others, a noncommutative approach to index theory, partly inspired by quantum theory, has emerged over the last two decades. This noncommutative index theory is described in the contribution of Alan Carey, John Phillips and Adam Rennie, beginning with a discussion of classical index theorems from a noncommutative point of view. This is followed by a review of K-theory and cyclic cohomology that culminates in a description of the local index formula in noncommutative geometry. The article concludes with an example that underlies recent applications of noncommutative geometry to Mumford curves.

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