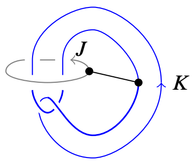
Preface

In this book, we explain how to count graph configurations to obtain invariants for 3-manifolds and knots in these 3-manifolds, and we investigate the properties of the obtained invariants.

The simplest of these invariants is the linking number of two disjoint knots in the ambient space $\mathbb{R}^3$. Gauss defined it in 1833 [33]. As we review in Section 1.2, this linking number counts configurations



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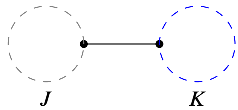

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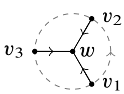
as the degree of an associated Gauss map.

Many mysterious knot invariants called “quantum invariants” were introduced in the mid-80s, starting with the Jones polynomial. Witten explained how to obtain many of them from the perturbative expansion of the Chern–Simons theory in a seminal article [121]. This physicist viewpoint led Guadagnini, Martellini, Mintchev [37] and Bar-Natan [8] to show in what sense a coefficient w_2 of the Jones polynomial counts configurations of the graphs



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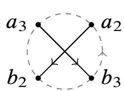

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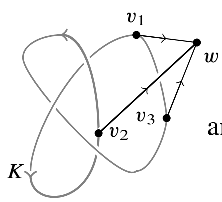

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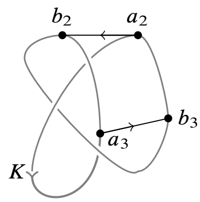

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The theory of Vassiliev invariants reviewed in Chapter 6 associates a degree in $(\mathbb{N} \cup \{\infty\})$ to a numerical knot invariant. The only knot invariants of degree 0 are the constant functions. The knot invariants of degree 2 are linear combinations of w_2 and the constant function that maps every knot to 1. The Jones polynomial can be renormalized into a series whose coefficients are finite-degree knot invariants. Altschüler and Freidel showed that every degree n real-valued knot invariant may be obtained by “counting” configurations of graphs with at most $2n$ vertices as explained in this book [3]. The knot invariants counting graph configurations mentioned above are assembled in a *universal Vassiliev invariant* $\mathcal{Z}(S^3, \cdot)$ valued in a product of vector spaces generated by some unitrivalent graphs called *Jacobi diagrams*. Kontsevich had constructed another universal Vassiliev invariant with similar properties called *the Kontsevich integral* [7,48]. The Kontsevich integral Z_K may be defined combinatorially from planar knot diagrams. It has been extensively studied. To my knowledge, the coincidence of the spatial invariant $\mathcal{Z}(S^3, \cdot)$ with Z_K is an open problem.

Developing the Witten approach further, Kontsevich outlined a way to count trivalent graphs in more general 3-manifolds and define a topological graded invariant $\mathcal{Z}$ for them [49]. These more general manifolds are the 3-dimensional $\mathbb{Q}$-spheres, simply called $\mathbb{Q}$-*spheres* in this book. They are the closed 3-manifolds with the same rational homology as the standard 3-dimensional sphere S^3. They include the 3-manifolds with the same $\mathbb{Z}$-homology as S^3 called $\mathbb{Z}$-*spheres*. A $\mathbb{Z}$-sphere is a closed orientable three-manifold in which knots bound orientable compact surfaces. The degree one part $\mathcal{Z}_1$ of the Kontsevich invariant $\mathcal{Z}$ of $\mathbb{Q}$-spheres is determined by a real-valued invariant Θ of $\mathbb{Q}$-spheres, which counts configurations of the graph $\ominus$ in the manifold.

Ohtsuki, Habiro, Goussarov, and others developed theories of finite type invariants of $\mathbb{Z}$-spheres analogous to the Vassiliev theory for knots in $\mathbb{R}^3$ [31, 87]. Kuperberg and Thurston showed why the Kontsevich invariant $\mathcal{Z}$ of $\mathbb{Z}$-spheres obtained by counting graphs in these manifolds is also *universal* with respect to the above theories [54]. This universality result implies that any real-valued invariant of $\mathbb{Z}$-spheres of degree $2n$ (with respect to one of the equivalent developed theories) is a combination of invariants counting configurations of graphs with at most $2n$ vertices.

In 1985, Casson had defined an invariant of $\mathbb{Z}$-*spheres*. The Casson invariant "counts" conjugacy classes of irreducible SU(2)-representations of the $\mathbb{Z}$-spheres fundamental groups. Their universality result allowed Kuperberg and Thurston to show that Θ and the Casson invariant are proportional. In particular, the Casson invariant also "counts" configurations of the graph $\ominus$. For a $\mathbb{Z}$-sphere R equipped with a basepoint ∞, the configurations are counted in a suitable compactification $C_2(R)$ of the space of ordered pairs of distinct points in the punctured manifold ($\check{R} = R \setminus \{\infty\}$). The set of counted configurations is the intersection in $C_2(R)$ of three transverse codimension-2 submanifolds called *propagating chains*, and $\Theta(R)$ is their algebraic intersection number. Dually, the invariant $\Theta(R)$ is the integral over $C_2(R)$ of the cube of a *propagating* closed 2-form. Propagating chains and propagating forms both represent the linking form on R. We call them *propagators*. They are the main ingredient used to count graph configurations in this book. They are associated with the graph edges. They are precisely defined in Chapter 3. When R is $\mathbb{Z}$-sphere, results of Pontrjagin and Rohlin in the 1950s [94] imply that the punctured $\check{R}$ can be equipped with a preferred homotopy class of parallelizations. For a general $\mathbb{Q}$-sphere, the invariant Θ is first introduced as an invariant of a pair (R, τ), where τ is a parallelization of $\check{R}$. It is next corrected with the help of a relative first Pontrjagin class to become an invariant of R. Chapter 4 contains the complete construction of Θ, and Chapter 5 establishes the needed properties of Pontrjagin classes.

Kuperberg and I associated explicit propagating chains to Morse functions and associated Morse flows. These propagators reviewed in Section 1.2.8 allowed me to express the Theta invariant in terms of Heegaard diagrams [69]. With this type of propagator, the "counted" graph configurations either map an edge of the graph into a flow line, or map the edge ends into descending manifolds or ascending manifolds of critical points of the Morse function. Fukaya proposed such a way of counting graphs [29]. Many authors, including Watanabe and Shimizu, further studied it.

In the book's second part, we define and study an invariant $\mathcal{Z}(R, L)$ for a link L in a $\mathbb{Q}$-*sphere* R. This invariant generalizes both $\mathcal{Z}(S^3, L)$ and $\mathcal{Z}(R) = \mathcal{Z}(R, \emptyset)$. Our definitions are more flexible than the original ones. We prove generalizations of the mentioned universality results in the book's fourth part.

To get more properties of $\mathcal{Z}$, we cut our pairs (R, L) of links L in $\mathbb{Q}$-spheres R into pieces called *tangles in* $\mathbb{Q}$-*cylinders*. These pieces can be composed in various ways. In the book's third part, we generalize $\mathcal{Z}$ to a functorial invariant of framed tangles in $\mathbb{Q}$-cylinders, and we prove that it behaves well under the various allowed compositions.

Our first chapter is a more complete and much longer preface to this book. It contains several introductions. Section 1.1 is a short summary for experts. Other readers can start with Section 1.2, a slow informal introduction based on examples from which a broad audience can get the flavor of the studied topics and hopefully become interested. Section 1.3 is an independent, more formal, mathematical overview of the contents. It is accessible to beginners in topology after the warm-up of Section 1.2. Section 1.4 describes the book organization. Section 1.5 outlines why I wrote this book and what I consider original and new.

Apart from this preface and the first chapter, which has some parts written for experts and is sometimes imprecise, the rest of the book is precise, detailed, and mostly self-contained. The only prerequisites are basic notions of algebraic topology and de Rham cohomology, surveyed in the appendices.

In 2018, Watanabe disproved a long-standing conjecture called the 4-dimensional Smale conjecture by constructing a topologically trivial S^4-bundle over S^2, which is not smoothly equivalent to the trivial bundle $S^4 \times S^2$ [117]. He distinguished his exotic $S^4 \times S^2$ from the standard $S^4 \times S^2$ using characteristic classes introduced by Kontsevich [49]. The involved Kontsevich–Watanabe characteristic class of a smooth topologically trivial S^4-bundle over S^2 counts configurations of the complete graph with four vertices in the total space of the bundle. The ideas and techniques used by Watanabe are similar to those presented in this book. Even though we only count graph configurations in dimension 3, this book can also serve as an introduction to the work of Watanabe and other articles about invariants counting graph configurations in higher dimensions.

I thank the referees for their careful reading and their helpful comments.

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September 2024