Preface

This book offers an introduction to the theory of adic spaces with applications to the geometry of automorphic forms. Non-archimedean geometry and adic spaces in particular have become indispensable tools in number theory and arithmetic geometry. The interest in these theories has surged in recent years, driven by ground-breaking advancements such as perfectoid spaces and recent developments in the Langlands program.

The book is comprised of expanded lecture notes for six mini-courses delivered by the contributing authors at the

Spring School on
Non-Archimedean Geometry and Eigenvarieties
in March 2023 at Heidelberg University.

The audience of the school consisted primarily of graduate students, advanced master's students and postdocs, most of whom were new to the field. To make the material accessible and beneficial to a wider audience, we decided to expand the notes of the individual courses and compile them into this book. Our aim was to create an accessible entry point into the world of adic spaces for beginners and to guide them from the foundations of the theory to its applications, ultimately leading them to modern research.

This book serves a dual purpose. The first purpose is to provide a focused, comprehensive and mostly self-contained introduction to the theory of adic spaces. By working through this book, the aim for the reader is to develop foundational skills, confidence in the language and intuition for the geometry of adic spaces.

The second purpose is to demonstrate the theory in action through applications. The applications that we decided to include in the school and in this book are *perfectoid spaces* and *eigenvarieties*. These topics are certainly an extra motivation to study the foundations of the theory of adic spaces and will help the reader appreciate the flexibility and potential of this approach to non-archimedean geometry. The ultimate aim for the reader is to develop the skills to use adic spaces in practice and to be able to move on to study research articles and work on open questions and problems in the field.

The original motivation for non-archimedean geometry is to provide an analogue over non-archimedean fields for the theory of complex analytic manifolds. Such a theory ideally also features a nice interplay with algebraic geometry in the spirit of Serre's GAGA. As is recalled at the beginning of the first chapter, three approaches to non-archimedean geometry have been particularly successful, namely, Tate's theory of rigid analytic spaces (which was historically the first), Berkovich's theory of Berkovich spaces and Huber's theory of adic spaces. Among these, the theory of adic spaces has emerged as the most suitable approach for many modern applications, which is why we focus on this theory.

Analogous to the construction of schemes in algebraic geometry, adic spaces are locally built from their rings of functions. Contrary to algebraic geometry, where we work with arbitrary commutative rings, in adic geometry we work with certain topological rings, called Huber rings. This distinction simultaneously enriches the theory and complicates the matter. From these Huber rings, we build affinoid adic spaces – the building blocks of adic spaces – using valuation theory. Adic spaces are then constructed by gluing these building blocks.

The resulting theory of adic spaces offers a compelling analogue to complex analytic geometry. Moreover, it encompasses the theories of rigid analytic spaces and of formal schemes and there is a strong relationship to algebraic geometry. Furthermore, adic spaces serve as the foundational framework for defining perfectoid spaces and they have proven to be the most effective setting for a flexible theory of eigenvarieties. These and other features and applications make the theory of adic spaces an important and powerful tool in modern arithmetic geometry.

We would like to elaborate further on the two applications that we selected for this book. Perfectoid spaces were introduced by Peter Scholze in 2011. They are a special class of adic spaces, namely, those that one can build out of so-called perfectoid rings. They have become perhaps the most significant driving force behind the growing interest in the theory of adic spaces. Perfectoid spaces have been used in a variety of groundbreaking results: to prove cases of the weight monodromy conjecture, to show the direct summand conjecture in commutative algebra, and as explained in Chapter 4, to develop p-adic Hodge theory for rigid analytic varieties. Moreover, they have had striking applications to the geometry of automorphic forms and the Langlands program. For instance, they have been instrumental in proving seminal results concerning the cohomology of (local and global) Shimura varieties and locally symmetric spaces and in attaching Galois representations to automorphic forms. Perfectoid spaces are playing an increasingly important role in the theory of p-adic automorphic forms and a vital role in the work of Fargues and Scholze on the geometrization of the local Langlands correspondence.

The second application of adic spaces that we have included in this book is to the theory of eigenvarieties. The study of p-adic families of automorphic forms was initiated in the 70s by work of Serre and Swinnerton–Dyer who studied congruences between modular forms. The pioneering work of Katz, Hida, Coleman and Mazur turned the topic into an important pillar in the arithmetic of modular forms. Non-archimedean geometry enters the picture in the construction of spaces of p-adic automorphic forms. Moreover, it provides a crucial tool to organise p-adic automorphic forms into p-adic geometric spaces. These spaces are called eigenvarieties - as they parametrise systems of Hecke *eigenvalues* of p-adic automorphic forms. Their rich geometry gives insights into p-adic and classical automorphic forms, links to the deformation theory of Galois representations and is an interesting topic of research. Coleman and Mazur constructed the first example – a space that became known as the Coleman–Mazur eigencurve. Buzzard then gave an axiomatic construction, the eigenvariety machine, that has since been used numerously to construct eigenvarieties.

The theory of p-adic automorphic forms and eigenvarieties has applications to the construction of p-adic L-functions, to modularity of Galois representations and the Fontaine–Mazur conjecture and to Iwasawa theory. It has also been a key tool in some recent developments in the Langlands program, for instance, in the work of Newton and Thorne on the symmetric power functoriality. New techniques from perfectoid geometry, locally analytic representation theory and the emerging perspective of a (categorical) p-adic Langlands program have introduced fresh impulses and tools. Originally the theory of eigenvarieties was developed in the language of rigid analytic spaces over non-archimedean base fields of characteristic zero. However, recent advances, particularly the work on the spectral halo by Andreatta, Iovita and Pilloni, have extended eigenvarieties to characteristic p local fields. These recent works have relied on the framework of adic spaces, which allow greater flexibility in the choice of the base spaces – a crucial ingredient for extending eigenvarieties. To our knowledge, we present the first lecture-course-style approach to eigenvarieties that is consistently phrased in the language of adic spaces instead of rigid analytic spaces.

This book is organised into two parts. The first three chapters build the foundation of the theory of adic spaces.

Chapter 1, written by John Bergdall, introduces Huber rings – the topological rings that underlie the theory of adic spaces. It then explains the valuation theory essential for the construction of adic spaces, covering the fundamental concept of valuation spectra and key constructions involving them. This chapter also includes an in-depth study of the closed unit disk – a central example in the theory.

Chapter 2, contributed by Katharina Hübner, focuses on constructing affinoid adic spaces and how to glue them into adic spaces. A critical aspect of the construction of an affinoid adic space is verifying a sheaf condition. This sheaf condition might fail for a general Huber ring – a significant nuisance in the theory. This chapter discusses this phenomenon and lists the classes of Huber rings where this sheaf condition is known to be satisfied. It also discusses important fundamental concepts such as analytic adic spaces or fibre products. Moreover, Chapter 2 provides links to the theory of rigid analytic spaces and formal schemes and to algebraic geometry.

Chapter 3, authored by Christian Johansson, delves into several key topics within the theory of adic spaces. This chapter covers fundamental concepts such as coherent sheaves and important properties of morphisms, including finiteness properties and flat and étale morphisms. It also includes a discussion of quasi-Stein spaces. Additionally, this chapter provides a section on the uniformization of curves and on Berkovich spaces. Chapter 3 is complemented by an appendix that offers background on Čech and sheaf cohomology as well as locally spectral spaces.

The second half of the book is devoted to the two applications that were previously discussed.

Chapter 4, contributed by Ben Heuer, introduces and studies perfectoid spaces. This chapter explains key technical results, such as tilting and almost purity, which make perfectoid spaces such a powerful tool. It then explains applications to p-adic Hodge theory of rigid analytic spaces.

Chapter 5, written by Judith Ludwig, develops Buzzard's eigenvariety machine in the setting of adic spaces. This chapter provides essential background from p-adic functional analysis, in particular, the spectral theory of compact operators on p-adic Banach spaces, which is crucial for the eigenvariety machine. It then explains the construction of the so-called spectral variety, develops some of its key properties and proceeds to the axiomatic, abstract construction of eigenvarieties. An appendix on pseudo-rigid spaces complements the chapter.

Chapter 6, written by James Newton, gives an introduction to p-adic automorphic forms and explains the construction of eigenvarieties in various different contexts. A special emphasis is put on the Coleman–Mazur eigencurve. This chapter also covers the construction of eigencurves for definite quaternion algebras and explains the very general construction of eigenvarieties using overconvergent cohomology. Additionally, it includes an overview and reference guide to the various different eigenvarieties that have been constructed.

Throughout the text, numerous examples and illustrations are included to help the reader understand new concepts and build intuition for the material. In addition, carefully designed exercises guide the reader throughout the book. By working on these exercises, readers can test their understanding, reinforce their learning and build skills in constructing arguments and proofs. Some exercises are encouraging the readers to explore other sources and thereby help gain an overview over the literature.

Readers are expected to have a solid background in algebraic geometry as well as some familiarity with p-adic analysis and topological rings. For the final chapter, a working knowledge of the theory of automorphic forms on reductive groups is required. The dependency of the chapters is illustrated in the diagram below. For readers who are new to the theory of adic spaces, we recommend beginning with the first three chapters, as they provide the foundational knowledge. The two applications are presented independently and can be read in any order. Readers who are familiar with adic spaces and are primarily interested in the applications can jump straight to the relevant chapters and refer back to the earlier chapters as needed.

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