Chapter 1

Introduction

Logarithmic Gromov–Witten theory, developed by the authors in [2, 15, 30], has proved a successful generalization of the notion of relative Gromov–Witten invariants developed in [47–49]. Relative Gromov–Witten invariants are invariants of pairs (X, D) where X is a non-singular variety and D is a smooth divisor on X; these invariants count curves with imposed contact orders with D at marked points. Logarithmic Gromov–Witten theory allows D instead to be normal crossings, or more generally, allows (X, D) to be a toroidal crossings variety.

1.1 Scope and motivation

The purpose of the present work is to extend logarithmic Gromov–Witten theory to admit *negative contact orders*. Working over a field $\Bbbk$, an example for how negative contact orders arise naturally is by restricting a normal crossings degeneration, such as

$$\pi : \mathbb{A}^2 = \operatorname{Spec} \Bbbk[z, w] \to \mathbb{A}^1 = \operatorname{Spec} \Bbbk[t], \quad \pi^\sharp(t) = zw,$$

to the irreducible component $C = V(w)$ of the central fiber $\pi^{-1}(0)$. Viewing π as a morphism of log spaces for the toric log structures on $\mathbb{A}^2$ and $\mathbb{A}^1$, denote by s_z, s_w, s_t the global sections of the log structure $\mathcal{M}_{\mathbb{A}^2}$ induced by z, w and $\pi^\sharp(t)$, respectively. The induced log structure $\mathcal{M}_C = \mathcal{M}_{\mathbb{A}^2}|_C$ on $C = \operatorname{Spec} \Bbbk[z]$ is generated by the restrictions of s_z, s_w, denoted by the same symbols. Note that the structure morphism $\mathcal{M}_C \to \mathcal{O}_C$ maps s_z to z, which has a first order zero at the origin $0 \in \mathbb{A}^1$ as given by a marked point, while s_w and s_t map to 0. The point is that viewed as a log space over $\mathbb{A}^1$, the equation $s_z s_w = s_t$ implies that away from $0 \in \mathbb{A}^1$, we have

$$s_w = z^{-1} \cdot s_t.$$

Such sections do not exist on log smooth curves over the standard log point. The power of z occurring in this equation reflects the negative contact order. Since z^{-1} is defined on the punctured curve $\mathbb{A}^1 \setminus \{0\}$, we call the resulting extension of log smooth curves, stable logarithmic maps and logarithmic Gromov–Witten theory *punctured curves*, *punctured (logarithmic) maps* and *punctured Gromov–Witten theory*.

Our motivation for studying punctured Gromov–Witten theory comes from three sources. First, as illustrated in the example, negative contact orders arise naturally when gluing a logarithmic stable map from its restrictions to closed subcurves, as desired in degeneration situations [3]. Note that in transverse situations, as achieved by the expanded degeneration technique in [48], negative contact orders can be avoided by turning a punctured map over a standard log point into a stable logarithmic map to an irreducible component of the target over the trivial log point; see [26, Sections 6, 7] for details. This simplification is not possible when an irreducible component of the curve maps into a deeper stratum. See [3, Section 5.2.4] for an example where no decomposition of the target splits any of the nodes into a pair of marked points with non-negative contact order. A treatment of gluing situations based on punctured maps is contained in Chapter 5.

The second motivation comes from mirror constructions and their link to symplectic cohomology, relating to the program on mirror symmetry of Gross and Siebert via toric degenerations. It turns out that the algorithmic construction of mirrors via wall structures in [29] admits a vast, intrinsic generalization by using punctured invariants [33]. Punctured invariants are used in this context to define the structure coefficients of the coordinate ring of the mirror degeneration, with the space of non-negative contact orders representing generators. The structure coefficients require punctured invariants with two positive and one negative contact order. The gluing techniques developed in Chapter 5 are the crucial ingredient in proving associativity of the resulting multiplicative structure. In [32], the gluing techniques for punctured invariants are also crucial in constructing a consistent wall structure in the intrinsic mirror symmetry setup, thus linking the mirror constructions in [29, 33] via [28]. Further, building on [8, 32] gives an algorithmic method of calculating certain one-pointed punctured invariants on blow-ups of toric varieties.

Another interesting related fact is the interpretation of punctured invariants as structure coefficients in some versions of symplectic cohomology. Thus punctured invariants provide an algebraic-geometric path to computing otherwise hard to compute symplectic invariants. See [7, 23, 24, 63, 66] for some steps in this direction.

The third motivation is from work of the second author on the logarithmic gauged linear sigma model. In the papers [17, 19], punctured maps are shown to be a key for computing the invariants of the logarithmic gauged linear sigma model of [18].[1] This provides the geometric foundation for calculating higher genus invariants of quintic 3-folds [34, 35], and for proving [55, Conjecture A.1] on the cycle of holomorphic differentials [16].

[1] In [19], punctured maps to a smooth boundary divisor with extra structure called R-maps, are studied. The moduli of punctured maps provide different virtually birational models over which effective formulas for computing higher genus Gromov–Witten invariants hold [17]. These crucial virtually birational models do not exist as moduli of rubber maps with expansions [25, 48].

1.2 Main features of punctured Gromov–Witten theory

Several aspects of the theory of punctured Gromov–Witten invariants appear to be straightforward generalizations from ordinary logarithmic Gromov–Witten theory. The formal similarity can, however, be quite misleading. In fact, finding the right setup and point of view took a very long time, and was only made possible by developing the theory simultaneously with the mentioned applications.

One major difference is the more singular and more interesting nature of the base space for moduli spaces of punctured maps. In ordinary Gromov–Witten theory, the natural base space is the Artin stack $\mathbf{M}$ of nodal curves. While non-separated, $\mathbf{M}$ is smooth, hence is locally pure dimensional. The relative obstruction theory of the moduli space of stable maps over $\mathbf{M}$ thus produces a virtual fundamental cycle by virtual pullback of the fundamental class $[\mathbf{M}]$. The picture in logarithmic Gromov–Witten theory is much the same, with $\mathbf{M}$ now replaced by the stack $\mathfrak{M} = \mathrm{Log}_{\mathbf{M}}$ of log smooth curves of the given genus and numbers of marked points over fine saturated (fs) log schemes. This stack is log smooth over the base field, hence is also locally pure-dimensional.

For punctured invariants, the analogue of $\mathfrak{M}$ is the stack $\breve{\mathfrak{M}}$ of logarithmic curves with punctures. One crucial feature of the deformation theory of punctured curves is that $\breve{\mathfrak{M}}$ is typically not pure-dimensional. In fact, the map $\breve{\mathfrak{M}} \to \mathbf{M}$ forgetting the log structure turns out to be only idealized logarithmically étale (Proposition 3.3). This means that locally in the smooth topology $\breve{\mathfrak{M}} \to \mathbf{M}$ is isomorphic to the composition of a closed embedding defined by a monomial ideal followed by a toric morphism of affine toric varieties with associated lattice homomorphism an isomorphism over $\mathbb{Q}$.

The induced stratified structure of punctured maps turns out to be captured by tropical geometry. The second main feature of punctured Gromov–Witten theory is thus the central role of tropical geometry, exceeding by far its increasingly recognized role in logarithmic Gromov–Witten theory. Working over a base space B, we first factor the log smooth target $X \to B$ over the *relative Artin fan* $\mathcal{X} \to B$ from [5, Corollary 3.3.5], an algebraic stack glued from quotients of toric charts for $X \to B$ by the fiberwise acting torus, see [3, Section 2.2]. Working with $\mathcal{X}$ as a target amounts to working with nodal curves and compatible families of tropical maps, thus making the theory of such punctured maps a combinatorially enriched version of the theory of stable curves. A better base space than $\breve{\mathfrak{M}}$ to work with is then the algebraic stack $\mathfrak{M}(\mathcal{X}/B)$ of punctured maps to $\mathcal{X}/B$. Indeed, the forgetful map

$$\mathfrak{M}(\mathcal{X}/B) \to \mathbf{M} \times B$$

is also idealized logarithmically étale (Theorem 3.25), but now with idealized structure easy to extract from the tropical geometry of the situation. In particular, Remark 3.31 gives a complete characterization of the strata of $\mathfrak{M}(\mathcal{X}/B)$ in terms of types of tropical maps. Due to its fundamental nature for punctured Gromov–Witten theory, we emphasize the role of tropical geometry throughout, including adapted presentations of material from [30] in Chapters 2, 3 and Appendix C.

A third feature of punctured Gromov–Witten theory developed here, but already relevant to ordinary logarithmic Gromov–Witten theory, is the introduction of *evaluation stacks* for imposing point conditions compatible with the virtual formalism. Since $X \to \mathcal{X}$ is smooth in the ordinary sense, we can choose a lift to X of the image of each marked point in $\mathcal{X}$ to arrive at an algebraic stack $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B)$ smooth over $\mathfrak{M}(\mathcal{X}/B)$ and such that the relative obstruction theory over $\mathfrak{M}(\mathcal{X}/B)$ arises from a relative obstruction theory over $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B)$. It is this *evaluation stack* $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B)$ that one needs to work with to impose conditions on the evaluations at the marked points rather than the product $X \times_B \cdots \times_B X$ in ordinary Gromov–Witten theory. Evaluation stacks also play a crucial role in our gluing formalism, see Section 5.3.

1.3 Statements of main results

For simplicity of the presentation of the main results we now assume $X \to B$ to be a projective log smooth morphism of log schemes and B the standard log point $\mathrm{Spec}(\mathbb{N} \to \Bbbk)$ or B log smooth over the trivial log point $\mathrm{Spec}\,\Bbbk$, where $\Bbbk$ is a field of characteristic 0. For more detailed statements we refer to the main body of the text.

Similar to logarithmic Gromov–Witten theory, as presented in [3, Section 2.5], we introduce (decorated) types of punctured maps and of (families of) tropical maps. Types restrict the combinatorics of punctured maps over geometric points as seen by their tropicalizations, such as the dual intersection graph, the contact orders of punctured and nodal points and the genera and the curve classes of its irreducible components (Definition 2.24). An appropriate global version of contact orders developed in Section 2.4 leads to the notion of global decorated type $\boldsymbol{\tau}$ that can be used to define moduli spaces of punctured maps *marked by* $\boldsymbol{\tau}$ (Definition 3.8). Theorem 3.10, Corollary 3.19 and Theorem 3.25 establish the basic properties of these moduli spaces, which can be summarized as follows.

Theorem A. *Let $\boldsymbol{\tau}$ be a decorated global type. Then the stacks $\mathfrak{M}(\mathcal{X}/B, \boldsymbol{\tau})$ and $\mathscr{M}(X/B, \boldsymbol{\tau})$ of $\boldsymbol{\tau}$-decorated basic stable punctured maps (Definition 3.8) to $\mathcal{X} \to B$ and to $X \to B$, respectively, are logarithmic algebraic stacks. Moreover, $\mathscr{M}(X/B, \boldsymbol{\tau})$ is Deligne–Mumford and proper over B.*[2] *If in addition X is simple* [3, Definition 2.1], *$\mathfrak{M}(\mathcal{X}/B, \boldsymbol{\tau})$ is idealized smooth over B.*

For the definition of punctured Gromov–Witten invariants we work over the evaluation stacks $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B, \tau)$, which lift the evaluations at a set $\mathbf{S}$ of punctured and

[2] We prove properness under the technical assumption that the log structure on X is Zariski and $\bar{\mathcal{M}}_X^{\mathrm{gp}}$ is globally generated. The latter assumption has meanwhile been removed in [39].

nodal points from $\mathcal{X}$ to X (Definition 5.14). We suppress $\mathbf{S}$ in the notation. The following theorem summarizes Propositions 4.2 and 4.5.

Theorem B. *For $\boldsymbol{\tau}$ a decorated global type let $\pi : \mathcal{C}(X/B, \boldsymbol{\tau}) \to \mathcal{M}(X/B, \boldsymbol{\tau})$ and $f : \mathcal{C}(X/B, \boldsymbol{\tau}) \to X$ be the universal curve and universal punctured map over the moduli space $\mathcal{M}(X/B, \boldsymbol{\tau})$ of $\boldsymbol{\tau}$-marked basic punctured maps to $X \to B$. Let $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B, \boldsymbol{\tau})$ be the corresponding evaluation stack for a subset $\mathbf{S}$ of punctured sections, and $Z \subset \underline{\mathcal{C}}(X/B, \boldsymbol{\tau})$ the closed substack defined by the union of the images of these sections. Then there is a canonical perfect relative obstruction theory*

$$\mathbb{G} \simeq (R\pi_* f^*\Theta_{X/B}(-Z))^\vee \to \mathbb{L}_{\mathcal{M}(X/B,\boldsymbol{\tau})/\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\boldsymbol{\tau})}$$

for the natural morphism $\varepsilon : \mathcal{M}(X/B, \boldsymbol{\tau}) \to \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B, \boldsymbol{\tau})$. Here $\Theta_{X/B}$ denotes the logarithmic tangent bundle of X over B.

A similar statement holds if $\mathbf{S}$ also contains nodal sections, see Proposition 4.5. Virtual pullback [50] now provides punctured Gromov–Witten invariants, with the basic correspondence the homomorphism

$$(\underline{\mathrm{ev}} \times p)_*\varepsilon^!_{\mathbb{G}} : A_*(\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B, \boldsymbol{\tau})) \to A_{*+d(g,k,A,n)}\Big(\prod_L \underline{Z}_L \times \mathcal{M}_{g,k}\Big)$$

on rational Chow groups defined in Definition 4.6. Here each $\underline{Z}_L$ is an evaluation stratum, the closed stratum of $\underline{X}$ that evaluation at a punctured section maps to by the choice of decorated global type $\boldsymbol{\tau}$, and $\mathcal{M}_{g,k}$ is the Deligne–Mumford stack of k-marked stable curves of genus g. A formula for the relative virtual dimension $d(g, k, A, n)$ is stated in (4.18).

The most challenging part of this paper was an efficient and practically useable treatment of gluing. While the final results may look straightforward, they rely on a number of careful choices and subtle points which became clear to us only after a long series of futile attempts.[3] Here we only summarize the results and refer to Remark 5.22 for some further discussion.

The formal setup for gluing takes a decorated global type $\boldsymbol{\tau}$ and splits the graph underlying $\boldsymbol{\tau}$ at a subset of edges, leading to a set $\{\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_r\}$ of decorated global types, with each split edge now producing a pair of legs in the graphs for the $\boldsymbol{\tau}_i$. Our first result on gluing reduces all gluing questions to the unobstructed evaluation stacks, as proved in Proposition 5.17 and Theorem 5.19.

[3] For some time our formalism only worked for gluing problems appearing in certain mirror constructions. We emphasize that the final results below are general and practical, as demonstrated in [71].

Theorem C. *There is a cartesian diagram*

$$\begin{array}{ccc} \mathcal{M}(X/B,\boldsymbol{\tau}) & \xrightarrow{\ \delta\ } & \prod_{i=1}^r \mathcal{M}(X/B,\boldsymbol{\tau}_i) \\ {\scriptstyle\varepsilon}\big\downarrow & & \big\downarrow{\scriptstyle \widehat{\varepsilon}=\prod_i \varepsilon_i} \\ \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\tau) & \xrightarrow{\ \delta^{\mathrm{ev}}\ } & \prod_{i=1}^r \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\tau_i) \end{array}$$

with horizontal arrows the splitting maps from Proposition 5.4, finite and representable by Corollary 5.15, and the vertical arrows the canonical strict morphisms.[4]

Moreover, if $\widehat{\varepsilon}^!$ and $\varepsilon^!$ denote Manolache's virtual pullback defined using the two given obstruction theories for the vertical arrows, then for $\alpha \in A_*(\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\tau))$, we have the identity

$$\widehat{\varepsilon}^! \delta^{\mathrm{ev}}_*(\alpha) = \delta_* \varepsilon^!(\alpha).$$

A numerical gluing formula follows from Theorem C for those Chow classes α on $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\boldsymbol{\tau})$ such that δ^{ev}_* can be written as a sum of products, see Corollary 5.20.[5] A straightforward consequence of Theorem C is a gluing formula for the degeneration situation from [3], see Corollary 5.26.

Our second result on gluing provides a description of the splitting map δ^{ev} in terms of an fs-fiber diagram, which apart from proving the properties stated in Theorem C, provides a route to using Theorem C in explicit computations. For the following statement the log stacks $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\boldsymbol{\tau})$, $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\boldsymbol{\tau}_i)$ have to be replaced by closely related log stacks $\widetilde{\mathfrak{M}}'^{\mathrm{ev}}(\mathcal{X}/B,\boldsymbol{\tau})$ and $\widetilde{\mathfrak{M}}'^{\mathrm{ev}}(\mathcal{X}/B,\boldsymbol{\tau}_i)$, which however have the same underlying reduced stacks, hence have identical Chow theories (Proposition 5.7).[6] These stacks come with logarithmic evaluation morphisms such as

$$\mathrm{ev}_{\mathbf{E}} : \widetilde{\mathfrak{M}}'^{\mathrm{ev}}(\mathcal{X}/B,\boldsymbol{\tau}) \to \prod_{E\in\mathbf{E}} X,$$

where $\mathbf{E}$ is the set of nodal sections to split. The following is Corollary 5.15 to which we refer for more details.

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[4]There is an entirely equivalent formalism allowing for disconnected punctured curves and disconnected types, in which case the products on the right form a single moduli stack corresponding to a disconnected decorated global type $\boldsymbol{\tau}$.

[5]Conversely, if there is no such decomposition, a numerical gluing formula cannot be achieved within Chow theory—a phenomenon already present in the classical case of a smooth gluing locus. A generally applicable gluing formula should therefore require working with a homology theory with a Künneth decomposition. It is possible that the formalism of virtual pullback in Borel–Moore homology developed in [40] may be useful.

[6]These stacks are closely related to Parker's moduli of cut curves introduced in [57].

Theorem D. *The splitting morphism* $\delta^{\mathrm{ev}} : \widetilde{\mathfrak{M}}'^{\mathrm{ev}}(\mathcal{X}/B, \boldsymbol{\tau}) \to \prod_i \widetilde{\mathfrak{M}}'^{\mathrm{ev}}(\mathcal{X}/B, \boldsymbol{\tau}_i)$ *fits into the cartesian diagram*

$$\begin{array}{ccc} \widetilde{\mathfrak{M}}'^{\mathrm{ev}}(\mathcal{X}/B, \boldsymbol{\tau}) & \xrightarrow{\delta^{\mathrm{ev}}} & \prod_{i=1}^{r} \widetilde{\mathfrak{M}}'^{\mathrm{ev}}(\mathcal{X}/B, \boldsymbol{\tau}_i) \\ {\scriptstyle \mathrm{ev}_{\mathbf{E}}}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_{\mathbf{L}}} \\ \prod_{E \in \mathbf{E}} X & \xrightarrow{\Delta} & \prod_{E \in \mathbf{E}} X \times_B X \end{array}$$

of fs log stacks. Here Δ *restricts to the diagonal morphism* $X \to X \times_B X$ *on each factor.*

We emphasize that the diagram in Theorem D is typically not cartesian on the level of underlying stacks due to the more subtle nature of fs fiber products. Theorem D is nevertheless a powerful tool for explicit computations. For example, under the assumption of toric gluing strata, Yixian Wu in [71] derives from Theorem D an explicit decomposition of the term $\delta^{\mathrm{ev}}_*[\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B, \boldsymbol{\tau})]$ appearing in Theorem C in terms of the strata in $\prod_i \mathfrak{M}(\mathcal{X}/B, \boldsymbol{\tau}_i)$.

1.4 Organization of the paper

Chapter 2 contains the basic definitions and related concepts concerning punctured curves and punctured maps, with Section 2.1 introducing pre-stable and stable punctured maps, and Section 2.2 along with Appendix C the tropical language, including the definition of types and the modified balancing condition. The subject of Section 2.4 is the discussion of contact orders in a simplified version sufficient for most applications, and the associated notion of global types. The more involved general picture concerning contact orders is discussed in Appendix A. Section 2.3 presents the concept of basicness for punctured maps, which while largely the same as for logarithmic stable maps, emphasizes the tropical point of view, and hence might be of some independent interest. Section 2.5 introduces the new phenomenon of puncturing log ideals that each family of punctured curves or punctured maps comes with. Section 2.6 discusses the generalization to targets with monodromy.

Chapter 3 deals with the moduli theory of punctured maps, proving Theorem A among other things. Section 3.1 introduces stacks of punctured curves, with the main result the idealized smoothness statement in Proposition 3.3, followed in Section 3.2 by definitions of various stacks of punctured maps marked by types. Sections 3.3 and 3.4 establish properness of the moduli spaces to projective targets by adapting the boundedness and stable reduction theorems from [30]. The topic of Section 3.5 is the idealized smoothness of the spaces $\mathfrak{M}(\mathcal{X}/B, \boldsymbol{\tau})$ and the induced stratified structure, all characterized in terms of tropical geometry.

Chapter 4 deals with obstruction theories, using the approach of [13]. Section 4.1 gives a careful treatment of functoriality as well as of compatibility of obstruction theories for maps of pairs. Section 4.2 applies this discussion to construct the desired relative obstruction theory for $\mathcal{M}(X/B,\boldsymbol{\tau}) \to \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\boldsymbol{\tau})$, in particular proving Theorem B. Another application of this discussion is to the virtual treatment of gluing, presented in Section 5.3. The section ends with the definition of punctured Gromov–Witten invariants in Section 4.3.

The last Chapter 5 contains our results on gluing. Section 5.1 introduces the splitting morphism, while Section 5.2 treats the converse operation of gluing, essentially proving Theorem D, maybe the hardest single result in the paper with a very long genesis. The virtual treatment of gluing, proving Theorem C, is the objective of Section 5.3. The last Section 5.4 applies our results to the degeneration situation of [3].

1.5 Relation to other work

We end this introduction by discussing related work. First, our approach owes a great deal to Brett Parker's program of exploded manifolds, [56, 58–62]. We have often found ourselves trying to translate Parker's results in the category of exploded manifolds into the category of log schemes. Indeed, some of the original versions of the definition of punctured invariants, as well as the approach to gluing, arose after discussions with Parker.

A gluing formula for logarithmic Gromov–Witten invariants without expansions in the case of a degeneration X_0 with smooth singular locus is due to Kim, Lho and Ruddat [44]. This case does not require punctured invariants or evaluation spaces, but is otherwise close in spirit to the present treatment. A gluing formula in a special case for certain rigid analytic Gromov–Witten invariants has been proved by Tony Yu [72].

After the earlier manuscript version of this paper was distributed, Mohammed Tehrani [22], in developing a symplectic analogue of stable logarithmic maps, found that punctures were naturally described in the theory. Even more recently, [21, 68] defined negative contact order Gromov–Witten invariants by a limiting version of orbifold Gromov–Witten invariants. However, as observed by Dhruv Ranganathan, the invariants as currently defined cannot coincide with logarithmic invariants as they do not satisfy the correct invariance properties under log étale modifications. Work of Battistella, Nabijou and Ranganathan [11] takes this into account and shows how genus zero logarithmic invariants can be recovered from the orbifold invariants after sufficient blowing up. Their work [12] considers the case of negative contact orders in the orbifold theory, and makes a somewhat more subtle comparison which involves the puncturing ideal defined in Section 2.5. We send the reader to those papers for more details.

Besides the immediate applications of punctures already mentioned above, punctures also have been used by Hülya Argüz in [7] to build a logarithmic analogue of certain tropical objects in the Tate elliptic curve related to Floer theory.

Finally, we also mention recent work of Dhruv Ranganathan [65] taking a different point of view on gluing in log Gromov–Witten theory using an approach closer in spirit to the expanded degeneration picture of Jun Li.

1.6 Conventions

All schemes and stacks are defined over an algebraically closed field $\Bbbk$ of characteristic 0. By a logarithmic algebraic stack, we mean an algebraic stack equipped with a log structure. We follow the convention that if X is a log scheme or stack, then $\underline{X}$ is the underlying scheme or stack. To unclutter notation, we nevertheless often write $\mathcal{O}_X$ instead of $\mathcal{O}_{\underline{X}}$, and f^* instead of $\underline{f}^*$ for the pullback by the schematic morphism underlying a log morphism $f : X \to Y$. Unless stated otherwise, $\mathcal{M}_X$ denotes the sheaf of monoids on X, and $\alpha_X : \mathcal{M}_X \to \mathcal{O}_X$ the structure map. The affine log scheme with a global chart defined by a homomorphism $Q \to R$ from a monoid Q to a ring R is denoted $\mathrm{Spec}(Q \to R)$. We use the notations $X \times_Z Y$, $X \times^{\mathrm{f}}_Z Y$, $X \times^{\mathrm{fs}}_Z Y$ to distinguish the fiber products in the category of all log schemes, and in the fine or the fine and saturated categories, respectively.

Throughout B denotes either a log point $\mathrm{Spec}(Q \to \Bbbk)$ with Q a toric monoid with $Q^\times = 0$, or an fs log scheme log smooth over $\mathrm{Spec}\,\Bbbk$.[7]

A *nodal curve* over a scheme $\underline{W}$ is a proper flat morphism $\underline{C} \to \underline{W}$ with all geometric fibers reduced of dimension one and with at worst nodes as singularities. A *pre-stable curve* is a nodal curve with all geometric fibers connected.

The category of rational polyhedral cones from [3, Section 2.1] is denoted **Cones**. An object σ of **Cones** comes with a lattice N_σ with $\sigma \subseteq (N_\sigma)_\mathbb{R} = N_\sigma \otimes_\mathbb{Z} \mathbb{R}$, and we denote by $\sigma_\mathbb{Z} = \sigma \cap N_\sigma$ the submonoid of integral points of σ. If P is a monoid, we write $P^\vee := \mathrm{Hom}(P, \mathbb{N})$, $P^* = \mathrm{Hom}(P, \mathbb{Z})$, and $P^\vee_\mathbb{R} = \mathrm{Hom}(P, \mathbb{R}_{\geq 0})$. We write $P^\times$ for the group of invertible elements of P. We write $\Bbbk[P]$ for the monoid ring of P with coefficients in the field $\Bbbk$, with $\Bbbk$-basis consisting of symbols z^p for $p \in P$.

[7] We only use these assumptions in the proof of Theorem 3.25 to assure that the reduced logarithmic strata are defined by logarithmic ideals. This theorem is at the heart of everything we do involving moduli spaces of punctured maps marked by a type.