## Chapter 2

# Punctured maps

## 2.1 Definitions

### 2.1.1 Puncturing

**Definition 2.1.** Let $Y = (\underline{Y}, \mathcal{M}_Y)$ be a fine and saturated logarithmic scheme with a decomposition $\mathcal{M}_Y = \mathcal{M} \oplus_{\mathcal{O}^\times} \mathcal{P}$. A *puncturing* of $Y$ along $\mathcal{P} \subset \mathcal{M}_Y$ is a fine sub-sheaf of monoids

$$\mathcal{M}_{Y^\circ} \subset \mathcal{M} \oplus_{\mathcal{O}^\times} \mathcal{P}^{\mathrm{gp}}$$

containing $\mathcal{M}_Y$ with a structure map $\alpha_{Y^\circ} : \mathcal{M}_{Y^\circ} \to \mathcal{O}_Y$ such that

(1) The inclusion $\mathrm{p}^\flat : \mathcal{M}_Y \to \mathcal{M}_{Y^\circ}$ is a morphism of logarithmic structures on $\underline{Y}$.

(2) For any geometric point $\bar{x}$ of $\underline{Y}$ let $s_{\bar{x}} \in \mathcal{M}_{Y^\circ,\bar{x}}$ be such that $s_{\bar{x}} \notin \mathcal{M}_{\bar{x}} \oplus_{\mathcal{O}^\times} \mathcal{P}_{\bar{x}}$. Representing $s_{\bar{x}} = (m_{\bar{x}}, p_{\bar{x}}) \in \mathcal{M}_{\bar{x}} \oplus_{\mathcal{O}^\times} \mathcal{P}^{\mathrm{gp}}_{\bar{x}}$, we have $\alpha_{Y^\circ}(s_{\bar{x}}) = \alpha_{\mathcal{M}}(m_{\bar{x}}) = 0$ in $\mathcal{O}_{Y,\bar{x}}$.

Denote by $Y^\circ = (\underline{Y}, \mathcal{M}_{Y^\circ})$. We will also call the induced morphism of logarithmic schemes $\mathrm{p} : Y^\circ \to Y$ a *puncturing* of $Y$ along $\mathcal{P}$, or call $Y^\circ$ a *puncturing* of $Y$ along $\mathcal{P}$. We refer to Figure 2.1 for illustration.

We say the puncturing is *trivial* if p is an isomorphism.

**Remark 2.2.** In all examples in this paper, $\mathcal{P}$ is a $DF(1)$ log structure, that is, there is a surjective sheaf homomorphism $\underline{\mathbb{N}} \to \overline{\mathcal{P}}$. In this case the condition $\alpha_{\mathcal{M}}(m_{\bar{x}}) = 0$ is redundant. Indeed, for $s_{\bar{x}} = (m_{\bar{x}}, p_{\bar{x}}) \notin \mathcal{M}_{\bar{x}} \oplus_{\mathcal{O}^\times} \mathcal{P}$, suppose $\alpha_{Y^\circ}(s_{\bar{x}}) = 0$. Note that the $DF(1)$ assumption implies that $p_{\bar{x}}^{-1} \in \mathcal{P}_{\bar{x}}$, so that $\alpha_{\mathcal{M}}(m_{\bar{x}}) = \alpha_Y(m_{\bar{x}}, 1) = \alpha_{Y^\circ}(s_{\bar{x}} \cdot p_{\bar{x}}^{-1}) = 0$. More generally, the same argument works if $\mathcal{P}$ is valuative.

For more general puncturings, the second vanishing condition $\alpha_{\mathcal{M}}(m_{\bar{x}}) = 0$ in Definition 2.1 (2) is not automatic, but is needed to obtain good behavior under base-change (Proposition 2.8). Our log stacks $\widetilde{\mathfrak{M}}'(\mathcal{X}/B, \tau)$ in Section 5.2.2 naturally carry such a more general puncturing. While these more general log structures have no further use in this paper, they may be of use elsewhere.

Note also that if $\mathcal{P}$ is a $DF(1)$ log structure and $\bar{y}$ is a geometric point of $\underline{Y}$, then

$$\overline{\mathcal{M}}_{Y,\bar{y}} \oplus \mathbb{N} \subseteq \overline{\mathcal{M}}_{Y^\circ,\bar{y}} \subset \overline{\mathcal{M}}_{Y,\bar{y}} \oplus \mathbb{Z}, \quad \overline{\mathcal{M}}_{Y^\circ,\bar{y}} \cap (\{0\} \times \mathbb{Z}_{<0}) = \emptyset. \tag{2.1}$$

We will see in Lemma 2.21 how such monoids can easily be encoded in the dual tropical picture.

**Remark 2.3.** Puncturings $\mathcal{M}^\circ$ of $\mathcal{M} \oplus_{\mathcal{O}^\times} \mathcal{P}$ are not unique. In a widely distributed early version of this manuscript as well as in [31], we found it instructive to work with

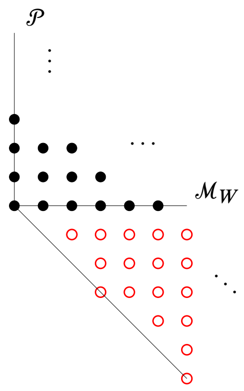


**Figure 2.1.** A puncturing $Y^\circ$ of a monoid $\mathcal{M} = \mathcal{M}_W$. Note that the part with negative projection in $\mathcal{P}^{\mathrm{gp}}$ (open circles) is not necessarily saturated.

a uniquely defined object $\mathcal{M}^{\mathcal{P}}$ we call here the *final puncturing*. It may be defined as the direct limit

$$\mathcal{M}^{\mathcal{P}} := \varinjlim_{\mathcal{M}^\circ \in \Lambda} \mathcal{M}^\circ,$$

over the collection $\Lambda$ of all puncturings of $\mathcal{M} \oplus_{\mathcal{O}^\times} \mathcal{P}$. This exists in the category of quasi-coherent, not necessarily coherent, logarithmic structures. It has the advantage of being independent of any choice. Its disadvantage, apart from not being finitely generated, is in that its behavior under base change is subtle.

We emphasize that

(1) all puncturings used in this paper, with the exception of the remark above, are fine, and in particular they are finitely generated.

(2) On the other hand, the puncturings we use are rarely saturated, even though the logarithmic structure they puncture are themselves saturated. The reason is that base change of a saturated puncturing can lead to a non-saturated puncturing. Imposing a saturation condition would therefore lead to a subtle fiberwise saturation procedure. Instead, we find that the notion of pre-stability of Definition 2.6 below suffices to control these logarithmic structures and their moduli.

**Remark 2.4.** In the introduction, we motivated punctures as arising from restrictions of log structures on log smooth curves to irreducible components. Indeed, this is one way of producing punctures: see Proposition 5.2 for details. However, since we allow fine rather than fine saturated log structures for the puncturing, it is clear that not all the punctures we consider are of this form. See also Lemma 2.21 for a description of the submonoids of $\overline{\mathcal{M}}_{Y,\bar{y}} \oplus \mathbb{Z}$ that can arise.

It is worth making a historical remark here. When we began this project, we first considered what we called "pre-nodal" log structures in which we allowed precisely those log structures coming via restriction from a log smooth curve. However, we found the moduli space of pre-nodal log maps was very poorly behaved, almost never Deligne–Mumford. The notion of punctured points along with the notion of pre-stability of Definition 2.6 resolved these technical difficulties, and made gluing possible.

### 2.1.2 Pre-stable punctured log structures

In case a puncturing is equipped with a morphism to another fine log structure there is a canonical choice of puncturing. The following proposition follows immediately from the definitions.

**Proposition 2.5.** *Let $X$ be a fine log scheme, and $Y$ as in Definition 2.1, with a choice of puncturing $Y^\circ$ and a morphism $f : Y^\circ \to X$. Let $\widetilde{Y}^\circ$ denote the puncturing of $Y$ given by the subsheaf of $\mathcal{M}_{Y^\circ}$ generated by $\mathcal{M}_Y$ and $f^\flat(f^*\mathcal{M}_X)$. Then*

(1) *We have $\mathcal{M}_{\widetilde{Y}^\circ}$ is a sub-logarithmic structure of $\mathcal{M}_{Y^\circ}$.*

(2) *There is a factorization*

$$\begin{array}{ccc} Y^\circ & \xrightarrow{\quad f \quad} & X. \\ & \searrow \quad \nearrow_{\tilde{f}} & \\ & \widetilde{Y}^\circ & \end{array}$$

(3) *Given $Y_1^\circ \to Y_2^\circ \to Y$ with both $Y_1^\circ$, $Y_2^\circ$ puncturings of $Y$, and compatible morphisms $f_i : Y_i^\circ \to X$, then $\widetilde{Y}_1^\circ = \widetilde{Y}_2^\circ$.*

**Definition 2.6.** A morphism $f : Y^\circ \to X$ from a puncturing of a log scheme $Y$ is said to be *pre-stable* if the induced morphism $Y^\circ \to \widetilde{Y}^\circ$ in the above proposition is the identity. In particular, one has $f = \tilde{f}$.

Proposition 2.5 yields the following criterion for pre-stability of a morphism from a punctured log scheme, see Figure 2.2.

**Corollary 2.7.** *A morphism $f : Y^\circ \to X$ is pre-stable if and only if the induced morphism of sheaves of monoids $f^*\overline{\mathcal{M}}_X \oplus \overline{\mathcal{M}}_Y \to \overline{\mathcal{M}}_{Y^\circ}$ is surjective.*

### 2.1.3 Pull-backs of puncturings

**Proposition 2.8.** *Let $Z$ and $Y$ be fs log schemes with log structures $\mathcal{M}_Z$ and $\mathcal{M}_Y$, and suppose given a morphism $g : Z \to Y$. Suppose also given an fs log structure*

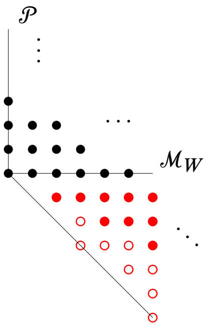


**Figure 2.2.** A morphism of the previous puncturing $Y^\circ$ which is not pre-stable, with $f^\flat \mathcal{M}_X$ generated by $(2, -1)$. The submonoid generated by $\mathcal{M}_Y$ and $f^\flat \mathcal{M}_X$, shown in solid dots, is a different puncturing $\widetilde{Y}^\circ$ which is pre-stable.

*$\mathcal{P}_Y$ on $\underline{Y}$ and an induced log structure $\mathcal{P}_Z := g^*\mathcal{P}_Y$ on $\underline{Z}$. Set*

$$Z' = \big(\underline{Z}, \mathcal{M}_Z \oplus_{\mathcal{O}_{\underline{Z}}^\times} \mathcal{P}_Z\big), \quad Y' = \big(\underline{Y}, \mathcal{M}_Y \oplus_{\mathcal{O}_{\underline{Y}}^\times} \mathcal{P}_Y\big).$$

*Further, let $Y^\circ$ be a puncturing of $Y'$ along $\mathcal{P}_Y$. Then there is a diagram*

$$\begin{array}{ccc} Z^\circ & \xrightarrow{g^\circ} & Y^\circ \\ \downarrow & & \downarrow \\ Z' & \xrightarrow{g'} & Y' \\ \downarrow & & \downarrow \\ Z & \xrightarrow[g]{} & Y \end{array}$$

*with all squares Cartesian in the category of underlying schemes, the lower square Cartesian in the category of fs log schemes, and the top square Cartesian in the category of fine log schemes. Furthermore, $Z^\circ$ is a puncturing of $Z'$ along $\mathcal{P}_Z$, and $g^\circ$ is pre-stable.*

*Proof.* We define $Z^\circ$ to be the fiber product $Z' \times^{\mathrm{f}}_{Y'} Y^\circ$ in the fine log category. The bottom square is Cartesian in all categories as $\mathcal{P}_Y$ is assumed saturated. Thus it is sufficient to show (1) the upper square is Cartesian in the ordinary category, that is, the underlying map of $Z^\circ \to Z'$ is the identity and (2) $Z^\circ$ is a puncturing of $Z'$.

Note that the fiber product $Z' \times_{Y'} Y^\circ$ in the category of log schemes is defined as $(\underline{Z}, \mathcal{M} := \mathcal{M}_{Z'} \oplus_{g^*\mathcal{M}_{Y'}} g^*\mathcal{M}_{Y^\circ})$. This pushout need not, in general, be integral, so we must integralize. Note there is a canonical isomorphism

$$\mathcal{M}^{\mathrm{gp}} = \mathcal{M}^{\mathrm{gp}}_{Z'} \oplus_{g^*\mathcal{M}^{\mathrm{gp}}_{Y'}} g^*\mathcal{M}^{\mathrm{gp}}_{Y^\circ} \cong \mathcal{M}^{\mathrm{gp}}_{Z'}$$

given by $(s_1, s_2) \mapsto s_1 \cdot (g')^\flat(s_2)$, where $(g')^\flat : g^*\mathcal{M}_{Y'}^{\mathrm{gp}} \to \mathcal{M}_{Z'}^{\mathrm{gp}}$ is induced by $g'$. The integralization $\mathcal{M}^{\mathrm{int}}$ of $\mathcal{M}$ is then the image of $\mathcal{M}$ in $\mathcal{M}^{\mathrm{gp}}$, which thus can be described as the subsheaf of $\mathcal{M}_{Z'}^{\mathrm{gp}}$ generated by $\mathcal{M}_{Z'}$ and $(g')^\flat(g^*\mathcal{M}_{Y^\circ})$. Note $\mathcal{M}_{Z'}$ and $(g')^\flat(g^*\mathcal{M}_{Y^\circ})$ both lie in $\mathcal{M}_Z \oplus_{\mathcal{O}_Z^\times} \mathcal{P}_Z^{\mathrm{gp}}$, and hence we can replace $\mathcal{M}^{\mathrm{gp}}$ with this subsheaf of $\mathcal{M}^{\mathrm{gp}}$ in describing $\mathcal{M}^{\mathrm{int}}$.

It is now sufficient to show that we can define a structure map $\alpha : \mathcal{M}^{\mathrm{int}} \to \mathcal{O}_Z$ compatible with the structure maps $\alpha_{Z'} : \mathcal{M}_{Z'} \to \mathcal{O}_Z$ and $\alpha_{Y^\circ} : g^*\mathcal{M}_{Y^\circ} \to \mathcal{O}_Z$. If $s \in \mathcal{M}^{\mathrm{int}}$ is of the form $s_1 \cdot (g')^\flat(s_2)$ for $s_1 \in \mathcal{M}_{Z'}$ and $s_2 \in g^*\mathcal{M}_{Y^\circ}$, then we define

$$\alpha(s) = \alpha_{Z'}(s_1) \cdot \alpha_{Y^\circ}(s_2).$$

We need to show this is well defined. If $s_2 \in g^*\mathcal{M}_{Y'}$, then $(g')^\flat(s_2) \in \mathcal{M}_{Z'}$, and thus as $g'$ is a log morphism,

$$\alpha(s) = \alpha_{Z'}(s_1) \cdot \alpha_{Y^\circ}(s_2) = \alpha_{Z'}(s_1) \cdot \alpha_{Z'}((g')^\flat(s_2)) = \alpha_{Z'}(s).$$

In particular, $\alpha(s)$ only depends on $s$, and not on the particular representation of $s$ as a product, provided that $s_2 \in g^*\mathcal{M}_{Y'}$.

On the other hand, if $s_2 \in (g^*\mathcal{M}_{Y^\circ}) \setminus (g^*\mathcal{M}_{Y'})$, then $\alpha_{Y^\circ}(s_2) = 0$ by definition of a puncturing. So in this case $\alpha(s) = 0$. Hence to check that $\alpha$ is well defined, it is enough to show that if $s = s_1 \cdot (g')^\flat(s_2) = s_1' \cdot (g')^\flat(s_2')$ with $s_2 \in g^*\mathcal{M}_{Y'}$ but $s_2' \notin g^*\mathcal{M}_{Y'}$, then $\alpha_{Z'}(s_1) \cdot \alpha_{Y^\circ}(s_2) = \alpha_{Z'}(s_1 \cdot (g')^\flat(s_2)) = 0$. Writing $s_i = (m_i, p_i)$, $s_i' = (m_i', p_i')$ using the descriptions $\mathcal{M}_{Z'} = \mathcal{M}_Z \oplus_{\mathcal{O}_Z^\times} \mathcal{P}_Z$, $g^*\mathcal{M}_{Y^\circ} \subset g^*\mathcal{M}_Y \oplus_{\mathcal{O}_Z^\times} \mathcal{P}_Z^{\mathrm{gp}}$, we note that we must have $m_1 g^\flat(m_2) = m_1' g^\flat(m_2')$. As $s_2' \notin g^*\mathcal{M}_{Y'}$, by Condition (2) of Definition 2.1 we necessarily have $\alpha_Y(m_2') = 0$. Hence $\alpha_Z(m_1' g^\flat(m_2')) = 0$, so $\alpha_Z(m_1 g^\flat(m_2)) = 0$. We deduce that $\alpha_{Z'}(s_1 (g')^\flat(s_2)) = 0$, as desired. This shows $\alpha$ is well defined.

Finally, it is clear from the above description that $Z^\circ$ is a puncturing. By Corollary 2.7, the pre-stability of $g^\circ$ follows from the surjectivity of

$$g^{-1}(\bar{\mathcal{M}}_{Y^\circ}) \oplus \bar{\mathcal{M}}_Z \to g^{-1}(\bar{\mathcal{M}}_{Y^\circ}) \oplus^{\mathrm{f}}_{g^{-1}(\bar{\mathcal{M}}_Y)} \bar{\mathcal{M}}_Z = \bar{\mathcal{M}}_{Z^\circ},$$

where $\oplus^{\mathrm{f}}$ denotes the fibered coproduct in the category of fine monoids. ∎

**Definition 2.9.** In the situation of Proposition 2.8, we say that $Z^\circ$ is the *pullback of the puncturing* $Y^\circ$.

**Corollary 2.10.** *Consider the situation of Proposition 2.8, and suppose in addition given a pre-stable morphism $f : Y^\circ \to X$. Then the composition $f \circ g^\circ : Z^\circ \to X$ is also pre-stable.*

*Proof.* This follows immediately from the definition of pre-stability and the construction of $Z^\circ$ in the proof of Proposition 2.8. ∎

### 2.1.4 Punctured curves

Throughout the paper, we will essentially only be interested in puncturing along logarithmic structures from designated marked points of logarithmic curves. Let $\pi : C \to W$ be a logarithmic curve in the sense of [41].

(1) The underlying morphism $\underline{\pi}$ is a family of ordinary pre-stable curves with pairwise disjoint sections $p_1, \ldots, p_n$ of $\underline{\pi}$ disjoint from the critical locus of $\underline{\pi}$.

(2) $\pi$ is a proper logarithmically smooth and integral morphism of fine and saturated logarithmic schemes.

(3) If $\underline{U} \subset \underline{C}$ is the non-critical locus of $\underline{\pi}$ then $\overline{\mathcal{M}}_C|_{\underline{U}} \cong \underline{\pi}^* \overline{\mathcal{M}}_W \oplus \bigoplus_{i=1}^n p_{i*}\mathbb{N}_W$.

Note that by (3), all marked points receive a non-trivial logarithmic structure. We write $\alpha_C : \mathcal{M}_C \to \mathcal{O}_C$ for the structure map of the logarithmic structure on $C$. We call a geometric point of $\underline{C}$ *special* if it is either a marked or a nodal point.

**Definition 2.11.** A *punctured curve* over a fine and saturated logarithmic scheme $W$ is given by the following data:

$$\left(C^\circ \xrightarrow{\mathrm{p}} C \xrightarrow{\pi} W, \mathbf{p} = (p_1, \ldots, p_n)\right) \tag{2.2}$$

where

(1) $C \to W$ is a logarithmic curve in the sense of [41] with its collection of pairwise disjoint sections $p_1, \ldots, p_n$ of the underlying curve as above.

(2) $C^\circ \to C$ is a puncturing of $C$ along $\mathcal{P}$, where $\mathcal{P}$ is the divisorial logarithmic structure on $\underline{C}$ induced by the divisor $\bigcup_{i=1}^n p_i(\underline{W})$.

When there is no danger of confusion, we may call $C^\circ \to W$ a punctured curve. Sections in $\mathbf{p}$ are called *punctured sections*, or simply *punctures*. If $\underline{W} = \operatorname{Spec} \kappa$ with $\kappa$ a field, we also speak of a *punctured point*. We also say $C^\circ$ is a puncturing of $C$ along the punctured sections $\mathbf{p}$.

If locally around a punctured point $p_i$ the puncturing is trivial, we say that the punctured point is a *marked point*. In this case, the theory will agree with the treatment of marked points in [2, 15, 30].

**Examples 2.12.** (1) Let $W = \operatorname{Spec} \Bbbk$ be the point with the trivial logarithmic structure, and $C$ be a non-singular curve over $W$. Choose a closed point $p \in C$ and a puncturing $\mathcal{M}_{C^\circ}$ of $C$ at $p$. Then $\mathcal{M}_{C^\circ} = \mathcal{P}$, as $\mathcal{M}_{C^\circ} \subset \mathcal{P}^{\mathrm{gp}}$ can have no sections $s$ with $\alpha_{C^\circ}(s) = 0$. Thus, in this case the only puncturing $C^\circ \to C$ is the trivial one.

(2) Let $W = \operatorname{Spec}(\mathbb{N} \to \Bbbk)$ be the standard logarithmic point, and $C$ be a non-singular curve over $W$, so that $\mathcal{M}_C = \mathcal{O}_C^\times \oplus \underline{\mathbb{N}}$, where $\underline{\mathbb{N}}$ denotes the constant sheaf on $C$ with stalk $\mathbb{N}$. Again choose a closed point $p \in C$ with defining ideal $(x)$. Let $\mathcal{M}_{C^\circ} \subset \pi^* \mathcal{M}_W \oplus_{\mathcal{O}_C^\times} \mathcal{P}^{\mathrm{gp}}$ be a puncturing. Let $s$ be a local section of $\mathcal{M}_{C^\circ}$

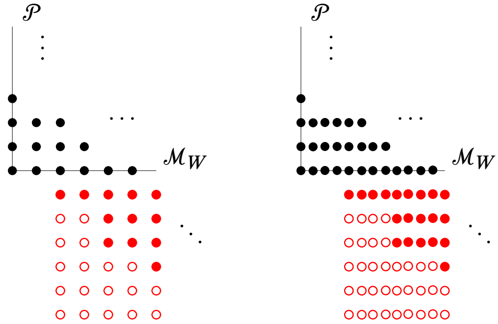


**Figure 2.3.** The solid puncturing on the left extends to $\Bbbk[\varepsilon]/(\varepsilon^2)$ but no further—the circled elements are the ones allowed for $k = 1$. Its pullback (see below) via $\mathcal{E}^2 = \varepsilon$ is pictured on the right—it is defined on $\Bbbk[\mathcal{E}]/(\mathcal{E}^4)$ but does not extend further.

near $p$. Write $s = ((\varphi, n), x^m)$ with $\varphi \in \mathcal{O}^\times_{C,p}$, $n \in \mathbb{N}$. If $m < 0$, then Condition (2) of Definition 2.1 implies that

$$\alpha_{\pi^*(\mathcal{M}_W)}(\varphi, n) = 0,$$

so we must have $n > 0$. Thus we see that

$$\overline{\mathcal{M}}_{C^\circ,p} \subset \{(n, m) \in \mathbb{N} \oplus \mathbb{Z} \mid m \geq 0 \text{ if } n = 0\}.$$

Conversely, any fine submonoid of the right-hand-side of the above inclusion which contains $\mathbb{N} \oplus \mathbb{N}$ can be realized as the stalk of the ghost sheaf at $p$ for a puncturing. Note the monoid on the right-hand side is not finitely generated, and is the stalk of the ghost sheaf of the final puncturing, see Remark 2.3.

(3) Let $\underline{W} = \operatorname{Spec} \Bbbk[\varepsilon]/(\varepsilon^{k+1})$, and let $W$ be given by the chart $\mathbb{N} \to \Bbbk[\varepsilon]/(\varepsilon^{k+1})$, $1 \mapsto \varepsilon$. Let $C_0$ be a non-singular curve over $\operatorname{Spec} \Bbbk$ with the trivial logarithmic structure, and let $C = W \times C_0$. Choose a section $p : W \to C$, with image locally defined by an equation $x = 0$. Condition (2) of Definition 2.1 now implies that a section $s$ of a puncturing $\mathcal{M}_{C^\circ}$ near $p$ takes the form $((\varphi, n), x^m)$ where $\varphi \in \mathcal{O}^\times_{C,p}$, and $0 \leq n \leq k$ implies $m \geq 0$. In particular,

$$\overline{\mathcal{M}}_{C^\circ,p} \subset \{(n, m) \in \mathbb{N} \oplus \mathbb{Z} \mid m \geq 0 \text{ if } n \leq k\},$$

and any fine submonoid of the right-hand side containing $\mathbb{N} \oplus \mathbb{N}$ can be realized as the stalk of the ghost sheaf at $p$ of a puncturing. See Figure 2.3.

### 2.1.5 Pull-backs of punctured curves

Consider a punctured curve $(C^\circ \to C \to W, \mathbf{p})$ and a morphism of fine and saturated logarithmic schemes $h : T \to W$. Denote by $(C_T \to T, \mathbf{p}_T)$ the pullback of the log curve $C \to W$ via $T \to W$. By Proposition 2.8, we obtain a commutative diagram

$$\begin{array}{ccc} C_T^\circ & \longrightarrow & C^\circ \\ {\scriptstyle \mathrm{p}_T}\downarrow & & \downarrow{\scriptstyle \mathrm{p}} \\ C_T & \longrightarrow & C \\ {\scriptstyle \pi_T}\downarrow & & \downarrow{\scriptstyle \pi} \\ T & \xrightarrow{h} & W \end{array} \tag{2.3}$$

where the bottom square is cartesian in the fine and saturated category, and the square on the top is cartesian in the fine category, and such that $C_T^\circ$ is a puncturing of the curve $C_T$ along $\mathrm{p}_T$. See again Figure 2.3.

**Definition 2.13.** We call $C_T^\circ \to T$ the *pullback* of the punctured curve $C^\circ \to W$ along $T \to W$.

### 2.1.6 Punctured maps

We now fix a morphism of fine and saturated logarithmic schemes $X \to B$.

**Definition 2.14.** A *punctured map to* $X \to B$ over a fine and saturated logarithmic scheme $W$ over $B$ consists of a punctured curve $(C^\circ \to C \to W, \mathbf{p})$ and a morphism $f$ fitting into a commutative diagram

$$\begin{array}{ccc} C^\circ & \xrightarrow{f} & X \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ W & \longrightarrow & B \end{array}$$

Such a punctured map is denoted by $(\pi : C^\circ \to W, \mathbf{p}, f)$ or $(C^\circ/W, \mathbf{p}, f)$.

The *pullback* of a punctured map $(C^\circ/W, \mathbf{p}, f)$ along a morphism of fine and saturated logarithmic schemes $T \to W$ is the punctured map $(C_T^\circ/T, \mathbf{p}_T, f_T)$ consisting of the pullback $C_T^\circ \to T$ of the punctured curve $C \to W$ and the pullback $f_T$ of $f$.

**Definition 2.15.** A punctured map $(C^\circ \to W, \mathbf{p}, f)$ is called *pre-stable* if $f : C^\circ \to X$ is pre-stable in the sense of Definition 2.6.

A pre-stable punctured map is called *stable* if its underlying map, marked by the punctured sections, is stable in the usual sense.

**Proposition 2.16.** *Let $(C^\circ/W, \mathbf{p}, f)$ be a punctured map over $W$.*

(1) *The locus of points of $W$ with pre-stable fibers forms an open sub-scheme of $\underline{W}$.*

(2) *If* $f : C^\circ \to X$ *is pre-stable, then the pullback* $f_T : C^\circ_T \to X$ *along any morphism of fine and saturated logarithmic schemes* $T \to W$ *is also pre-stable.*

*Proof.* The map $f : C^\circ \to X$ induces a morphism of fine logarithmic structures

$$f^\flat \oplus \mathrm{p}^\flat : f^*\mathcal{M}_X \oplus_{\mathcal{O}^\times_C} \mathcal{M}_C \to \mathcal{M}_{C^\circ}.$$

The pre-stability of $f$ is equivalent to the condition that $f^\flat \oplus \mathrm{p}^\flat$ is surjective by Corollary 2.7. Statement (1) can be proved by applying Lemma 2.17 to the neighborhood of each puncture. Statement (2) follows immediately from Corollary 2.10. ■

**Lemma 2.17.** *Let* $\underline{Y}$ *be a scheme, and* $\psi^\flat : \mathcal{M} \to \mathcal{N}$ *be a morphism of fine log structures on* $\underline{Y}$. *Then the locus* $\underline{Y}' \subset \underline{Y}$ *over which* $\psi^\flat$ *is surjective forms an open subscheme of* $\underline{Y}$.

*Proof.* We thank the anonymous referee for suggesting the following simplified proof. Since both $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{O}^\times_{\underline{Y}}$-torsors over $\overline{\mathcal{M}}$ and $\overline{\mathcal{N}}$ respectively, the surjectivity of $\psi^\flat$ is equivalent to the surjectivity of the induced morphism $\overline{\mathcal{M}} \to \overline{\mathcal{N}}$ of ghost sheaves. Since the statement is local on $\underline{Y}$, we may assume that $\overline{\mathcal{N}}$ is globally generated.

Suppose $y \in \underline{Y}$ is a geometric point over which $\overline{\mathcal{M}}_y \to \overline{\mathcal{N}}_y$ is surjective. Then each global section of $\overline{\mathcal{N}}$ lifts to a section of $\overline{\mathcal{M}}$ in an étale neighborhood of $y$. Since $\Gamma(\underline{Y}, \overline{\mathcal{N}})$ is finitely generated, there is a common étale neighborhood of $y$ over which all the global sections of $\overline{\mathcal{N}}$ lift to $\overline{\mathcal{M}}$. This finishes the proof. ■

The most interesting aspect of punctured curves is the appearance of negative contact orders, defined as follows.

**Definition 2.18.** The *contact order* of a punctured map $(C^\circ/W, \mathbf{p}, f)$ to $X \to B$ over a log point $W = \mathrm{Spec}(Q \to \kappa)$ at $p \in \mathbf{p}$ is the composition

$$u_p : \overline{\mathcal{M}}_{X,\underline{f}(p)} \xrightarrow{f^\flat} \overline{\mathcal{M}}_{C,p} \to Q \oplus \mathbb{Z} \xrightarrow{\mathrm{pr}_2} \mathbb{Z} \tag{2.4}$$

with the second map the canonical inclusion. We say that the contact order $u_p$ is *negative* if $u_p(\overline{\mathcal{M}}_{X,\underline{f}(p)}) \not\subseteq \mathbb{N}$.

The difference with the case of logarithmic stable maps [30, Definition 1.8] is the appearance of $\mathbb{Z}$ instead of $\mathbb{N}$. The tropical interpretation of this condition will be discussed in Section 2.2 below. Note that if $(C^\circ/W, \mathbf{p}, f)$ is pre-stable, the contact order at $p \in \mathbf{p}$ is negative if and only if $p$ is not a marked point.

**Example 2.19.** Here is a simple example featuring a negative contact order. Let $\underline{X}$ be a smooth surface, $\underline{D} \subseteq \underline{X}$ a non-singular rational curve with self-intersection $-1$

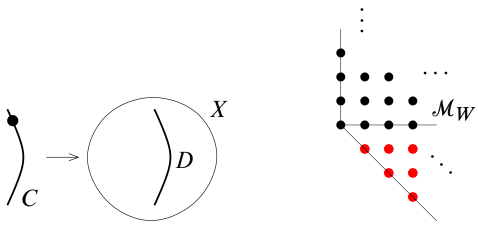


**Figure 2.4.** The $(-1)$-curve and its monoid.

inducing the divisorial log structure $X$ on $\underline{X}$. Let $C \to W$ be the punctured curve of Example 2.12 (2), with $C \cong \mathbb{P}^1$. Let $\underline{f} : \underline{C} \to \underline{X}$ be an isomorphism of $\underline{C}$ with $\underline{D}$. This can be enhanced to a punctured map $C^\circ \to X$ as follows.

We first define $\bar{f}^\flat : \underline{f}^* \overline{\mathcal{M}}_X = \underline{\mathbb{N}} \to \overline{\mathcal{M}}_{C^\circ} \subseteq \overline{\mathcal{E}} = \underline{\mathbb{N}} \oplus \mathbb{Z}_p$ by $1 \mapsto (1, -1)$, where $\mathbb{Z}_p$ denotes the sky-scraper sheaf at $p$ with stalk $\mathbb{Z}$. Note that the inverse image of $1 \in \Gamma(X, \overline{\mathcal{M}}_X)$ under the projection map $\mathcal{M}_X \to \overline{\mathcal{M}}_X$ is the $\mathcal{O}_X^\times$-torsor contained in $\mathcal{M}_X$ corresponding to the line bundle $\mathcal{O}_X(-D)$, and thus $1 \in \Gamma(C, \underline{f}^* \overline{\mathcal{M}}_X)$ similarly yields the $\mathcal{O}_C^\times$-torsor corresponding to $\mathcal{O}_C(1)$, using $-D^2 = 1$. On the other hand, the torsor contained in $\mathcal{M}_{C^\circ}$ corresponding to $(1, 0)$ is the torsor of $\mathcal{O}_C$, and the torsor corresponding to $(0, 1)$ is the torsor of the ideal $\mathcal{O}_C(-p)$. Hence $(1, -1) \in \Gamma(C, \overline{\mathcal{M}}_{C^\circ})$ corresponds to $\mathcal{O}_C(1)$. Choosing an isomorphism of torsors then lifts the map $\bar{f}^\flat$ to a map $f^\flat : \underline{f}^* \mathcal{M}_X \to \mathcal{M}_{C^\circ}$ inducing a morphism $f : C^\circ \to X$ (Figure 2.4).

Note this morphism does not lift to $C' \to W' = \mathrm{Spec}(\Bbbk[\varepsilon]/(\varepsilon^2))$ as in Example 2.12 (3), since we cannot even lift $\bar{f}^\flat$ at the level of ghost sheaves. Indeed, $(1, -1)$ is not a section of the ghost sheaf of $(C')^\circ$.

**Remark 2.20** (Geometric implication of negative contact orders). Let $f : C^\circ / W \to X$ be a punctured map with $W = \mathrm{Spec}(Q \to \Bbbk)$. Suppose $p \in C$ is a punctured point which is not a marked point, and let $C'$ be the irreducible component containing $p$, with generic point $\eta$. Then, intuitively, $C'$ has negative order of tangency with certain strata in $X$, and this forces $C'$ to be contained in those strata.

Explicitly, let $P_p = \overline{\mathcal{M}}_{X, \underline{f}(p)}$ and let $u_p : P_p \to \mathbb{Z}$ be as in Definition 2.2. Then if $\delta \in P_p$ with $u_p(\delta) < 0$, we must have $\mathrm{pr}_1 \circ \bar{f}_p(\delta) \neq 0$, as there is no element of $\overline{\mathcal{M}}_{C^\circ, p} \subset Q \oplus \mathbb{Z}$ of the form $(0, n)$ with $n < 0$. Thus if $\chi : P_p \to \overline{\mathcal{M}}_{X, \underline{f}(\eta)}$ denotes the generization map, we must have $u_p^{-1}(\mathbb{Z}_{<0}) \cap \chi^{-1}(0) = \emptyset$. This restricts the strata in which $f(C')$ can lie.

For example, if $X = (\underline{X}, D)$ for a simple normal crossings divisor $D$ with irreducible components $D_1, \ldots, D_n$, then $P_p = \bigoplus_{i : f(p) \in D_i} \mathbb{N}$. The value $u_p$ on the generator of $P_p$ corresponding to $D_i$ is the contact order with $D_i$. Then $f(C')$ must lie in the intersection of those $D_i$ that have negative contact order at $p$.

A critical aspect of this phenomenon is discussed in Section 2.5, see especially Proposition 2.52 and Example 2.54.

## 2.2 The tropical interpretation

We now introduce the tropical picture, which gives the underlying organizing language for punctured Gromov–Witten theory. We assume familiarity with the discussion in ordinary logarithmic Gromov–Witten theory as presented in [3, Section 2]. We review in Section 2.2.1 the notations and basic concepts briefly while discussing the modifications needed for including non-trivial punctures.

### 2.2.1 Tropical punctured maps

In Appendix C we define tropicalization as a functor associating to a fine log algebraic stack a generalized cone complex $\Sigma(X)$. There is one stratum of $|\Sigma(X)|$ for each logarithmic stratum of $X$, the latter defined as a maximal connected locally closed subset $Z \subseteq |\underline{X}|$ with $\overline{\mathcal{M}}_X|_Z$ locally constant. For each logarithmic stratum $Z$ we choose, once and for all, a geometric point $\bar{x}_Z$ with image in $Z$. Then $\Sigma(X)$ is defined as the diagram with only one cone

$$\sigma_Z = \mathrm{Hom}\big(\overline{\mathcal{M}}_{X,\bar{x}_Z}, \mathbb{R}_{\geq 0}\big) \tag{2.5}$$

for each logarithmic stratum $Z$, along with all its faces, and arrows induced by all sequences of generization morphisms and all face inclusions, including inverses of those that are isomorphisms. Note that due to monodromy, $\Sigma(X)$ may contain non-trivial arrows $\sigma \to \sigma$. The group

$$\mathrm{Aut}_{\Sigma(X)}(\sigma) = \{\sigma \to \sigma \text{ arrow in } \Sigma(X)\}$$

is a subgroup of the permutation group of the set of rays of $\sigma$, hence is always finite. Note that the map

$$\sigma_Z / \mathrm{Aut}_{\Sigma(X)}(\sigma_Z) \to |\Sigma(X)|$$

induced from $\sigma_Z \to |\Sigma(X)|$ may still not be injective due to arrows from strata of $X$ whose closure intersect the closure of $Z$ and that are not induced by monodromy on $Z$. Accordingly, the image of $\sigma$ in $|\Sigma(X)|$ may be a finite quotient even on its interior.

By abuse of notation, $\Sigma(X)$ denotes both the distinguished presentation or the equivalence class as a generalized cone complex. When writing $\sigma \in \Sigma(X)$ we refer to the chosen presentation, so there is a unique logarithmic stratum $Z \subseteq X$ with $\sigma = \sigma_Z$. For any geometric point $\bar{x}$ with image in $Z$ we have the cone

$$\sigma_{\bar{x}} = \mathrm{Hom}\big(\overline{\mathcal{M}}_{X,\bar{x}}, \mathbb{R}_{\geq 0}\big)$$

together with an isomorphism

$$\sigma_Z \to \sigma_{\bar{x}},$$

but this isomorphism is only unique up to pre-composition with arrows $\sigma_Z \to \sigma_Z$ in $\Sigma(X)$. In other words, the isomorphism $\sigma_Z \to \sigma_{\bar{x}}$ is unique up to the action of the monodromy group $\mathrm{Aut}_{\Sigma(X)}(\sigma_Z)$ of the logarithmic stratum $Z$.

For $\sigma \in \Sigma(X)$ we denote by

$$X_\sigma = \big\{x \in \underline{X} \mid \text{ there exists an arrow } \sigma \to \sigma_{\bar{x}} \text{ in } \Sigma(X)\big\} \subseteq \underline{X} \tag{2.6}$$

the closed set of points $x \in \underline{X}$ with $\sigma$ connected to $\sigma_{\bar{x}} = \mathrm{Hom}(\overline{\mathcal{M}}_{X,\bar{x}}, \mathbb{R}_{\geq 0})$ by a sequence of generizations and inverses of invertible generizations of the stalks of $\overline{\mathcal{M}}_X$. We endow $X_\sigma$ with the reduced induced scheme structure. In practice, say when $X$ is log smooth over a log point, $X_\sigma$ is the closure of the logarithmic stratum given by $\sigma \in \Sigma(X)$. For brevity, we refer to the $X_\sigma$ as *strata* of $X$, but note that from the point of view of stratified spaces, and differing from the use in Appendix C, these are at best closures of strata. Note also that for $\sigma = \{0\}$ we obtain $X_{\{0\}} = \underline{X}$ assuming $\Sigma(X)$ connected, even if there is no geometric point $\bar{x}$ of $X$ with $\overline{\mathcal{M}}_{X,\bar{x}} = 0$.

A stable logarithmic map $(C/W, \mathbf{p}, f)$ gives rise via functoriality of the tropicalization functor $\Sigma$ to the diagram

$$\begin{array}{ccc} \Sigma(C) & \xrightarrow{\Sigma(f)} & \Sigma(X) \\ {\scriptstyle \Sigma(\pi)}\downarrow & & \downarrow \\ \Sigma(W) & \longrightarrow & \Sigma(B) \end{array} \tag{2.7}$$

We will almost exclusively consider such diagrams in which $W$ is covered by a single chart and $\Sigma(W)$ has a single maximal cone $\omega = (\mathcal{M}^\vee_{W,\bar{w}})_\mathbb{R}$ for $\bar{w}$ some geometric point of $\underline{W}$. Then it is shown in [3, Proposition 2.25] that $\Sigma(\pi)$ along with the genera of the irreducible components of the geometric fiber $C_{\bar{w}}$ is a (family of) abstract tropical curves over $\omega$, also written $(G, \mathbf{g}, \ell)$. Here $G$ is the dual intersection graph of $C_{\bar{w}}$ with sets $V(G)$, $E(G)$, $L(G)$ of vertices, edges and legs, and the maps

$$\mathbf{g} : V(G) \to \mathbb{N}, \quad \ell : E(G) \to \mathrm{Hom}(\omega_\mathbb{Z}, \mathbb{N}) \setminus \{0\},$$

record the genera of the irreducible components of $C_{\bar{w}}$ and the lengths of edges as functions on $\omega$, respectively, see [3, Definition 2.19]. If $G$ arises from the tropicalization of a log curve over a geometric logarithmic point, we denote elements of $V(G)$, $E(G)$, $L(G)$ both by their graph-theoretic notations as vertices $v$, edges $E$, and legs $L$, or the corresponding algebraic geometric notations as generic points $\eta$, nodes $q$, and marked points $p$. By abuse of notation, we view homomorphisms $\omega_\mathbb{Z} \to \mathbb{N}$ also as homomorphisms $\omega \to \mathbb{R}_{\geq 0}$ respecting the integral structure. Conversely, from $(G, \mathbf{g}, \ell)$, the cone complex

$$\Gamma = \Gamma(G, \ell)$$

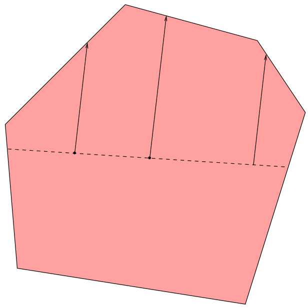

**Figure 2.5.** The length of a bounded leg varies piecewise linearly under linear variations of the adjacent vertex. The figure shows the intersection of the situation with an affine hyperplane.

recovering $\Sigma(C)$ has one copy of $\omega$ for each $v \in V(G)$, a cone

$$\omega_E = \{(s,\lambda) \in \omega \times \mathbb{R}_{\geq 0} \mid \lambda \leq \ell(E)(s)\} \tag{2.8}$$

for each edge $E \in E(G)$, and a copy of $\omega \times \mathbb{R}_{\geq 0}$ for each leg. Note that legs have infinite lengths for any parameter $s \in \omega_{\mathbb{R}}$ when viewing $\Gamma$ as a family of metric graphs.

The only change in the punctured setup is that a leg may now have finite length. Indeed, if $L \in L(G)$ corresponds to a non-trivial puncture with puncturing submonoid $Q^\circ \subset Q \oplus \mathbb{Z}$, then $(Q^\circ)^\vee_{\mathbb{R}} = \omega_L$ with

$$\omega_L = \{(s,\lambda) \in \omega \times \mathbb{R}_{\geq 0} \mid \lambda \leq \ell(L)(s)\} \tag{2.9}$$

defined in analogy with (2.8) by a length function $\ell(L) : \omega \to \mathbb{R}_{\geq 0}$ with $\ell(L) \neq 0$. Note, however, that $\ell(L)$ is now only piecewise linear as illustrated in Figure 2.5. Here a continuous function $\ell : \omega \to \mathbb{R}_{\geq 0}$ on $\omega \in$ **Cones** is *piecewise linear* if there exists a fan subdivision of $\omega$ such that $\ell$ is the restriction of a linear form on each cone of the fan. For the following relation to monoids recall (2.1) from Remark 2.2.

**Lemma 2.21.** *Let $Q$ be a sharp toric monoid and $\omega = Q^\vee_{\mathbb{R}}$. Assume further that $Q^\circ \subseteq Q \oplus \mathbb{Z}$ is a finitely generated submonoid with $Q \oplus \mathbb{N} \subsetneq Q^\circ$, $Q^\circ \cap (\{0\} \times \mathbb{Z}_{<0}) = \emptyset$. Then there exists a nonzero, concave, piecewise linear function*

$$\ell : \omega \to \mathbb{R}_{\geq 0}$$

*with rational slopes such that*

$$(Q^\circ)^\vee_{\mathbb{R}} = \{(s,\lambda) \in \omega \times \mathbb{R}_{\geq 0} \mid 0 \leq \lambda \leq \ell(s)\}. \tag{2.10}$$

*Each such $\ell : \omega \to \mathbb{R}_{\geq 0}$ arises in this fashion, and two submonoids $Q_1^\circ, Q_2^\circ \subseteq Q \oplus \mathbb{Z}$ with $Q_i \oplus \mathbb{N} \subsetneq Q_i^\circ$, $Q_i^\circ \cap (\{0\} \times \mathbb{Z}_{<0}) = \emptyset$, $i = 1, 2$, lead to the same $\ell$ if and only if $(Q_1^\circ)^{\mathrm{sat}} = (Q_2^\circ)^{\mathrm{sat}}$.*

*Proof.* Let $(s,\lambda)\in(Q^\circ)^\vee_{\mathbb{R}}$. Then since $Q\oplus\mathbb{N}\subseteq Q^\circ$, necessarily $s\in\omega=Q^\vee_{\mathbb{R}}$ and $\lambda\geq 0$. Conversely, $(s,0)\in(Q^\circ)^\vee_{\mathbb{R}}$ for all $s\in\omega$, and in fact,

$$\omega\times\{0\}\subseteq(Q^\circ)^\vee_{\mathbb{R}}$$

is a facet. Since $Q^\circ\neq Q\oplus\mathbb{N}$ no ray of $(Q^\circ)^\vee_{\mathbb{R}}$ is vertical, that is, agrees with $\mathbb{R}_{\geq 0}\cdot(0,1)$. Thus the union of the maximal cells of $\partial(Q^\circ)^\vee_{\mathbb{R}}$ neither contained in $\omega\times\{0\}$ nor in $\partial\omega\times\mathbb{R}$ form the graph of a piecewise linear function $\ell:\omega\to\mathbb{R}_{\geq 0}$ as in the statement of the lemma. Convexity of $(Q^\circ)^\vee_{\mathbb{R}}$ implies that $\ell$ is concave. Finally, $\ell\neq 0$ for otherwise $(0,-1)\in Q^\circ_{\mathbb{R}}$, contradicting $Q^\circ\cap(\{0\}\times\mathbb{Z}_{<0})=\emptyset$.

Conversely, given a nonzero, concave, piecewise linear $\ell:\omega\to\mathbb{R}_{\geq 0}$ with rational slopes, the cone $\sigma$ on the right-hand side of (2.10) contains $\omega\times\{0\}$ as a facet. Hence

$$\sigma^\vee\subseteq\omega^\vee\times\mathbb{R}=Q_{\mathbb{R}}\times\mathbb{R}\quad\text{and}\quad Q_{\mathbb{R}}\times\mathbb{R}_{\geq 0}\subseteq\sigma^\vee.$$

The case $Q_{\mathbb{R}}\times\mathbb{R}_{\geq 0}=\sigma^\vee$ does not occur since $\sigma\neq\omega\times\mathbb{R}_{\geq 0}$ by finiteness of the values of $\ell$. Moreover, $\ell\neq 0$ implies $\sigma$ is a full-dimensional cone, and hence $(0,-1)\notin\sigma^\vee$, or $\sigma^\vee\cap(\{0\}\times\mathbb{Z}_{<0})=\emptyset$. This shows that knowing $\ell$ retrieves the convex hull of $Q^\circ$ in $Q_{\mathbb{R}}\times\mathbb{R}$, hence the set of integral points of its saturation $(Q^\circ)^{\mathrm{sat}}$. ∎

**Definition 2.22.** (1) A (family of) *punctured tropical curves* over a cone $\omega\in\mathbf{Cones}$ is a graph $G$ together with two maps

$$\mathbf{g}:V(G)\to\mathbb{N},\quad \ell:E(G)\cup L^\circ(G)\to\mathrm{Map}(\omega,\mathbb{R}_{\geq 0})$$

for some subset $L^\circ(G)\subseteq L(G)$, with $\ell(E)\in\mathrm{Hom}(\omega_{\mathbb{Z}},\mathbb{N})\setminus\{0\}$ for $E\in E(G)$ and $\ell(L):\omega\to\mathbb{R}_{\geq 0}$ for $L\in L^\circ(G)$ nonzero, concave, piecewise linear, with rational slopes. We refer to elements of $L^\circ(G)$ as *finite or punctured legs*, all other legs as *infinite or marked*.

(2) A (family of) *punctured tropical maps* over $\omega\in\mathbf{Cones}$ is a map of generalized cone complexes $h:\Gamma\to\Sigma(X)$ for $\Gamma=\Gamma(G,\ell)$ the cone complex defined by a punctured tropical curve $(G,\mathbf{g},\ell)$ over $\omega$.

For readability and as in [3] throughout, we assume for the rest of this subsection that $\Sigma(X)$ is simple [3, Definition 2.1]. This means that for each $\sigma\in\Sigma(X)$ the map $\sigma\to|\Sigma(X)|$ is injective. We will treat the general case in Section 2.6. As in [3, Proposition 2.26], it then follows readily from the definitions that the tropicalization of a punctured map to $X$ over a logarithmic point $\mathrm{Spec}(Q\to\kappa)$ with $\kappa$ algebraically closed is a punctured tropical map over $Q^\vee_{\mathbb{R}}$.

Given a punctured tropical map, one extracts associated discrete data as in [3, Remark 2.22]. These are an *image cone* map

$$\boldsymbol{\sigma}:V(G)\cup E(G)\cup L(G)\to\Sigma(X)\tag{2.11}$$

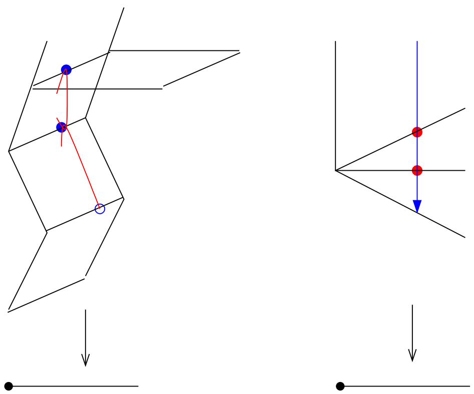

**Figure 2.6.** A curve in the fiber of a one-parameter family of surfaces (a threefold) and its tropicalization. There are two components, represented by vertices; one node represented by an edge; one regular marked point represented by an infinite leg and one puncture represented by a finite leg, which, by pre-stability, extends exactly as far as the cone allows.

associating to each object of $G$ the (distinguished representative of the) minimal cone of $\Sigma(X)$ it maps to, and, referring again to the notation in Section 1.6, *contact orders*

$$u_q = u_E \in N_{\boldsymbol{\sigma}(E)}, \quad u_p = u_L \in N_{\boldsymbol{\sigma}(L)} \tag{2.12}$$

for edges $E = E_q \in E(G)$ and for legs $L = L_p \in L(G)$, respectively.

Contact orders are defined by the image of the tangent vector $(0, 1)$ in the tangent space $N_\omega \times \mathbb{Z}$ of $\omega_E$ or $\omega_L$ under $h$. The contact order for an edge $E$ depends, up to sign, on a choice of orientation on $E$, which we suppress in the notation. For legs, this definition is consistent with the definition of contact orders of punctured maps in Definition 2.18.

Note that the contact order $u_p \in N_{\boldsymbol{\sigma}(L_p)}$ of a marked point $p \in C_{\bar{w}}$ lies in $\boldsymbol{\sigma}(L_p)$. Conversely, a non-trivial puncture is forced by a leg $L = L_p$ if for any parameter value $s \in \omega$, the line segment $h(\{s\} \times [0, \ell(L)(s)])$ inside the image cone $\boldsymbol{\sigma}(L) \in \Sigma(X)$ does not extend to a half-line.

There is a simple tropical interpretation of pre-stability saying that images of legs extend as far as possible inside their image cones. See Figure 2.6 for an illustration. We call such tropical punctured maps *pre-stable*.

**Proposition 2.23.** *Let $(C^\circ/W, \mathbf{p}, f)$ be a pre-stable punctured map over a log point $W = \mathrm{Spec}(Q \to \kappa)$ and $h = \Sigma(f) : \Gamma(G, \ell) \to \Sigma(X)$ its tropicalization. For each finite leg $L \in L^\circ(G)$, we write $\omega_L \subseteq \omega \times \mathbb{R}_{\geq 0}$ as in (2.9). Then for all $s \in \omega$, we have*

$$h(s, \ell(L)(s)) = h(s, 0) + \ell(L)(s) \cdot u_L \in \partial\boldsymbol{\sigma}(L),$$

*while $h(s, \ell(L)(s)) + \varepsilon u_L \notin \boldsymbol{\sigma}(L)$ for all $\varepsilon > 0$.*

*Proof.* Let $\bar{p} \to C$ be the punctured point defined by $L$, and write $\omega = Q_{\mathbb{R}}^{\vee}$, $\sigma = P_{\mathbb{R}}^{\vee}$ for $P = \overline{\mathcal{M}}_{X,\underline{f}(\bar{p})}$. The map $h_L : \omega_L \to \sigma$ defined by $h$ is dual to

$$\bar{f}^{\flat}_{\underline{f}(\bar{p})} : P \to \overline{\mathcal{M}}_{C,\bar{p}} = Q^{\circ} \subset Q \oplus \mathbb{Z}.$$

By pre-stability, $Q^{\circ}$ is generated by $Q \oplus \mathbb{N}$ and by the image of $\bar{f}^{\flat}_{\underline{f}(\bar{p})}$. Dually, we obtain

$$\omega_L = (Q^{\circ})^{\vee}_{\mathbb{R}} = (\omega \times \mathbb{R}_{\geq 0}) \cap h_L^{-1}(\sigma).$$

Now $\omega_L$ is the convex hull of $\omega \times \{0\}$ and of $\{(s, \ell(L)(s)) \in \omega \times \mathbb{R}_{\geq 0}\}$, the graph of $\ell(L)$ as a map $\omega \to \mathbb{R}_{\geq 0}$. This shows that no point $(s, \ell(L)(s))$ maps to an interior point of $\sigma$, and the line segment in $\sigma$ connecting $h(s, 0)$ with $h(s, \ell(L)(s))$ can not be extended, as claimed. ■

Note that while $\omega_L^{\vee} \cap (N_{\omega} \times \mathbb{Z})^*$ only computes the saturation of $Q^{\circ}$, the tropical picture also contains the map $P \to Q \oplus \mathbb{Z}$. In the pre-stable case, $Q^{\circ}$ is then the submonoid generated by the image of this map and by $Q \oplus \mathbb{N}$, so can be fully computed tropically.

### 2.2.2 Types of punctured maps

As in [3, Definition 2.23] for stable logarithmic maps, we now capture the combinatorics underlying punctured maps and their tropicalization by the notion of *type*.

**Definition 2.24.** (1) The *type of a family of tropical punctured maps* $h : \Gamma = \Gamma(G, \ell) \to \Sigma(X)$ over $\omega \in$ **Cones** is the tuple

$$\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u})$$

consisting of the associated genus-decorated connected graph $(G, \mathbf{g})$, the image cone map $\boldsymbol{\sigma}$ from (2.11) and the collection $\mathbf{u} = \{u_q, u_p\}_{p,q} = \{u_E, u_L\}_{E,L}$ of contact orders from (2.12). In particular, for $x \in E(G) \cup L(G)$ we require $u_x \in N_{\boldsymbol{\sigma}(x)}$. We also sometimes write $\mathbf{u}(x)$ instead of $u_x$ when referring to a contact order given by a type rather than by a punctured map.

(2) The *type of a punctured map* $(C/W, \mathbf{p}, f)$ *to* $X$ *at a geometric point* $\bar{w}$ *of* $\underline{W}$ is the type of the associated tropical map $\Gamma \to \Sigma(X)$ over $\omega = (\overline{\mathcal{M}}^{\vee}_{W,\bar{w}})_{\mathbb{R}}$.

Thus the type records the combinatorial data associated to $h : \Gamma \to \Sigma(X)$, but forgets the length function $\ell : E(G) \cup L^{\circ}(G) \to \mathrm{Map}(\omega, \mathbb{R}_{\geq 0})$.

For a punctured map over a logarithmic point, one sometimes also wants to keep the curve classes $\mathbf{A}(v) = \underline{f}_*([\underline{C}(v)])$ for $\underline{C}(v) \subset \underline{C}$ the irreducible component of $\underline{C}$ given by $v \in V(G)$. Here $\mathbf{A}(v)$ is a class of curves in singular homology of the corresponding stratum $X_{\boldsymbol{\sigma}(v)}$, or in some other appropriate monoid of curve classes, written $H_2^+(X_\sigma)$ for $\sigma \in \Sigma(X)$ in any case.[1] We refer to [33, Basic setup 1.6] for a listing of the properties of $H_2^+$ assumed throughout. Adding this information, one arrives at the notion of *decorated type*

$$\boldsymbol{\tau} = (\tau, \mathbf{A}) = (G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u}, \mathbf{A}). \tag{2.13}$$

Finally, just as in logarithmic Gromov–Witten theory, generization of punctured maps gives rise to contraction morphisms of graphs: Let $(C^\circ/W, \mathbf{p}, f)$ be a punctured map to $X$ and let $\bar{w}' \to \bar{w}$ be a specialization arrow of geometric points of $W$. Denote by $h : \Gamma = \Gamma(G, \ell) \to \Sigma(X)$ and $h' : \Gamma' = \Gamma(G', \ell') \to \Sigma(X)$ the tropicalizations of the strict restrictions of $(C^\circ/W, \mathbf{p}, f)$ to $\bar{w}, \bar{w}'$. Then as in [3, eq. (2.15)], generization defines a contraction morphism of the associated decorated graphs

$$\phi : (G, \mathbf{g}) \to (G', \mathbf{g}'),$$

given by contracting those edges $E = E_q \in E(G)$ with corresponding node $\bar{q} \to \underline{C}_{\bar{w}}$ not contained in the closure of the nodal locus of $\underline{C}_{\bar{w}'}$. By abuse of notation we write $\phi$ also for the maps

$$V(G) \to V(G'), \quad L(G) \to L(G'), \quad E(G) \setminus E_\phi \xrightarrow{\text{bij}} E(G')$$

defining $\phi$. Here $E_\phi \subseteq E(G)$ is the subset of contracted edges. Analogous to [3, Definition 2.24] there is a corresponding natural notion of contraction morphism of (decorated) types of tropical punctured maps

$$\begin{aligned} \tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u}) &\to \tau' = (G', \mathbf{g}', \boldsymbol{\sigma}', \mathbf{u}'), \\ \boldsymbol{\tau} = (G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u}, \mathbf{A}) &\to \boldsymbol{\tau}' = (G', \mathbf{g}', \boldsymbol{\sigma}', \mathbf{u}', \mathbf{A}'). \end{aligned} \tag{2.14}$$

Under such contraction morphisms, legs never get contracted. Moreover, identifying $L(G) = L(G')$, the contact order $\mathbf{u}(L) \in N_{\boldsymbol{\sigma}(L)}$ of a leg of $G$ is the image of $\mathbf{u}'(L) \in N_{\boldsymbol{\sigma}'(L)}$ under the inclusion of lattices $N_{\boldsymbol{\sigma}'(L)} \to N_{\boldsymbol{\sigma}(L)}$ induced by the face map $\boldsymbol{\sigma}'(L) \to \boldsymbol{\sigma}(L)$. An analogous statement applies to contact orders of non-contracted edges.

**Proposition 2.25.** *Let $(C^\circ/W, \mathbf{p}, f)$ be a stable punctured map to $X$ over some logarithmic scheme $W$ and $(\tau_{\bar{w}}, \mathbf{A}_{\bar{w}})$ with $\tau_{\bar{w}} = (G_{\bar{w}}, \mathbf{g}_{\bar{w}}, \boldsymbol{\sigma}_{\bar{w}}, \mathbf{u}_{\bar{w}})$ its decorated type at the geometric point $\bar{w} \to W$ according to Definition 2.24 and (2.13).*

*Then if $\bar{w}' \to \bar{w}$ is a specialization arrow of geometric points of $W$, the map*

$$(\tau_{\bar{w}}, \mathbf{A}_{\bar{w}}) \to (\tau_{\bar{w}'}, \mathbf{A}_{\bar{w}'})$$

*induced by generization is a contraction morphism.*

[1] The notation allows *defining* $H_2^+(X_\sigma) := H_2^+(X)$ for all $\sigma \in \Sigma(X)$, by interpreting classes of curves in a stratum $X_\sigma$ as classes of curves in $X$.

*Proof.* The proof is essentially identical to [3, Lemma 2.30], noting that the proof of [30, Lemma 1.11] also works for contact orders at punctures. ∎

### 2.2.3 The balancing condition

The above discussion fits well with the tropical balancing condition at vertices of the dual graph of $C^\circ$. In fact, the statement [30, Proposition 1.15] holds unchanged as there is no balancing condition at the endpoint of a leg $L \in L(G)$. As we will need the balancing condition to prove boundedness, we review this statement here. We note that the balancing conditions discussed here are heavily used in applications such as [33], [27, Section 4] or [32], as balancing severely limits the possible combinatorial types.

Suppose given a stable punctured map $(C^\circ/W, \mathbf{p}, f)$ with $W = \operatorname{Spec}(\mathbb{N} \to \kappa)$ the standard log point over an algebraically closed field, and denote by $(G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u})$ its type. Let $g : \widetilde{D} \to C$ be the normalization of an irreducible component $D$ with generic point $\eta$ of $C$. One then obtains, with $\overline{\mathcal{M}} = \underline{f}^*\overline{\mathcal{M}}_X$, composed maps

$$\tau_\eta^X : \Gamma(\widetilde{D}, g^*\overline{\mathcal{M}}) \to \operatorname{Pic} \widetilde{D} \xrightarrow{\deg} \mathbb{Z}$$

$$\tau_\eta^C : \Gamma(\widetilde{D}, g^*\overline{\mathcal{M}}_{C^\circ}) \to \operatorname{Pic} \widetilde{D} \xrightarrow{\deg} \mathbb{Z}$$

with the first map on each line given by taking a section of the ghost sheaf to the corresponding $\mathcal{O}_{\widetilde{D}}^\times$-torsor, the inverse image of this section in $g^*\mathcal{M}$ or $g^*\mathcal{M}_{C^\circ}$. These are compatible: the pullback of $f^\flat$ to $\widetilde{D}$, $\varphi : g^*\mathcal{M} \to g^*\mathcal{M}_{C^\circ}$, induces $\overline{\varphi} : g^*\overline{\mathcal{M}} \to g^*\overline{\mathcal{M}}_{C^\circ}$ and a commutative diagram

$$\begin{array}{ccc} \Gamma(\widetilde{D}, g^*\overline{\mathcal{M}}) & \xrightarrow{\overline{\varphi}} & \Gamma(\widetilde{D}, g^*\overline{\mathcal{M}}_{C^\circ}) \\ & \searrow^{\tau_\eta^X} & \downarrow{\scriptstyle \tau_\eta^C} \\ & & \mathbb{Z} \end{array}$$

The map $\tau_\eta^X$ is given by $\underline{f}$ and $\mathcal{M}$, so depends on the logarithmic geometry of $f : C^\circ \to X$; however if $\underline{f}$ contracts $D$, then $\tau_\eta^X = 0$. On the other hand, $\tau_\eta^C$ is determined completely by the geometry of $D \subseteq C$ and $g^*\overline{\mathcal{M}}_{C^\circ}$ as follows. We use the notation in [30, Section 1.4]. For each point $q \in D$ over a node of $\underline{C}$ we have $\overline{\mathcal{M}}_{C^\circ,\bar{q}} = S_{e_q}$, the submonoid of $\mathbb{N}^2$ generated by $(0, e_q)$, $(e_q, 0)$ and $(1, 1)$. The generization map $\chi_q : \overline{\mathcal{M}}_{C^\circ,\bar{q}} \to \overline{\mathcal{M}}_{C^\circ,\bar{\eta}} = \mathbb{N}$ is given by projection to the second coordinate: $\chi_q(a, b) = b$. In what follows, we use $q$ always to denote points over nodes and $p$ to denote punctured points. We then have

$$\Gamma\big(\widetilde{D}, g^*\overline{\mathcal{M}}_{C^\circ}\big) \subseteq \big\{(n_q)_{q\in\widetilde{D}} \mid n_q \in S_{e_q} \text{and } \chi_q(n_q) = \chi_{q'}(n_{q'}) \text{ for } q, q' \in \widetilde{D}\big\} \oplus \bigoplus_{p\in\widetilde{D}} \mathbb{Z}.$$

This inclusion induces an equality at the level of groups. The equation $\chi_q(n_q) = \chi_{q'}(n_{q'})$ allows us to write $b = b_q = \chi_q(n_q)$ independent of $q$. We then obtain, with proof identical to that of [30, Lemma 1.14].

**Lemma 2.26.** $\tau_\eta^C(((a_q, b)_{q\in\widetilde{D}}, (n_p)_{p\in\widetilde{D}})) = -\sum_{p\in\widetilde{D}} n_p + \sum_{q\in\widetilde{D}} \frac{b-a_q}{e_q}$,

The equation $\tau_\eta^X = \tau_\eta^C \circ \varphi$ is a formula in $N_D := \Gamma(\widetilde{D}, g^*\overline{\mathcal{M}}^{\mathrm{gp}})^*$, which is described in [30, eqs. (1.12), (1.13)] as follows. Let $\Sigma \subset \widetilde{D}$ be the set of points $x$ in $\widetilde{D}$ mapping to a special point of $C$. Thus $\Sigma$ can be identified with the subset of $E(G) \cup L(G)$ of edges or legs adjacent to the vertex $v$ corresponding to $\eta$. For any point $x \in \widetilde{D}$, we write $P_x := \overline{\mathcal{M}}_{X,g(x)}$. Then

$$N_D = \varinjlim_{x\in\widetilde{D}} P_x^* = \Big(\bigoplus_{x\in\Sigma} P_x^*\Big)\Big/\sim$$

where for any $a \in P_\eta^*$ and any $x, x' \in \Sigma$,

$$(0, \ldots, 0, \iota_{x,\eta}(a), 0, \ldots, 0) \sim (0, \ldots, 0, \iota_{x',\eta}(a), 0, \ldots, 0).$$

Here $\iota_{x,\eta} : P_\eta^* \to P_x^*$ is the dual of generization, and the non-zero entries lie in the position indexed by $x$ and $x'$ respectively. Thus an element of $N_D$ is represented by a choice of tangent vector $n_x \in N_{\boldsymbol{\sigma}(x)} = P_x^*$, one for each preimage $x \in \widetilde{D}$ of a special point of $C$; and two such choices are identified if they can be related by repeatedly subtracting a tangent vector in $N_{\boldsymbol{\sigma}(v)} = P_\eta^*$ from one of the $n_x$ and adding it to another.

We then have, exactly as in [30, Proposition 1.15], the balancing condition:

**Proposition 2.27.** *Suppose $(C^\circ/W, \mathbf{p}, f)$ is a stable punctured map to $X/B$ with $W = \mathrm{Spec}(\mathbb{N} \to \kappa)$ a standard log point. Let $D \subseteq \underline{C}$ be an irreducible component with generic point $\eta$ and $\Sigma \subset \widetilde{D}$ the preimage of the set of special points. If $\tau_\eta^X \in \Gamma(\widetilde{D}, g^*\overline{\mathcal{M}}^{\mathrm{gp}})^*$ is represented by $(\tau_x)_{x\in\Sigma}$, then*

$$(u_x)_{x\in\Sigma} + (\tau_x)_{x\in\Sigma} = 0$$

*in $N_D = \Gamma(\widetilde{D}, g^*\overline{\mathcal{M}}^{\mathrm{gp}})^*$.*

**Remark 2.28.** With regard to the above interpretation of elements of $N_D$ in terms of the type of $(C^\circ/W, \mathbf{p}, f)$, Proposition 2.27 says the following. The degree data of the $\mathcal{O}_C^\times$-torsors contained in $g^*\mathcal{M}$ defines a tuple of tangent vectors $\tau_x \in N_{\boldsymbol{\sigma}(x)}$, one for each edge or leg $x \in E(G) \cup L(G)$ adjacent to the vertex $v$ corresponding to $\eta$, well-defined up to trading elements of $N_{\boldsymbol{\sigma}(v)}$ via the embedding $N_{\boldsymbol{\sigma}(v)} \hookrightarrow N_{\boldsymbol{\sigma}(x)}$ defined by the face morphism $\boldsymbol{\sigma}(v) \to \boldsymbol{\sigma}(x)$. Then (1) $\tau_x + u_x$ lies in the image of $P_\eta^* \to P_x^*$, and (2) the traditional tropical balancing condition holds in $P_\eta^*$ for $\tau_x + u_x$, $x$ running over the set of special points.

Traditional tropical geometry arises for the case that $X$ is a toric variety with its toric log structure. Then $\mathcal{M}_X^{\mathrm{gp}}$ is the sheaf of rational functions that are invertible on the big torus. Monomial functions define trivial $\mathcal{O}_X^\times$-subtorsors of $\mathcal{M}_X^{\mathrm{gp}}$. Denoting by $N$ the cocharacter lattice of the torus, we thus have a canonical monomorphism

$$N^* \to \Gamma\big(X, \mathcal{M}_X^{\mathrm{gp}}\big) \to \Gamma\big(X, \overline{\mathcal{M}}_X^{\mathrm{gp}}\big)$$

with composition with $\tau_\eta^X$ identically zero. Composing the equation displayed in Proposition 2.27 with the induced map $N_D \to N$ then yields the traditional balancing condition $\sum_x \bar{u}_x = 0$ for $\bar{u}_x$ the image of $u_x$ under the embedding $N_{\sigma(x)} \to N$.

The following is an encapsulation of balancing which gives easy to use restrictions on curve classes realized by punctured maps with given contact orders. For the statement we denote by $\mathcal{L}_s^\times$ the torsor corresponding to $s \in \Gamma(X, \overline{\mathcal{M}}_X^{\mathrm{gp}})$, that is, the inverse image of $s$ under the homomorphism $\mathcal{M}_X^{\mathrm{gp}} \to \overline{\mathcal{M}}_X^{\mathrm{gp}}$, and write $\mathcal{L}_s$ for the corresponding line bundle. Furthermore, the germ of $s$ at $f(p_i)$ lies in $P_{p_i}^{\mathrm{gp}} = \overline{\mathcal{M}}_{X,f(p_i)}^{\mathrm{gp}}$ and hence defines a homomorphism $P_{p_i}^* \to \mathbb{Z}$, which we write as $\langle \cdot, s\rangle$.

**Proposition 2.29.** *Suppose given a punctured map $(C^\circ/W, \mathbf{p}, f)$ for $W$ a log point. Let $(G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u})$ be the type of this map, and let $\underline{D} \subseteq \underline{C}$ be an irreducible component of the domain, corresponding to $v \in V(G)$. Let $p_1, \ldots, p_n \in \mathbf{p}$ be the punctured points of $C^\circ$ contained in $\underline{D}$, and let $q_1, \ldots, q_m$ be the nodes of $\underline{C}$ contained in $\underline{D}$ but which are not nodes of $\underline{D}$. This gives rise to contact orders $u_{p_i}$, $u_{q_j}$, noting that for the contact orders of the nodes, we orient the corresponding edge away from $v$. Then we have*

$$\deg\big(\underline{f}^*\mathcal{L}_s\big)\big|_{\underline{D}} = -\sum_{i=1}^n \langle u_{p_i}, s\rangle - \sum_{i=1}^m \langle u_{q_i}, s\rangle.$$

*Proof.* First, by making a base-change, we can assume $W$ is the standard log point. Note $\underline{f}^*\mathcal{L}_s$ must be isomorphic to the line bundle $\mathcal{L}_{\bar{f}^\flat(s)}$ associated to the torsor corresponding to $\bar{f}^\flat(s)$.

Now the total degree of $\mathcal{L}_{\bar{f}^\flat(s)}$ can be calculated using Lemma 2.26 and details of the proof of [30, Proposition 1.15]. Let $g : \underline{\widetilde{D}} \to \underline{C}$ be the normalization of $\underline{D}$, and let $\eta$ be the generic point of $\underline{D}$. Then

$$\begin{aligned}\deg(f \circ g)^*\mathcal{L}_s &= \deg g^*\mathcal{L}_{\bar{f}^\flat(s)} = \tau_\eta^C(\varphi(s))\\ &= \sum_{q \in \widetilde{D}} \frac{1}{e_q}(\langle V_\eta, s\rangle - \langle V_{\eta_q}, s\rangle) - \sum_{p_i \in \widetilde{D}} \langle u_{p_i}, s\rangle,\end{aligned}$$

in the notation of [30, Lemma 1.14, Proposition 1.15], and the last equality coming from the proof of [30, Proposition 1.15]. Here $V_\eta : P_\eta \to \mathbb{N}$ is the map $\bar{f}^\flat : \overline{\mathcal{M}}_{X,f(\eta)} \to \overline{\mathcal{M}}_{C,\eta}$, and similarly $V_{\eta_q}$, where $\eta_q$ is the generic point of the other branch of $C$ at the node $q$. By [30, eq. (1.9)], $\frac{1}{e_q}(V_\eta - V_{\eta_q}) = -u_q$, where $u_q$ is the contact order of the node $q$ with corresponding edge oriented away from $v$. Note that self-nodes of $\underline{D}$ appear twice in this sum, with opposite sign, and hence cancel. This then yields the desired formula. ■

**Corollary 2.30.** *Suppose given a punctured curve $(C^\circ/W, \mathbf{p}, f)$ with $W$ a log point, $\mathbf{p} = \{p_1, \ldots, p_n\}$. Then we have*

$$\deg \underline{f}^* \mathcal{L}_s = -\sum_{i=1}^{n} \langle u_{p_i}, s \rangle.$$

*Proof.* This is obtained from the previous proposition by summing over all irreducible components of $\underline{C}$. ■

## 2.3 Basicness

A key concept in logarithmic moduli problems is the existence of *basic* or *minimal* logarithmic structures. The existence of such distinguished logarithmic structures on the base space of families is a necessary condition for a logarithmic moduli problem to be represented by a logarithmic algebraic stack. A good notion of basicness should be an open property, and hence is typically defined by a condition at geometric points.

The definition of basic stable logarithmic maps from [30, Section 1.5] is based on universality of the associated family of tropical maps. The original definition in [30, Definition 1.20] phrases this property in terms of the dual monoids and only indicates the tropical interpretation in [30, Remark 1.18]. A proof of the equivalence of the definitions in the present notation is given in [3, Proposition 2.28]. This equivalence of descriptions really only reflects the anti-equivalence between the categories of fs monoids and of rational polyhedral cones. In the following, we freely use this equivalence of categories when referring to material from [30].

The definition of basicness in the punctured case is formally the same as for stable logarithmic maps. Here we take the concrete, tropical view. For readability, we again assume that $X$ is simple, deferring the general discussion to Section 2.6.

**Definition 2.31.** A pre-stable punctured map $(C/W, \mathbf{p}, f)$ is *basic at a geometric point $\bar{w}$ of $\underline{W}$* if the associated family of tropical maps

$$h : \Gamma = \Gamma(G, \ell) \to \Sigma(X)$$

over $(\overline{\mathcal{M}}_{W,\bar{w}})^\vee_{\mathbb{R}}$ is universal among tropical maps of the same type $(G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u})$. This means that each family of stable tropical maps of type $(G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u})$ over some cone $\omega$ arises by pullback from $h : \Gamma \to \Sigma(X)$ via a unique map $\omega \to (\overline{\mathcal{M}}_{W,\bar{w}})^\vee_{\mathbb{R}}$ in **Cones**. Basicness without specifying $\bar{w}$ refers to basicness at all geometric points.

The monoids $\overline{\mathcal{M}}_{W,\bar{w}}$ obtained from basic punctured maps also formally have the same description as for stable logarithmic maps described in [30, Proposition 1.19] and [3, Proposition 2.28]. We provide a full proof of this description emphasizing the tropical perspective.

**Proposition 2.32.** *Let $(C^\circ/W, \mathbf{p}, f)$ be a basic, pre-stable punctured map over a logarithmic point $\operatorname{Spec}(Q \to \kappa)$ with $\kappa$ an algebraically closed field, and let $(G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u})$ be its type. For each generic point $\eta \in \underline{C}$ with $v = v_\eta \in V(G)$ the associated vertex write*

$$P_\eta = \overline{\mathcal{M}}_{X,\underline{f}(\eta)} = (\boldsymbol{\sigma}(v)_{\mathbb{Z}})^\vee.$$

*Then the map*

$$Q^\vee \to \Big\{ ((V_\eta)_\eta, (\ell_q)_q) \in \prod_\eta P_\eta^\vee \times \prod_q \mathbb{N} \mid V_\eta - V_{\eta'} = \ell_q \cdot \mathbf{u}(q) \Big\} \tag{2.15}$$

*given by the duals of $(\overline{\pi_\eta^\flat})^{-1} \circ \bar{f}_\eta^\flat : P_\eta \to Q$ and of the classifying map $\prod_q \mathbb{N} \to Q$ of the log smooth curve $C/W$, is an isomorphism. Here $q$ runs over the set of nodes of $\underline{C}$ and, in the equation, $\eta$, $\eta'$ are the generic points of the adjacent branches, with the order chosen as in the definition of* $\mathbf{u}$.

*Proof.* Denote by $\omega \in$ **Cones** the cone defined by the right-hand side of (2.15). We first construct a tropical punctured map

$$h_0 : \Gamma = \Gamma(G, \ell_0) \to \Sigma(X)$$

over $\omega$ as follows. Define

$$\ell_0(E) : \omega_{\mathbb{Z}} \to \mathbb{N}, \quad h_0(v) : \omega_{\mathbb{Z}} \to P_\eta^\vee \tag{2.16}$$

for $E = E_q \in E(G)$ and $v = v_\eta \in V(G)$ as the projections to the $q$-th factor in $\prod_q \mathbb{N}$ and to $P_\eta^\vee = \boldsymbol{\sigma}(v)_{\mathbb{Z}}$, respectively. For an edge $E = E_q$ with adjacent vertices $v$, $v'$ and associated cone $\omega_E$ from (2.8), the map $h_0$ is defined by

$$\begin{aligned}
(h_0)_E : \omega_E &\to (P_q^\vee)_{\mathbb{R}} = \boldsymbol{\sigma}(E), \\
&(s, \lambda) \mapsto h_0(v)(s) + \lambda \cdot \mathbf{u}(E) = h_0(v')(s) + (\ell_0(E) - \lambda)(-\mathbf{u}(E)),
\end{aligned}$$

with the sign of $\mathbf{u}(E)$ chosen according to the orientation of $E$. In this definition, we view $h_0(v(s))$, $h_0(v'(s))$ as elements of $(P_q^\vee)_{\mathbb{R}}$ via the face inclusions $P_\eta^\vee$, $P_{\eta'}^\vee \to P_q^\vee$. The equality holds by the relation in the definition of $\omega$ by the right-hand side of equation (2.15). In particular, $(h_0)_E$ restricts to $h_0(v)$, $h_0(v')$ on its two faces defined by $v$, $v'$.

Finally, for a leg $L = L_p \in L(G)$ with adjacent vertex $v \in V(G)$, the length function $\ell_0(L)$ and the map $(h_0)_L$ defined on $(\omega_L)_{\mathbb{Z}}$ is uniquely determined by $h_0(v)$

and by the contact order $\mathbf{u}(L)$ via pre-stability (Proposition 2.23). This finishes our construction of a pre-stable tropical punctured map $h_0$ over $\omega$.

Conversely, if $h : \Gamma = \Gamma(G, \ell) \to \Sigma(X)$ is a tropical punctured map of type $(G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u})$ over some cone $\omega' \in$ **Cones**, the map

$$\omega' \to \omega, \quad s \mapsto (h(v_\eta(s)), \ell(E_q))_{\eta,q},$$

with $v_\eta : \omega' \to \Gamma$ the section of $\Gamma \to \omega'$ defined by $v_\eta \in V(G')$, is readily seen to be the unique morphism in **Cones** producing $h$ by pullback from $h_0$. ∎

**Definition 2.33.** The fs monoid $Q$ defined by (2.15) is called the *basic monoid associated to the type* $\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u})$, while its dual $Q^\vee \in$ **Cones** (or $Q^\vee_{\mathbb{R}}$ with the integral structure understood) is called the *associated basic cone*.

Note that while the definition of the basic monoid makes sense for all types, the length function $\ell_0(E)$ constructed in (2.16) in the proof of Proposition 2.32 may be zero for some edge $E$. In this case, the universal tropical domain $\Gamma(G, \ell_0)$ in the proof of Proposition 2.32 is not the domain of a tropical punctured map according to Definition 2.22. The basic monoid is therefore only meaningful if there exists at least one tropical punctured map of the given type.[2] Observe also that just as marked points do not enter the definition of basicness, there is no role for punctures in the statement of Proposition 2.32.

**Proposition 2.34.** *Let* $(C^\circ/W, \mathbf{p}, f)$ *be a pre-stable punctured map. Then*

$$\Omega := \big\{\bar w \in |\underline{W}| \mid \{\bar w\} \times_{\underline{W}} (C^\circ/W, \mathbf{p}, f) \text{ is basic}\big\}$$

*is an open subset of* $|\underline{W}|$.

*Proof.* This is identical to [30, Proposition 1.22]. ∎

**Proposition 2.35.** *Any pre-stable punctured map to* $X \to B$ *arises as the pullback from a basic pre-stable punctured map to* $X \to B$ *with the same underlying ordinary pre-stable map. Both the basic pre-stable punctured map and the morphism are unique up to a unique isomorphism.*

*Proof.* The proof is almost identical to [30, Proposition 1.24]. Let $(\pi : C \to W, \mathbf{p}, f)$ be a pre-stable punctured map over $B$. For each geometric point $\bar w \to \underline{W}$ one obtains a tropical punctured map

$$h_{\bar w} : \Gamma_{\bar w} \to \Sigma(X)$$

over $\omega_{\bar w} = (\overline{\mathcal{M}}_{W,\bar w})^\vee_{\mathbb{R}}$, of some type $(G_{\bar w}, \mathbf{g}_{\bar w}, \boldsymbol{\sigma}_{\bar w}, \mathbf{u}_{\bar w})$. By Proposition 2.25, generization $\bar w \in \mathrm{cl}(\bar w')$ (i.e. existence of a specialization arrow $\bar w' \to \bar w$ as in Appendix C)

[2] The analogue of this statement in [30] is the condition $\underline{\mathrm{GS}}(\overline{\mathcal{M}}) \neq \emptyset$.

leads to a contraction morphism (2.14)

$$(G_{\bar w}, \mathbf{g}_{\bar w}, \boldsymbol{\sigma}_{\bar w}, \mathbf{u}_{\bar w}) \to (G_{\bar w'}, \mathbf{g}_{\bar w'}, \boldsymbol{\sigma}_{\bar w'}, \mathbf{u}_{\bar w'}).$$

This contraction morphism induces an embedding of $\Gamma_{\bar w'}$ as a subcomplex of $\Gamma_{\bar w}$ such that $h_{\bar w'}$ becomes the restriction of $h_{\bar w}$. These maps are compatible with the classifying maps to the dual of the respective basic monoids in Proposition 2.32, producing a cartesian diagram of pre-stable tropical punctured maps.

As in the proof of [30, Proposition 1.24], this situation produces monoid sheaves $\overline{\mathcal{M}}^{\mathrm{bas}}_{C^\circ}$, $\overline{\mathcal{M}}^{\mathrm{bas}}_{W}$ on $\underline{C}$ and $\underline{W}$, respectively, and a commutative diagram

$$\begin{array}{ccccc} \underline{f}^*\overline{\mathcal{M}}_X & \longrightarrow & \overline{\mathcal{M}}^{\mathrm{bas}}_{C^\circ} & \longrightarrow & \overline{\mathcal{M}}_{C^\circ} \\ & & \uparrow & & \uparrow \\ & & \underline{\pi}^*\overline{\mathcal{M}}^{\mathrm{bas}}_{W} & \longrightarrow & \underline{\pi}^*\overline{\mathcal{M}}_W \end{array} \tag{2.17}$$

In case $B$ has a non-trivial log structure, all morphisms are compatible with morphisms from the pullback of $\overline{\mathcal{M}}_B$. Continuing as in [30, Proposition 1.24], we can now define the desired basic log structures by fiber product:

$$\mathcal{M}^{\mathrm{bas}}_{W} = \mathcal{M}_W \times_{\overline{\mathcal{M}}_W} \overline{\mathcal{M}}^{\mathrm{bas}}_{W}, \quad \mathcal{M}^{\mathrm{bas}}_{C^\circ} = \mathcal{M}_{C^\circ} \times_{\overline{\mathcal{M}}_{C^\circ}} \overline{\mathcal{M}}^{\mathrm{bas}}_{C^\circ}.$$

Each of these defines a log structure with the structure map being the composition of the projection to the first factor followed by the structure map for that log structure. The pair of induced morphisms

$$\pi_{\mathrm{bas}} : C^\circ_{\mathrm{bas}} = (\underline{C}, \mathcal{M}^{\mathrm{bas}}_{C^\circ}) \to W_{\mathrm{bas}} = (\underline{W}, \mathcal{M}^{\mathrm{bas}}_{W}), \quad f_{\mathrm{bas}} : C^\circ_{\mathrm{bas}} \to X$$

have tropicalizations at any geometric point $\bar w$ of $\underline{W}$ given by the universal pre-stable tropical punctured map to $\Sigma(X)$ over $\Sigma(B)$ of type $(G_{\bar w}, \mathbf{g}_{\bar w}, \boldsymbol{\sigma}_{\bar w}, \mathbf{u}_{\bar w})$. Thus $(C^\circ_{\mathrm{bas}}/W_{\mathrm{bas}}, \mathbf{p}, f)$ is a basic punctured map to $X$. By the construction by fiber products of monoid sheaves, it follows that $f_{\mathrm{bas}}$ commutes with the morphisms to $B$, and that $(C^\circ/W, \mathbf{p}, f)$ is the pullback of $(C^\circ_{\mathrm{bas}}/W_{\mathrm{bas}}, \mathbf{p}, f)$ by $W \to W_{\mathrm{bas}}$. The constructed basic punctured map is also pre-stable since $(C^\circ/W, \mathbf{p}, f)$ is and by the definition of $\mathcal{M}^{\mathrm{bas}}_{C^\circ}$ as a fiber product. Finally, the universal property of the basic monoid with regard to pre-stable tropical punctured maps in Proposition 2.32 implies uniqueness. ■

**Remark 2.36.** Following [30], our construction of the basic pre-stable punctured map in the proof of Proposition 2.35 argues pointwise and uses compatibility with generizations to obtain the universal diagram of ghost sheaves. However, the existence of an étale sheaf with the stated stalks and generization maps is never checked, notably in the proof of [30, Lemma 1.23]. We use this occasion to close this gap.

The basic monoids and generization homomorphisms define a contravariant functor

$$\mathbf{Pt}(W) \to \mathbf{Mon}, \quad \bar{w} \mapsto Q_{\bar{w}} \tag{2.18}$$

from the category of geometric points $\mathbf{Pt}(W)$ with specialization arrows, recalled at the beginning of Appendix C, to the category of monoids. A specialization arrow $\bar{w} \to \bar{w}'$ maps to an epimorphism of monoids $Q_{\bar{w}'} \to Q_{\bar{w}}$ given by localization at a face and subsequently dividing out the subgroup of invertible elements. In any case, from a functor as in (2.18) one can define an étale sheaf $\overline{\mathcal{M}}^{\mathrm{bas}}$ by associating to an étale map $h : U \to X$ the monoid

$$\overline{\mathcal{M}}^{\mathrm{bas}}(U) = \mathrm{colim}_{\bar{w} \to h} Q_{\bar{w}},$$

together with the obvious restriction maps. Here the colimit is taken over all factorizations of $\bar{w} : \mathrm{Spec}\, \kappa(\bar{w}) \to X$ over $h$. The gap in [30] concerns the implicit claim that for a geometric point $\bar{w}$ of $X$ the natural map

$$Q_{\bar{w}} \to \overline{\mathcal{M}}^{\mathrm{bas}}_{\bar{w}}$$

is an isomorphism.

This claim is étale local in $\underline{W}$. Hence we can assume that the given (non-basic) log structure $\mathcal{M}_W$ on $W$ is a Zariski log structure with a global chart that is neat at some geometric point $\bar{w}$. We may also assume that the logarithmic stratum containing $\bar{w}$ lies in the closure of all other strata, and that the restriction map

$$\Gamma(W, \overline{\mathcal{M}}_W) \to \overline{\mathcal{M}}_{W,\bar{w}}$$

is an isomorphism. By [52, Proposition II.2.1.2] we obtain a continuous map

$$g : |\underline{W}| \to S = \mathrm{Spec}\, \overline{\mathcal{M}}_{W,\bar{w}}$$

from the topological space underlying $\underline{W}$ to the monoidal scheme of prime ideals of $\overline{\mathcal{M}}_{W,\bar{w}}$, a finite topological space, together with an isomorphism

$$g^{-1}\overline{\mathcal{M}}_S \to \overline{\mathcal{M}}_W.$$

Here $\overline{\mathcal{M}}_S$ is the structure sheaf of $\mathrm{Spec}\, \overline{\mathcal{M}}_{W,\bar{w}}$, a sheaf of sharp monoids.[3]

Note that a finite topological space is an Alexandrov space. Thus a subset is closed iff it is closed under specialization, and sheaves (of sets, say) are indeed given by contravariant functors from the category of points to **Sets**, see e.g. [46, Section 2].

[3] We have reinterpreted the statement in [52] as a statement for Kato fans to avoid dealing with invertible elements, which are irrelevant for our discussion.

The universal property of basic monoids provides a monoid homomorphism

$$Q_{\bar w} \to \overline{\mathcal{M}}_{W,\bar w},$$

hence a morphism of monoid spectra

$$k : S = \operatorname{Spec} \overline{\mathcal{M}}_{W,\bar w} \to S_{\mathrm{bas}} = \operatorname{Spec} Q_{\bar w}.$$

Compatibility of the basic monoids and their universal property with generization now shows first that $(k \circ g)^{-1} \overline{\mathcal{M}}_{S_{\mathrm{bas}}}$ is a sheaf of monoids with stalks equal to the basic monoids on $W$ and having the expected generization homomorphisms, hence defines $\overline{\mathcal{M}}_W^{\mathrm{bas}}$, and second that the composition

$$\overline{\mathcal{M}}_W^{\mathrm{bas}} = (k \circ g)^{-1} \overline{\mathcal{M}}_{S_{\mathrm{bas}}} \to g^{-1} \overline{\mathcal{M}}_S \to \overline{\mathcal{M}}_W$$

stalkwise restricts to the classifying homomorphisms for $\overline{\mathcal{M}}_W$.

A similar argument on $C$ provides the remaining parts of Diagram (2.17).

**Proposition 2.37.** *An automorphism $\varphi : C^\circ/W \to C^\circ/W$ of a basic pre-stable punctured map $(C^\circ/W, \mathbf{p}, f)$ with $\underline{\varphi} = \mathrm{id}_{\underline{C}^\circ}$ is trivial.*

*Proof.* This is identical to [30, Proposition 1.25]. ∎

## 2.4 Global contact orders and global types

A fundamental ingredient in the definition of logarithmic Gromov–Witten invariants is the global specification of contact orders at the marked points. The local behaviour of contact orders in families of stable logarithmic maps is captured by the notion of morphism of types (2.14), implying that generization leads to the possible propagation of contact orders via face inclusions in $\Sigma(X)$. The global definition can be subtle in the presence of monodromy, as the following examples show.

**Example 2.38.** This example is modeled on the well-known toric construction of the Tate curve. Let $Y$ be the three-dimensional toric variety (not of finite type) defined by the fan consisting of the collection of three-dimensional cones

$$\Sigma^{[3]} = \left\{\mathbb{R}_{\geq 0}(n,0,1) + \mathbb{R}_{\geq 0}(n+1,0,1) + \mathbb{R}_{\geq 0}(n,1,1) + \mathbb{R}_{\geq 0}(n+1,1,1) \mid n \in \mathbb{Z}\right\}$$

and their faces. Projection onto the third coordinate yields a toric morphism $Y \to \mathbb{A}^1$. After a base-change

$$\widehat{Y} = Y \times_{\mathbb{A}^1} \operatorname{Spec} \Bbbk[\![t]\!] \to \operatorname{Spec} \Bbbk[\![t]\!],$$

one may divide out $\widehat{Y}$ by the action of $\mathbb{Z}$ defined as follows. This action is generated by an automorphism of $\widehat{Y}$ induced by an automorphism of $Y$ defined over $\mathbb{A}^1$. This

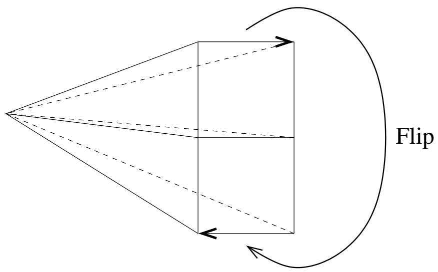


**Figure 2.7.** Tropicalization of a Zariski logarithmic scheme with contact order monodromy: $\ell = 2$.

automorphism is given torically via the linear transformation $\mathbb{Z}^3 \to \mathbb{Z}^3$ given by the matrix

$$\begin{pmatrix} 1 & 0 & \ell \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

where $\ell$ is a fixed positive integer. We then define $X = \widehat{Y}/\mathbb{Z}$, with log structure induced by the toric log structure on $Y$ (Figure 2.7).

Then $X \to \operatorname{Spec} \Bbbk[\![t]\!]$ is a degeneration of the total space of a $\mathbb{G}_m$-torsor over an elliptic curve, the torsor corresponding to a 2-torsion element of the Picard group of the elliptic curve. As long as $\ell \geq 2$, $X$ has a Zariski log structure. Further, $\Sigma(X)$ is a cone over a Möbius strip composed of $\ell$ squares. If one takes $u = (0, 1, 0) \in \sigma^{\mathrm{gp}}$ for any three-dimensional cone in $\Sigma(X)$, then propagating $u$ via chains of face inclusions identifies $u$ with $-u$ due to the twist in the Möbius strip.

**Example 2.39.** A variant of the previous example that we learnt from Jonathan Wise also produces monodromy of infinite order.

Let $\sigma \subset \mathbb{R}^4$ be the cone generated by the following column vectors:

$$\begin{aligned} &v_1 = (0,0,0,1)^t, \quad v_2 = (0,1,0,1)^t, \quad v_3 = (0,0,1,1)^t, \quad v_4 = (0,1,1,1)^t, \\ &v_5 = (1,0,1,1)^t, \quad v_6 = (1,1,1,1)^t, \quad v_7 = (2,1,0,1)^t, \quad v_8 = (2,2,0,1)^t. \end{aligned}$$

The linear transformation of $\mathbb{R}^4$ with matrix

$$A = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

fulfills

$$Av_1 = v_7, \quad Av_2 = v_8, \quad Av_3 = v_5, \quad Av_4 = v_6.$$

Thus $A\tau_1 = \tau_2$ for the two facets

$$\tau_1 = \langle v_1, v_2, v_3, v_4\rangle, \quad \tau_2 = \langle v_5, v_6, v_7, v_8\rangle$$

of $\sigma$.

Now $\tau_1^{\mathrm{gp}} \cap \tau_2^{\mathrm{gp}}$ is the lattice spanned by

$$x = 2v_3 - v_1 = 2v_6 - v_8 = (0, 0, 2, 1), \quad y = v_2 - v_1 = v_6 - v_5 = (0, 1, 0, 0).$$

The restriction of $A$ to this sublattice is a shear transformation, hence is of infinite order:

$$Ax = x - y, \quad Ay = y.$$

It is not hard to define a log structure on the nodal cubic curve $\underline{X}$ with $\overline{\mathcal{M}}_{X,q}^{\vee} \simeq \sigma \cap \mathbb{Z}^4$ at the node $q$, and the generization maps to the two branches of $C$ at $q$ dual to the inclusions $\tau_1, \tau_2 \hookrightarrow \sigma$. Then $X = (\underline{X}, \mathcal{M}_X)$ has infinite monodromy.

By pulling back $\mathcal{M}_X$ to a two-nodal curve of arithmetic genus 1, with the map to $X$ contracting a $\mathbb{P}^1$, produces an example with Zariski log structure and infinite monodromy.

Note that the feature of infinite monodromy can not be seen from the underlying topological space of the tropicalization $\Sigma(X)$. In fact, as a topological space, $\sigma$ is the cone over a polyhedron $\Xi \subset \mathbb{R}^3$ that is the convex hull of two disjoint facets with four vertices each, the intersections of $\tau_1$, $\tau_2$ with the affine hyperplane $x_4 = 1$ for $x_1, \ldots, x_4$ the coordinates on $\mathbb{R}^4$. Thus $|\Sigma(X)|$ is the cone over the cell complex obtained from $\Xi$ by identifying these two facets. But replacing $v_7$, $v_8$ with $(2, 0, 0, 1)^t$, $(2, 1, 0, 1)^t$ and adapting $A$ accordingly produces an example with homeomorphic $|\Sigma(X)|$ and without monodromy.

In the presence of monodromy as in Examples 2.38 and 2.39, the naïve definition of global contact orders by a reduced subscheme $\underline{Z} \subset \underline{X}$ and a section $s \in \Gamma(\underline{Z}, (\overline{\mathcal{M}}_X|_{\underline{Z}})^*)$ not extending to any larger subscheme from [30, Definition 3.1] does not work. We provide here an alternative treatment based on a notion of tangent vectors for the generalized cell complex $\Sigma(X)$ that suffices for the definition of finite type moduli spaces and of certain punctured Gromov–Witten invariants also in cases with monodromy. Some applications such as gluing (Theorem 5.8) in rare cases may require the more refined definition presented in Appendix A. For the sake of simplicity of presentation, we merely indicate what has to be modified to treat such rare cases.

### 2.4.1 Global contact orders

For $\sigma \in \Sigma(X)$ denote by $\Sigma_\sigma(X)$ the star of $\sigma$, considered as the category $\Sigma(X)$ under $\sigma$, i.e., the category with objects face embeddings $\sigma \to \sigma'$ in $\Sigma(X)$ and arrows given by morphisms $\sigma' \to \sigma''$ commuting with the given morphisms from $\sigma$. Thus the star $\Sigma_\sigma(X)$ is formed by all cones $\sigma_{\bar{x}} = \mathrm{Hom}(\bar{\mathcal{M}}_{X,\bar{x}}, \mathbb{R}_{\geq 0})$ with $\bar{x}$ running over the geometric points of $X_\sigma$. Associating to $(\sigma \to \sigma') \in \Sigma_\sigma(X)$ the free abelian group $N_{\sigma'}$, viewed as a set, gives a diagram in the category of sets indexed by $\Sigma_\sigma(X)$. This diagram can be viewed as the diagram of integral tangent vectors of $\Sigma_\sigma(X)$. Taking the colimit in the category of sets provides a set of homomorphisms $\bar{\mathcal{M}}_{X,\bar{x}} \to \mathbb{Z}$ for geometric points $\bar{x}$ of $X_\sigma$ compatible with all generization homomorphisms. Elements of this colimit therefore provide a way to specify compatible sets of contact orders along the stratum $X_\sigma$ independently of monodromy.

**Definition 2.40.** Let $\sigma \in \Sigma(X)$ and $\mathfrak{N}_\sigma : \Sigma_\sigma(X) \to$ **Sets** be the diagram in the category of sets mapping $\sigma \to \sigma'$ to $N_{\sigma'}$. A *global contact order for* $\sigma \in \Sigma(X)$, or for the corresponding stratum $X_\sigma \subseteq \underline{X}$, is an element $\bar{u}$ of

$$\mathfrak{C}_\sigma(X) := \mathrm{colim}^{\mathbf{Sets}} \mathfrak{N}_\sigma = \mathrm{colim}^{\mathbf{Sets}}_{\sigma \to \sigma'} N_{\sigma'},$$

the *set of contact orders for* $\sigma$. For $\sigma' \in \Sigma_\sigma(X)$, or for a geometric point $\bar{x}$ of $X_\sigma$, we denote by

$$\iota_{\sigma\sigma'} : N_{\sigma'} \to \mathfrak{C}_\sigma(X), \quad \iota_{\sigma\bar{x}} : N_{\sigma_{\bar{x}}} \to \mathfrak{C}_\sigma(X)$$

the canonical maps.

A *global contact order* is a contact order for some $\sigma \in \Sigma(X)$. The set of global contact orders is denoted $\mathfrak{C}(X) := \coprod_{\sigma \in \Sigma(X)} \mathfrak{C}_\sigma(X)$.

We say a contact order $\bar{u}$ for $\sigma \in \Sigma(X)$ has *finite monodromy* if for all $(\sigma \to \sigma') \in \Sigma_\sigma(X)$ the set $\iota^{-1}_{\sigma\sigma'}(\bar{u}) \subseteq N_{\sigma'}$ is finite.

A global contact order $\bar{u} \in \mathfrak{C}_\sigma(X)$ is *monodromy-free* if for all $(\sigma \to \sigma') \in \Sigma_\sigma(X)$ there exists at most one $u \in N_{\sigma'}$ with $\bar{u} = \iota_{\sigma\sigma'}(u)$.

To be explicit, we spell out the definition of $\iota_{\sigma\bar{x}}$ for $\bar{x}$ a geometric point of $X_\sigma$. Let $Z \subseteq X$ be the smallest logarithmic stratum containing the image of $\bar{x}$. Then since $Z \cap X_\sigma \neq \emptyset$, the definition of $\Sigma(X)$ provides an isomorphism

$$\sigma_{\bar{x}} = \mathrm{Hom}\big(\bar{\mathcal{M}}_{X,\bar{x}}, \mathbb{R}_{\geq 0}\big) \xrightarrow{\simeq} \mathrm{Hom}\big(\bar{\mathcal{M}}_{X,\bar{x}_Z}, \mathbb{R}_{\geq 0}\big) = \sigma_Z$$

together with a face map $\sigma \to \sigma_Z$, unique up to arrows $\sigma \to \sigma$ and $\sigma_Z \to \sigma_Z$ in $\Sigma(X)$. Then $\iota_{\sigma\bar{x}}$ is defined by composing the induced isomorphism of lattices $N_{\sigma_{\bar{x}}} \simeq N_{\sigma_Z}$ with $\iota_{\sigma\sigma_Z}$. The definition of $\mathfrak{C}_\sigma(X)$ is designed to make all maps $\iota_{\sigma\bar{x}}$ independent of choices. In particular, a contact order as in (2.4) and (2.12) has an associated global contact order.

Note that if $\bar{\mathcal{M}}_X$ has monodromy along $X_\sigma$, there is a non-trivial group $G$ of arrows $\sigma \to \sigma$ in $\Sigma(X)$. In this case, the map $\iota_{\sigma\sigma} : N_\sigma \to \mathfrak{C}_\sigma(X)$ factors over the quotient $N_\sigma \to N_\sigma/G$ of the induced linear action of $G$ on $N_\sigma$. In particular, two tangent vectors $u, u' \in N_\sigma$ define the same global contact order $\bar{u} = \iota_{\sigma\sigma}(u) = \iota_{\sigma\sigma}(u')$ if they are related by monodromy along $X_\sigma$.

Given a punctured map $(C^\circ/W, \mathbf{p}, f)$ to $X$ and $s : \underline{W} \to \underline{C}$ a punctured or nodal section, each geometric point $\bar{w}$ of $\underline{W}$ has an associated contact order $u_{s(\bar{w})}$ at $s(\bar{w})$, giving the contact orders $u_p, u_q$ of (2.12) of the associated tropicalization:

$$u_{s(\bar{w})} : \overline{\mathcal{M}}_{X,\underline{f}(s(\bar{w}))} \to \mathbb{Z}.$$

Recall also that the contact order for a node, defined in [30, eq. (1.8)], depends on the choice of an ordering of the two branches of $\underline{C}_{\bar{w}}$ through the node $q$, just as $u_E = u_q$ depends on the choice of orientation of the edge $E$. Now for any $\sigma \in \Sigma(X)$ with $\mathrm{im}(\underline{f} \circ s) \subseteq X_\sigma$ and any $\bar{w} \to \underline{W}$, we obtain the induced global contact order

$$u_s^\sigma(\bar{w}) = \iota_{\sigma \underline{f}(s(\bar{w}))}(u_{s(\bar{w})}) \tag{2.19}$$

The following lemma shows that fixing global contact orders in families of punctured maps is both an open and closed condition. In particular, prescribing global contact orders for strata, formalized in the notion of marking below (Definition 3.4), works well in moduli problems.

**Lemma 2.41.** *Let $(C^\circ/W, \mathbf{p}, f)$ be a punctured map, $s : \underline{W} \to \underline{C}$ a punctured or nodal section, and $\sigma \in \Sigma(X)$ with $\mathrm{im}(\underline{f} \circ s) \subseteq X_\sigma$. Then the function $\bar{w} \mapsto u_s^\sigma(\bar{w})$ from (2.19), associating to a geometric point $\bar{w}$ of $\underline{W}$ the global contact order of $(C_{\bar{w}}^\circ/\bar{w}, \mathbf{p}_{\bar{w}}, f_{\bar{w}})$ for $\sigma$, is locally constant.*

*Proof.* The existence of neat charts for the punctured map $f : C^\circ \to X$ [52, Theorem III.1.2.7] shows that the composition

$$s^{-1} f^{-1} \overline{\mathcal{M}}_X \to s^{-1} \overline{\mathcal{M}}_{C^\circ} \to \underline{\mathbb{Z}},$$

is a morphism of constructible sheaves of sets. See also [52, Theorem II.2.5.4]. This composition defines the contact order as a function on $\underline{W}$. Hence the subset of $\underline{W}$ with $f$ of a given contact order is a constructible set. It remains to show that contact orders are compatible with generization. Consider a specialization $\bar{w}'$ of $\bar{w}$, with $\underline{f} \circ s(\bar{w}') = \bar{x}'$ a specialization of $\underline{f} \circ s(\bar{w}) = \bar{x}$. By Proposition 2.25 the face embedding $N_{\sigma_{\bar{x}}} \to N_{\sigma_{\bar{x}'}}$ dual to generization, which is an arrow in $\mathfrak{N}_\sigma$, maps the contact order $u_{\bar{x}} \in N_{\sigma_{\bar{x}}}$ to $u_{\bar{x}'} \in N_{\sigma_{\bar{x}'}}$. Hence $\iota_{\sigma\bar{x}}(u_{\bar{x}}) = \iota_{\sigma\bar{x}'}(u_{\bar{x}'})$, as needed. ■

**Definition 2.42.** Let $(C^\circ/W, \mathbf{p}, f)$ be a punctured map, and $s : \underline{W} \to \underline{C}$ a punctured or nodal section with $\mathrm{im}(\underline{f} \circ s) \subseteq X_\sigma$ for some $\sigma \in \Sigma(X)$. Then $(C^\circ/W, \mathbf{p}, f)$ is said to have *global contact order* $\bar{u} \in \mathfrak{C}_\sigma(X)$ *for* $\sigma$ *along* $s$ if for each geometric point $\bar{w}$ of $\underline{W}$ the function in (2.19) fulfills $u_s^\sigma(\bar{w}) = \bar{u}$.

**Remark 2.43.** A previous version of this paper contained a notion of evaluation stratum for a global contact order. This was meant as the analogue of the pullback via $X \to \mathcal{A}_X$ of the image of $\mathcal{Z}_\sigma \to \mathcal{A}_X$ in the notion of contact orders based on the

Artin fan of $X$ developed in Appendix A.2. We decided to remove this part for several reasons.

First, the given treatment was ad hoc since unlike in the notion based on Artin fans, there is no good functorial characterization of schematic evaluation strata based on families of punctured curves. This lack of a universal property is due to possible obstructions to deformations of punctured maps not coming from obstructions to deformations of the evaluation point.

Second, contact orders are naturally selected after fixing a reference stratum, see Section 3.2 below. In the most important case of realizable types of punctured maps (Definition 2.44 (2) below), the reference stratum already defines a reduced closed subscheme of the evaluation stratum for the given contact order. Thus defining a non-reduced evaluation stratum is pointless in this case. Indeed, so far there has not been any use of non-reduced evaluation strata in practice, and notably not in the applications mentioned in the introduction.

Third, should there ever be a need to define a non-reduced evaluation stratum, it can easily be defined via the theory of contact orders developed in Appendix A.

### 2.4.2 Global types

As emphasized throughout the paper, a central aspect of the theory of punctured maps involves the underlying combinatorics in terms of tropical geometry. On the level of moduli spaces, this aspect is captured by the notion of marking by tropical types.

For this purpose, we need a global version of the type of punctured maps (Definition 2.24). Crucially we replace contact orders by the global contact orders from Definition 2.40. For readability, we again restrict to the case of simple $X$ first. The discussion of the additional data needed for the general case is contained in Section 2.6.

**Definition 2.44.** (1) A *global type (of a family of tropical punctured maps to* $\Sigma(X)$*)* is a tuple

$$\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \bar{\mathbf{u}})$$

consisting of a genus-decorated connected graph $(G, \mathbf{g})$ and two maps

$$\boldsymbol{\sigma} : V(G) \cup E(G) \cup L(G) \to \Sigma(X), \quad \bar{\mathbf{u}} : E(G) \cup L(G) \to \mathfrak{C}(X)$$

with $\bar{\mathbf{u}}(x) \in \mathfrak{C}_{\boldsymbol{\sigma}(x)}$ for each $x \in E(G) \cup L(G)$. A (type of) punctured maps has an *associated global type* by replacing the contact orders by the associated global contact orders. *Morphisms of global types* are defined analogously to morphisms of types of tropical punctured maps in (2.14).

If the composition of $\bar{\mathbf{u}}$ with the natural map $\mathfrak{C}(X) \to \mathfrak{C}(B)$ equals 0, we say $\tau$ is a global type for $X/B$ or relative $B$.

(2) A global type $\tau$ is *realizable*[4] if there exists a tropical map to $\Sigma(X)$ with associated global type $\tau$.

(3) A *decorated global type* $\boldsymbol{\tau} = (\tau, \mathbf{A})$ of tropical punctured maps is obtained by adding a curve class $\mathbf{A}$ as in (2.13).

(4) A *class of tropical punctured maps* for a connected $X$ is a decorated global type with a graph $G$ with only one vertex $v$, no edges, and all strata $\boldsymbol{\sigma}(x) = \{0\}$. We write a class of tropical punctured maps as $\beta = (g, \bar{\mathbf{u}}, A)$ with $g \in \mathbb{N}$, $A \in H_2^+(X)$ and $\bar{\mathbf{u}} : L(G) \to \mathfrak{C}_{\{0\}}(X)$. The *class of a decorated global type* is the class of tropical punctured maps obtained by contracting all edges and keeping the set of legs, but with associated strata $0 \in \Sigma(X)$ and each global contact order the image under the canonical map

$$\mathfrak{C}_{\boldsymbol{\sigma}(L)}(X) \to \mathfrak{C}_{\{0\}}(X).$$

For a class $\beta$ of a global type we write $\underline{\beta} = (g, k, A)$ with $k = |L(G)|$ for the class of the underlying ordinary stable map.

If $X$ is disconnected, one takes one class of tropical punctured map for each connected component of $X$.

We will often drop the adjective *"tropical"* and refer to a global type, decorated global type, or class of punctured maps.

The following lemma will only be used in the proof of Proposition 3.24, which in turn is only used in the dimension formulas of Proposition 3.30.

**Lemma 2.45.** *Let $(G, \mathbf{g}, \boldsymbol{\sigma}, \bar{\mathbf{u}})$ be a realizable global type, and assume all logarithmic strata $Z_\sigma \subseteq X$ for $\sigma \in \operatorname{im}(\boldsymbol{\sigma})$ are monodromy-free. Then there is a unique type $\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u})$ of punctured maps with associated global type $(G, \mathbf{g}, \boldsymbol{\sigma}, \bar{\mathbf{u}})$.*

*Proof.* Indeed, realizability implies in particular that for each $x \in E(G) \cup L(G)$, the contact order $u_x \in \mathfrak{C}_{\boldsymbol{\sigma}(x)}(X)$ lies in the image of the natural map $N_{\boldsymbol{\sigma}(x)} \to \mathfrak{C}_{\boldsymbol{\sigma}(x)}(X)$. However, it follows immediately from the definition of $\mathfrak{C}_\sigma(X)$ that the map $N_\sigma \to \mathfrak{C}_\sigma(X)$ is injective for each $\sigma \in \Sigma(X)$. ∎

A sufficient condition for the absence of monodromy in Lemma 2.45 is of course that $X$ is simple.

**Remark 2.46** (Relation with types). There are two differences of the notion of global type to the notion of type in Definition 2.24. First, contact orders are replaced by global contact orders. Second, the requirement $\bar{\mathbf{u}}(x) \in \mathfrak{C}_{\boldsymbol{\sigma}(x)}(X)$ for $x \in E(G) \cup L(G)$ does not imply $u_x \in N_{\boldsymbol{\sigma}(x)}$. The lack of the latter condition for edges makes it impossible to define a basic monoid just depending on a global type.

[4] The term signifies that the combinatorial data underlies a tropical object. It should not be confused with realizability in tropical algebraic geometry, which signifies that a tropical object is the tropicalization of an algebraic object.

However, some useful discrete data remain. For simplicity we assume $X$ is simple again, deferring the discussion of the general case to Section 2.6.4. Consider a tropical punctured map $\Gamma \to \Sigma(X)$, with associated type $\tau' = (G', \mathbf{g}', \boldsymbol{\sigma}', \mathbf{u}')$, basic monoid $Q_{\tau'}$ as in (2.15), and dual monoid $Q_{\tau'}^\vee$ underlying the corresponding moduli of tropical maps. We have an associated *global* type $\overline{\tau}' = (G', \mathbf{g}', \boldsymbol{\sigma}', \bar{\mathbf{u}}')$ as in Definition 2.44 (1) obtained by replacing the contact orders $\mathbf{u}'(x)$ with their images in $\mathfrak{C}_{\boldsymbol{\sigma}(x)}(X)$.

Now fix a contraction morphism $\phi : \overline{\tau}' \to \tau$ to a global type $\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \bar{\mathbf{u}})$, with set of contracted edges $E_\phi$. We claim that there is a well-defined face $Q_{\tau\tau'}^\vee$ of $Q_{\tau'}^\vee$, see (2.20), with dual localization (2.21), not requiring a morphism of types lifting $\overline{\tau}' \to \tau$. Fix a point of $Q_{\tau'}^\vee$ given as a tuple $(V_v, \ell_E)_{v \in V(G), E \in E(G)}$. Then $(V_v, \ell_E)_{v,E} \in Q_{\tau\tau'}^\vee$ if and only if

(1) the position $V_v$ of any vertex $v$ maps to the cell $\boldsymbol{\sigma}(\phi(v))$ associated to $\phi(v) \in V(G)$ by $\tau$, and
(2) if $E \in V(G')$ is an edge contracted by $\phi$ then $\ell_E = 0$.

Here we replaced generic points $\eta$ and nodal points $q$ in (2.15) by vertices $v \in V(G')$ and edges $E \in E(G')$. It is critical that $\boldsymbol{\sigma}(\phi(v))$ is a well-defined face of $\boldsymbol{\sigma}(v)$. This is where we use the simplicity assumption. Define $Q_{\tau\tau'}$ as the dual of this face, given precisely as:

$$Q_{\tau\tau'}^\vee = \big\{(V_v, \ell_E) \in Q_{\tau'}^\vee \mid \forall v \in V(G') : V_v \in \boldsymbol{\sigma}(\phi(v)) \quad \forall E \in E_\phi : \ell_E = 0\big\}. \tag{2.20}$$

We then obtain a localization morphism

$$\chi_{\tau\tau'} : Q_{\tau'} \to Q_{\tau\tau'}, \tag{2.21}$$

just as for basic monoids associated to types of tropical punctured maps [3, Definition 2.31 (3)]. The difference is that now both $Q_{\tau\tau'}$ and $\chi_{\tau\tau'}$ depend not only on the morphism $\phi : \overline{\tau}' \to \tau$ of global types, but also on the lift of $\overline{\tau}'$ to a type $\tau'$ of tropical punctured maps.

## 2.5 Puncturing log-ideals

The punctured points which are not marked points impose extra important constraints on the possible deformations of a punctured curve, hence of punctured stable maps, captured by an ideal in the base monoid. This is a key new feature of the theory which we now describe.

### 2.5.1 Review of idealized log schemes

We review here the notion of idealized log schemes from [52], as this notion is considerably less common in the literature.

Given a sheaf of monoids $\mathcal{M}$ on a scheme $X$, we use the term *log-ideal* for a sheaf of monoid ideals $\mathcal{K} \subseteq \mathcal{M}$. The sheaf of monoid ideals $\mathcal{K}$ is said to be *coherent* (see [52, Proposition II.2.6.1]) if locally on $X$, $\mathcal{K}$ is generated by a finite set of sections.

An *idealized log scheme* is data $(X, \mathcal{M}_X, \alpha_X, \mathcal{K}_X)$ where $(X, \mathcal{M}_X, \alpha_X)$ is an ordinary log scheme, with $\alpha_X : \mathcal{M}_X \to \mathcal{O}_X$ the structure map, and $\mathcal{K}_X \subseteq \mathcal{M}_X$ a log-ideal such that $\mathcal{K}_X \subseteq \alpha_X^{-1}(0)$. A *morphism of idealized log schemes* $f : (X, \mathcal{K}_X) \to (Y, \mathcal{K}_Y)$ is a morphism $f : X \to Y$ of log schemes such that

$$f^\flat(f^{-1}(\mathcal{K}_Y)) \subseteq \mathcal{K}_X.$$

See [52, Definition III.1.3.1].

If $f : X \to Y$ is a morphism of log schemes and $\mathcal{K}_Y \subseteq \mathcal{M}_Y$ is a log-ideal, we adopt the notation of [52] by writing $f^\bullet(\mathcal{K}_Y) \subseteq \mathcal{M}_X$ as the log-ideal generated by $f^\flat(f^{-1}(\mathcal{K}_Y))$. We say a morphism $f : X \to Y$ of idealized log schemes is *idealized strict* [52, Definition III.1.3.2] if $\mathcal{K}_X = f^\bullet\mathcal{K}_Y$.

If $W$ is a fine log scheme and $\mathcal{K} \subseteq \mathcal{M}_W$ is a log-ideal, then $\mathcal{K}$ is invariant under the multiplicative action of $\mathcal{O}_W^\times$, and the quotient $\overline{\mathcal{K}} = \mathcal{K}/\mathcal{O}_W^\times$ is a log-ideal in $\overline{\mathcal{M}}_W$. As the stalks of $\overline{\mathcal{M}}_W$ are finitely generated monoids, the stalks of $\overline{\mathcal{K}}$ are then finitely generated ideals.

**Lemma 2.47.** *Let $(W, \mathcal{M}_W)$ be a fine log scheme and $\mathcal{K} \subseteq \mathcal{M}_W$ a log-ideal. Then the following are equivalent:*

(1) *$\mathcal{K}$ is a coherent sheaf of ideals;*

(2) *for any geometric points $\bar{x}$, $\bar{y}$ of $W$ with $\bar{y} \to \bar{x}$ a specialization arrow, the stalk $\mathcal{K}_{\bar{y}}$ is generated by the image of the generization map $\mathcal{K}_{\bar{x}} \to \mathcal{K}_{\bar{y}}$.*

*Proof.* (1)⇒(2): Suppose $\mathcal{K}$ is a coherent sheaf of ideals. Then given geometric points as in the statement of the lemma, there is an open neighborhood $U$ of $\bar{x}$ and a finite set of sections $S \subseteq \Gamma(U, \mathcal{M}_W)$ generating $\mathcal{K}|_U$. In particular, $\bar{y}$ lifts to a geometric point of $U$ and hence $\mathcal{K}_{\bar{x}}$ and $\mathcal{K}_{\bar{y}}$ are both generated by $S$. In particular, the generization map $\mathcal{K}_{\bar{x}} \to \mathcal{K}_{\bar{y}}$ is surjective.

(2)⇒(1): Suppose the generatedness statement always holds. Since $\mathcal{M}_W$ is fine, for any geometric point $\bar{x}$ of $W$, one may find an étale neighborhood $U$ with a chart $\phi : Q \to \mathcal{M}_W|_U$ inducing an isomorphism $Q \to \overline{\mathcal{M}}_{W,\bar{x}}$. Let $K \subseteq Q$ be the inverse image of $\overline{\mathcal{K}}_{\bar{x}}$ under this isomorphism, and let $S \subseteq K$ be a finite generating set. Then $\phi(S)$ provides a subset of $\Gamma(U, \mathcal{M}_W)$, necessarily generating an ideal subsheaf $\mathcal{K}'$ of $\mathcal{K}$. However, because of the assumed surjectivity, it follows immediately that $\mathcal{K}' = \mathcal{K}$. ∎

Many notions in log geometry have idealized versions. In particular, there are notions of idealized log étale and idealized log smooth morphisms, defined using idealized versions of formal lifting. We send the reader to [52, Section IV.3.1] for details.

Morally, an idealized log smooth morphism is one modeled on a morphism between torus invariant subschemes of toric varieties; alternatively it is a morphism $X \to Y$ such that there is a closed substack $\mathcal{Z}_{X/Y}$ of a relative Artin fan $\mathcal{A}_{X/Y}$ [5, Corollary 3.3.5] defined by a monomial ideal such that the induced morphism $X \to \mathcal{A}_{X/Y}$ factors through a smooth morphism $X \to \mathcal{Z}_{X/Y}$. See Proposition B.2 for precise statements as needed in this paper.

**Proposition 2.48.** *If $X \to B$ is log smooth, and $B$ is log smooth over $\Bbbk$ or is a log point, then every stratum $X_\sigma$ of $X$ is idealized log smooth over $B$, where $\sigma \in \Sigma(X)$. Here we endow $X_\sigma$ with its reduced induced subscheme structure, and with the log structure induced by the closed embedding $X_\sigma \hookrightarrow X$.*

*Proof.* Since the statement is étale local in $B$, we may assume there exists a global chart $B \to A_Q = \operatorname{Spec} \Bbbk[Q]$. Note also that by Proposition C.11, $X_\sigma$ is irreducible, hence is set-theoretically the closure of a geometric generic point $\overline{\eta}$ of $X_\sigma$.

Define the log ideal $\mathcal{K} \subseteq \mathcal{M}_{X_\sigma}$ on $X_\sigma$ by

$$\mathcal{K}(U) := \left\{ s \in \mathcal{M}_{X_\sigma}(U) \mid \alpha_{X_\sigma}(s) = 0 \right\}.$$

To check that $(X_\sigma, \mathcal{K}) \to (B, \emptyset)$ is idealized log smooth near a point $x \in X_\sigma$, we consider a chart for $X \to B$ as in Proposition B.4, an étale neighborhood $h : U \to X$ of $x$ fitting into a commutative diagram

$$\begin{array}{ccccc} U & \xrightarrow{g} & B \times_{\mathcal{A}_Q} \mathcal{A}_P & \longrightarrow & \mathcal{A}_P \\ & \searrow & \downarrow & & \downarrow \\ & & B & \longrightarrow & \mathcal{A}_Q, \end{array}$$

with all horizontal arrows strict, $g : U \to B \times_{\mathcal{A}_Q} \mathcal{A}_P$ smooth, $P^\times = \{0\}$, and a lift $\tilde{x}$ of $x$ to $U$ mapping to the closed (deepest) stratum of $\mathcal{A}_P$. Then we obtain an isomorphism $\psi : P \to \overline{\mathcal{M}}_{X,\bar{x}} = (\sigma_{\bar{x}}^\vee)_{\mathbb{Z}}$. Each specialization arrow $\overline{\eta} \to \bar{x}$ defines a face inclusion $\sigma \to \sigma_{\bar{x}}$, hence a closed reduced substack $\mathcal{Z} \subset \mathcal{A}_P$ with

$$h(g^{-1}(B \times_{\mathcal{A}_Q} \mathcal{Z})) \subseteq \mathcal{S}_\sigma,$$

where $\mathcal{Z}_\sigma$ is the logarithmic stratum of $X$ with closure $X_\sigma$. Thus if $F_i \subseteq P$ denotes the dual faces of $P$ defined by such specializations, then by the definitions of $\mathcal{K}$ and $X_\sigma$,

$$\psi\Big(P \setminus \bigcup_i F_i\Big) = \overline{\mathcal{K}}_{\bar{x}} \subseteq \overline{\mathcal{M}}_{X,\bar{x}}. \tag{2.22}$$

Note this gives an alternative, stalkwise definition of the log ideal $\mathcal{K}$, using the reasoning in Remark 2.36.

To show the claim on idealized smoothness, it thus remains to show that the preimage in $U$ of the closed reduced substacks of $\mathscr{A}_P$ are reduced for then the subscheme of $U$ defined by $P \setminus \bigcup_i F_i$ agrees with $h^{-1}(X_\sigma)$.

Now a closed reduced substack $\mathcal{Z} \subseteq \mathscr{A}_P$ maps onto a closed reduced substack $\mathcal{T}$ of $\mathscr{A}_Q$, which by our assumptions on $B$ pulls back to a reduced subscheme $S \subseteq B$. Therefore $B \times_{\mathscr{A}_Q} \mathcal{Z} = S \times_{\mathcal{T}} \mathcal{Z}$ is reduced since $S \to \mathcal{T}$ is smooth, and so is its preimage in $U$. ■

### 2.5.2 Log-ideals of punctured curves

Let $(\pi : C^\circ \to W, \mathbf{p})$ be a punctured curve. For each of the punctures $p : \underline{W} \to \underline{C}$ consider the composition

$$v_p : p^*\mathcal{M}_{C^\circ} \to \overline{\mathcal{M}}_W \oplus \underline{\mathbb{Z}} \to \underline{\mathbb{Z}} \tag{2.23}$$

of fine monoid sheaves, with the first map induced by the canonical monoid inclusion $p^*\overline{\mathcal{M}}_{C^\circ} \to \overline{\mathcal{M}}_W \oplus \underline{\mathbb{Z}}$ and the second map the projection. Denote by $\mathcal{I}_p \subseteq p^*\mathcal{M}_{C^\circ}$ the sheaf of ideals generated by $(v_p)^{-1}(\underline{\mathbb{Z}}_{<0})$.

**Definition 2.49.** The *puncturing log-ideal* $\mathcal{K}_W \subseteq \mathcal{M}_W$ of the punctured curve $(\pi : C^\circ \to W, \mathbf{p})$ is the ideal sheaf

$$\bigcup_p (\pi^\flat)^{-1}(\mathcal{I}_p) \subseteq \mathcal{M}_W,$$

with $p$ running over all punctures.

In the context of the definition we abuse notation when writing $\pi^\flat$ for the composition

$$\mathcal{M}_W \xrightarrow{\pi^\flat} \pi_*\mathcal{M}_{C^\circ} \to \pi_* p_* p^*\mathcal{M}_{C^\circ} = p^*\mathcal{M}_{C^\circ},$$

where as usual $p^*\mathcal{M}_{C^\circ}$ denotes the pullback log structure, while the right arrow is induced by the adjunction unit morphism $1 \to p_* p^{-1}$ of the associated abelian sheaves.

We sometimes also refer to the quotient $\overline{\mathcal{K}}_W$ of $\mathcal{K}_W$ by $\mathcal{O}_W^\times$ as the puncturing log-ideal, but will then write $\overline{\mathcal{K}}_W \subseteq \overline{\mathcal{M}}_W$ for clarity.

An illustration for the definition is contained in Figure 2.8.

This picture indicates an equivalent way to identify $\overline{\mathcal{K}}_W$. For the stalkwise characterization we may do a strict base change to a geometric point of $\underline{W}$ and hence assume $W$ is a log point. For a marking $p$ on a component of $C^\circ$ with generic point $\eta$, consider the generization map $\phi_{p,\eta} : \overline{\mathcal{M}}_{C^\circ,p} \to \overline{\mathcal{M}}_{C^\circ,\eta} \simeq \overline{\mathcal{M}}_W$. Identify $\overline{\mathcal{M}}_W$ as a submonoid of $\overline{\mathcal{M}}_{C^\circ,p}$ via $\pi^\flat$, making $\phi_{p,\eta}$ an idempotent homomorphism on $\overline{\mathcal{M}}_{C^\circ,p}$ with image $\overline{\mathcal{M}}_W$. An element $m \in \overline{\mathcal{M}}_W$ is in $\overline{\mathcal{K}}_W$ if and only if there is a marking $p$

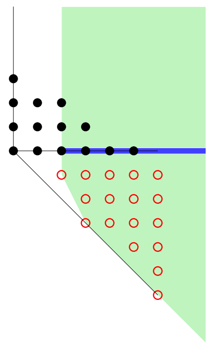

**Figure 2.8.** An idealized punctured point (ideal lightly shaded) and the resulting log ideal (the horizontal shaded ray). If there are several punctures, one takes the ideal generated by these horizontal regions.

and an element $n \in (u_p)^{-1}(\mathbb{Z}_{<0})$ such that $\phi_{p,\eta}(n) = m$, where $u_p : \overline{\mathcal{M}}_{C^\circ,p} \to \mathbb{Z}$ is the contact order associated to the identity morphism. Indeed, if there is $n \in (u_p)^{-1}(\underline{\mathbb{Z}}_{<0})$ and $n' \in \overline{\mathcal{M}}_{C^\circ,p}$ with $\pi^\flat(m) = n + n'$ then, writing $n'' = n + \phi_{p\eta}(n')$ we have $m = \phi_{p\eta}(n'')$; conversely, if $m = \phi_{p\eta}(n'')$ with $u_p(n'') = -b < 0$ then, using the notation of (2.23), we have $\pi^\flat(m) = n'' + b \cdot (0, 1)$.

**Lemma 2.50.** *The puncturing log-ideal $\mathcal{K}_W$ of a punctured curve $(\pi : C^\circ \to W, \mathbf{p})$ is coherent.*

*Proof.* We verify the characterization of Lemma 2.47. Let $\bar{x}, \bar{y} \to W$ with $\bar{x} \in \mathrm{cl}(\bar{y})$. Fix a generization map $\chi_{\bar{x}\bar{y}} : \overline{\mathcal{K}}_{\bar{x}} \to \overline{\mathcal{K}}_{\bar{y}}$ and let $m_{\bar{y}} \in \overline{\mathcal{K}}_{\bar{y}}$. We wish to construct $m_{\bar{x}} \in \overline{\mathcal{K}}_{\bar{x}}$ with $\chi_{\bar{x}\bar{y}}(m_{\bar{x}}) = m_{\bar{y}}$.

We refer to the following commutative diagram of generizations and contact orders:

$$\begin{array}{ccc}
\overline{\mathcal{M}}_{W,\bar{x}} = \overline{\mathcal{M}}_{C^\circ,\eta_{\bar{x}}} & \xrightarrow{\chi_{\bar{x}\bar{y}} = \chi_{\eta_{\bar{x}}\eta_{\bar{y}}}} & \overline{\mathcal{M}}_{C^\circ,\eta_{\bar{y}}} = \overline{\mathcal{M}}_{W,\bar{y}} \\
\uparrow \phi_{p_{\bar{x}}\eta_{\bar{x}}} & & \uparrow \phi_{p_{\bar{y}}\eta_{\bar{y}}} \\
\overline{\mathcal{M}}_{C^\circ,p_{\bar{x}}} & \xrightarrow{\chi_{p_{\bar{x}}p_{\bar{y}}}} & \overline{\mathcal{M}}_{C^\circ,p_{\bar{y}}} \\
\downarrow u_{p_{\bar{x}}} & & \downarrow u_{p_{\bar{y}}} \\
\mathbb{Z} & = & \mathbb{Z}
\end{array}$$

Note that $m_{\bar{y}} \in \overline{\mathcal{K}}_{\bar{y}}$ means that there is a puncture $p_{\bar{y}}$ lying on a component with generic point $\eta_{\bar{y}}$ of $C_{\bar{y}}$ and an element $m_{p_{\bar{y}}} \in (u_{p_{\bar{y}}})^{-1}(\underline{\mathbb{Z}}_{<0})$ whose generization is $\phi_{p_{\bar{y}}\eta_{\bar{y}}}(m_{p_{\bar{y}}}) = m_{\bar{y}}$.

Since $\mathcal{M}_{C^\circ}$ is coherent, there is an element $m_{p_{\bar{x}}} \in \overline{\mathcal{M}}_{C^\circ,\bar{x}}$ such that

$$\chi_{p_{\bar{x}}p_{\bar{y}}}(m_{p_{\bar{x}}}) = m_{p_{\bar{y}}}.$$

Note that $u_{p_{\bar{y}}} \circ \chi_{p_{\bar{x}} p_{\bar{y}}} = u_{p_{\bar{x}}}$, see Lemma 2.41. This implies $m_{p_{\bar{x}}} \in (u_{p_{\bar{x}}})^{-1}(\underline{\mathbb{Z}}_{<0})$. Write $m_{\bar{x}} := \phi_{p_{\bar{x}}\eta_{\bar{x}}}(m_{p_{\bar{x}}})$. By definition $m_{\bar{x}} \in \bar{\mathcal{K}}_{\bar{x}}$.

We obtain that $m_{\bar{y}} = \phi_{p_{\bar{y}}\eta_{\bar{y}}} \circ \chi_{p_{\bar{x}} p_{\bar{y}}}(m_{p_{\bar{x}}}) = \chi_{\eta_{\bar{x}}\eta_{\bar{y}}}\phi_{p_{\bar{x}}\eta_{\bar{x}}}(m_{p_{\bar{x}}}) = \chi_{\eta_{\bar{x}}\eta_{\bar{y}}}(m_{\bar{x}}) = \chi_{\bar{x}\bar{y}}(m_{\bar{x}})$, as needed. ∎

Puncturing log-ideals behave well under pull-backs.

**Proposition 2.51.** *Let $(\pi : C^\circ \to W, \mathbf{p})$ be a punctured curve, $(\pi_T : C_T^\circ \to T, \mathbf{p}_T)$ its pullback via $h : T \to W$ and $\mathcal{K}_W$, $\mathcal{K}_T$ the respective puncturing log-ideals. Then $\mathcal{K}_T = h^\bullet \mathcal{K}_W$.*

*Proof.* Denote by $g : C_T^\circ \to C^\circ$ the pullback of $h$ to the curves. By coherence of $\mathcal{K}_W$ and $\mathcal{K}_T$ it suffices to check that for each geometric point $\bar{t} \to T$, the image of $\bar{\mathcal{K}}_{W,\underline{h}(\bar{t})}$ under $\bar{h}_{\bar{t}}^\flat$ generates $\bar{\mathcal{K}}_{T,\bar{t}}$. Denote by $\bar{w} = h(\bar{t})$. For a puncture $p$ of $C^\circ$ consider the commutative diagram

$$\begin{array}{ccccccc}
\bar{\mathcal{M}}_{W,\bar{w}} & \xrightarrow{\bar{\pi}^\flat} & \bar{\mathcal{M}}_{C^\circ,p(\bar{w})} & \longrightarrow & \bar{\mathcal{M}}_{W,\bar{w}} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z} \\
\downarrow{\scriptstyle \bar{h}^\flat} & & \downarrow{\scriptstyle \bar{g}^\flat} & & \downarrow & & \downarrow{\scriptstyle =} \\
\bar{\mathcal{M}}_{T,\bar{t}} & \xrightarrow{\bar{\pi}_T^\flat} & \bar{\mathcal{M}}_{C_T^\circ,p_T(\bar{t})} & \longrightarrow & \bar{\mathcal{M}}_{T,\bar{t}} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z}
\end{array}$$

The two left squares are cocartesian in the category of fine monoids by the definition of pullback of punctured curves. This shows first that $\bar{g}^\flat(\bar{\mathcal{I}}_{p,\bar{w}})$ generates $\bar{\mathcal{I}}_{p_T,\bar{t}}$, and in turn that $\bar{h}^\flat((\bar{\pi}^\flat)^{-1}(\bar{\mathcal{I}}_{p,\bar{w}}))$ generates $(\bar{\pi}_T^\flat)^{-1}(\bar{\mathcal{I}}_{p_T,\bar{t}})$. Taking the union over all punctures finishes the proof. ∎

Here comes the crucial vanishing property putting restrictions on deformations of punctured curves.

**Proposition 2.52.** *Let $(C^\circ/W, \mathbf{p})$ be a punctured curve and $\mathcal{K}_W \subseteq \mathcal{M}_W$ its puncturing log-ideal. Then it holds*

$$\alpha_W(\mathcal{K}_W) = 0.$$

*Proof.* Let $\mathcal{I}_p = v_p^{-1}(\mathbb{Z}_{<0}) \subseteq p^*\mathcal{M}_{C^\circ}$ be the ideal sheaf defined after (2.23). Definition 2.1 (2) implies $(p^*\alpha_{C^\circ})(\mathcal{I}_p) = 0$. Pulling back via $\pi^\flat : \mathcal{M}_W \to p^*\mathcal{M}_{C^\circ}$ thus yields

$$\alpha_W((\pi^\flat)^{-1}(\mathcal{I}_p)) = (p^*\alpha_{C^\circ})(\mathcal{I}_p) = 0.$$

The claimed vanishing follows by taking the union over the punctures $p$. ∎

Proposition 2.52 demonstrates the announced statement that the base of a family of punctured curves is naturally an idealized log scheme (or stack).

**Corollary 2.53.** *For a punctured curve $(C^\circ/W, \mathbf{p})$ with $\mathcal{K}_W$ its puncturing log-ideal, the triple $(W, \mathcal{M}_W, \mathcal{K}_W)$ is a coherent idealized log scheme.*

**Example 2.54.** Let $(C^\circ/W, \mathbf{p})$ be a punctured curve over the logarithmic point $W = \operatorname{Spec}(Q \to \Bbbk)$, with $Q = \mathbb{N}^2$, $\underline{C}$ a smooth and connected curve and with only one punctured point $p$ with

$$\bar{\mathcal{M}}_{C^\circ,p} = (Q \oplus \mathbb{N}) + \mathbb{N}\cdot(a,0,-1) + \mathbb{N}\cdot(0,b,-1) \subset Q \oplus \mathbb{Z},$$

for some $a, b \in \mathbb{N} \setminus \{0\}$. Then the puncturing log-ideal $\bar{\mathcal{K}}_W$ is generated by $(a,0)$, $(0,b)$. This implies that if we view $W$ as the strict closed subspace of $\mathbb{A}^2 = \operatorname{Spec} \Bbbk[t_1, t_2]$ with its toric log structure, then the maximal subscheme of $\mathbb{A}^2$ to which $(C/W, \mathbf{p})$ extends is given by the ideal $(t_1^a, t_2^b) \subset \Bbbk[t_1, t_2]$.

### 2.5.3 Log-ideals of punctured maps

We define puncturing log-ideals only for pre-stable punctured maps.[5]

**Definition 2.55.** The *puncturing log ideal* $\mathcal{K}_W$ of a pre-stable punctured map $(C^\circ/W, \mathbf{p}, f)$ is the puncturing log-ideal of the punctured domain curve $(C^\circ/W, \mathbf{p})$, as defined in Definition 2.49.

It is clear from the definition and Proposition 2.51 that puncturing log ideals of punctured maps are stable under base change, and they also enjoy the vanishing property $\alpha_W(\mathcal{K}_W) = 0$ from Proposition 2.52.

We finish this subsection by giving a tropical interpretation in the spirit of Proposition 2.23 of the radical of the puncturing log-ideal $\mathcal{K}_W$ of a pre-stable punctured map, see Proposition 2.57. This interpretation is based on the following technical result concerning monoid ideals.

**Lemma 2.56.** *Suppose given a sharp toric monoid $Q$, and a collection of sharp toric monoids $P_{p_1}, \ldots, P_{p_r}$ along with monoid homomorphisms $\varphi_{p_i} : P_{p_i} \to Q \oplus \mathbb{Z}$ with $u_{p_i} := \mathrm{pr}_2 \circ \varphi_{p_i}$. Let $\mathrm{ev}_i := (\mathrm{pr}_1 \circ \varphi_{p_i})^t : Q^\vee_{\mathbb{R}} \to (P_{p_i})^\vee_{\mathbb{R}}$. Let the ideal $I \subset Q$ be the monoid ideal*

$$I = \bigcup_{i=1}^{r} \big\langle \mathrm{pr}_1 \circ \varphi_{p_i}(m) \mid m \in P_{p_i} \text{ and } u_{p_i}(m) < 0 \big\rangle.$$

*For $\sigma$ a face of the cone $Q^\vee_{\mathbb{R}}$, let $A_\sigma = \operatorname{Spec} \Bbbk[\sigma^\perp \cap Q]$ be the closed toric stratum of* $\operatorname{Spec} \Bbbk[Q]$ *corresponding to $\sigma$. Then there is a decomposition*

$$\operatorname{Spec} \Bbbk[Q]/\sqrt{I} = \bigcup_{\sigma} A_\sigma$$

[5] If $(C^\circ/W, \mathbf{p}, f)$ has associated pre-stable map $(\tilde{C}^\circ/W, \mathbf{p}, \tilde{f})$ (Proposition 2.5), the ideal $\mathcal{K}_W$ of Definition 2.49 associated to $C^\circ/W$ may strictly include the corresponding ideal associated to $\tilde{C}^\circ/W$.

*where the union is over all faces $\sigma$ of $Q^\vee_{\mathbb{R}}$ such that if $x \in \mathrm{Int}(\sigma)$, then $\mathrm{ev}_i(x) + \varepsilon u_{p_i} \in (P_{p_i})^\vee_{\mathbb{R}}$ for $\varepsilon > 0$ sufficiently small and $1 \le i \le r$.*[6]

*Proof.* Let $I_{p_i} \subset Q$ be the monoid ideal

$$I_{p_i} = \big\langle \mathrm{pr}_1 \circ \varphi_{p_i}(m) \mid m \in P_{p_i} \text{ satisfies } u_{p_i}(m) < 0 \big\rangle.$$

Of course $V(I) = \bigcap_i V(I_{p_i})$. We first show that if $\sigma$ satisfies the given condition, then $A_\sigma \subseteq V(I_{p_i})$ for each $i$. The monomial ideal defining $A_\sigma$ is $Q \setminus (\sigma^\perp \cap Q)$, so it is enough to show that $\sigma^\perp \cap I_{p_i} = \emptyset$. Choose an $x \in \mathrm{Int}(\sigma)$. Let $q \in I_{p_i}$ be a generator of $I_{p_i}$, that is, there exists an $m \in P_{p_i}$ such that $q = \mathrm{pr}_1(\varphi_{p_i}(m))$ and $u_{p_i}(m) < 0$. Since $m \in P_{p_i}$ and $\mathrm{ev}_i(x) + \varepsilon u_{p_i} \in (P_{p_i})^\vee_{\mathbb{R}}$ for some $\varepsilon > 0$, we have

$$0 \le \langle \mathrm{ev}_i(x) + \varepsilon u_{p_i}, m \rangle.$$

Thus $\langle u_{p_i}, m \rangle < 0$ implies $\langle \mathrm{ev}_i(x), m \rangle > 0$, or $\langle x, \mathrm{pr}_1(\varphi_{p_i}(m)) \rangle = \langle x, q \rangle > 0$, as desired.

Conversely, suppose that $A_\sigma \subseteq V(I)$ for some face $\sigma$ of $Q^\vee_{\mathbb{R}}$, but there exists an $i$ and some $x \in \mathrm{Int}(\sigma)$ such that $\mathrm{ev}_i(x) + \varepsilon u_{p_i} \notin (P_{p_i})^\vee_{\mathbb{R}}$ for any $\varepsilon > 0$. Then there exists an $m \in P_{p_i}$ such that $\langle \mathrm{ev}_i(x) + \varepsilon u_{p_i}, m \rangle < 0$ for all $\varepsilon > 0$. Since $\langle \mathrm{ev}_i(x), m \rangle \ge 0$, we must have $\langle \mathrm{ev}_i(x), m \rangle = 0$ and $u_{p_i}(m) < 0$. Thus $q = \mathrm{pr}_1(\varphi_{p_i}(m))$ lies in $I_{p_i}$. We have

$$\langle x, q \rangle = \langle \mathrm{ev}_i(x), m \rangle = 0,$$

so $q \in \sigma^\perp$. In particular, $z^q$ does not vanish on $A_\sigma$, contradicting $A_\sigma \subseteq V(I)$. ∎

**Proposition 2.57.** *Let $(C^\circ/W, \mathbf{p}, f)$ be a punctured map to $X$ over the logarithmic point $W = \mathrm{Spec}(Q \to \kappa)$,*

$$h : \Gamma = \Gamma(G, \ell) \to \Sigma(X)$$

*the associated tropical curve over $\omega = Q^\vee_{\mathbb{R}}$, and $(G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u})$ its type. Denote by $\sqrt{\bar{\mathcal{K}}_W} \subset Q$ the radical of the puncturing log-ideal of $(C^\circ/W, \mathbf{p}, f)$.*

*Then a face $Q' \subseteq Q$ lies in $Q \setminus \sqrt{\bar{\mathcal{K}}_W}$ if and only if for any punctured leg $L \in L(G)$ it holds*[7]

$$\ell(L)((Q')^\perp \cap \omega_{\mathbb{Z}}) \neq 0.$$

*In other words, $Q'$ determines a face $(Q')^\perp \cap \omega_{\mathbb{Z}}$ of $\omega_{\mathbb{Z}}$, and each point of this face corresponds to a tropical map. Thus we require that the length function $\ell(L)$ of each punctured leg be non-vanishing on this face of $\omega_{\mathbb{Z}}$.*

---

[6] See Example 2.58 below.

[7] Again, see Example 2.58 below.

*Proof.* By pre-stability, $\bar{\mathcal{K}}_W$ is generated by those $q \in Q$ such that there exists a puncture $p_i \to \underline{C}$ of $C^\circ$ and $m \in \bar{\mathcal{M}}_{X,\underline{f}(p_i)}$ with $\bar{f}^\flat(m) = (q, a)$ and $a = u_{p_i}(m) < 0$. Thus $\bar{\mathcal{K}}_W = I$ in Lemma 2.56 applied with $P_{p_i} = \bar{\mathcal{M}}_{X,\underline{f}(p_i)}$. Using the characterization of punctured legs in the pre-stable case in Proposition 2.23, the statement to be proved is then a reformulation of the conclusion of Lemma 2.56 in terms of tropical maps. ∎

Phrased more geometrically, the conclusion of Proposition 2.57 says that exactly those faces of the basic cone of a tropical punctured map (Definition 2.33) can possibly arise from a generization of punctured maps if the puncturing legs remain of positive length.

We end this section with an example highlighting the fact that the natural base spaces in punctured Gromov–Witten theory are possibly reducible spaces due to the puncturing ideals. See Theorem 3.25 and Remark 3.27 for the general picture underlying this phenomenon.

**Example 2.58.** *Algebraic setup.* Take $B = \operatorname{Spec} \Bbbk$, and consider $X$ a smooth surface with log structure coming from a smooth rational curve $D \subseteq X$ with $D^2 = 2$. Consider a type of punctured maps of genus 0, underlying curve class $[D]$, and four punctures, $p_1, \ldots, p_4$, with contact orders $-1, -1, 2$ and 2 respectively. Consider a punctured curve $f : C^\circ \to X$ where $C = C_1 \cup C_2 \cup C_3$ has three irreducible components and two nodes $q_1 = C_1 \cap C_2$, $q_2 = C_1 \cap C_3$. We assume $p_1, p_3 \in C_2$, $p_2, p_4 \in C_3$. Finally, $\underline{f}$ identifies $C_1$ with $D$ and contracts $C_2$ and $C_3$. Orienting the node $q_i$ from $C_1$ to $C_i$, it is not difficult to check such a curve exists with $u_{q_1} = u_{q_2} = 1$ (Figure 2.9).

*The tropical curve.* The corresponding tropical curve $\Gamma$ has three vertices, $v_1, v_2, v_3$, edges $E_{q_1}, E_{q_2}$, and legs $E_{p_1}, \ldots, E_{p_4}$. The moduli space of tropical curves of this type is $\mathbb{R}^3_{\geq 0}$, with coordinates $\rho, \ell_1, \ell_2$, where $\rho$ gives the distance of the image of $v_1$ from the origin of $\Sigma(X) = \mathbb{R}_{\geq 0}$, and $\ell_1, \ell_2$ give the lengths of the edges $E_{q_1}, E_{q_2}$. In particular, the basic monoid for this punctured log curve is $Q = \mathbb{N}^3$, generated by $\rho, \ell_1, \ell_2$.

*The punctured ideal.* In this case we may easily calculate the puncturing ideal (Definition 2.55). We have contributions from each of the two punctures. Using the definition, we note that at the puncture $p_i$, $i = 1$ or 2, the map $\varphi_{\bar{\eta}} \circ \chi_{\eta,p_i} : P_{p_i} = \mathbb{N} \to Q$ is dual to $\mathrm{ev}_i : Q^\vee_{\mathbb{R}} \to (P_{p_i})^\vee = \mathbb{R}_{\geq 0}$ evaluating the tropical curve parametrized by a point at $Q^\vee_{\mathbb{R}}$ at $v_2$ or $v_3$, see Lemma 2.56. Thus for $m \in Q^\vee_{\mathbb{R}}$, $\mathrm{ev}_i(m) = \rho(m) + \ell_i(m)$. Dually $\varphi_{\bar{\eta}} \circ \chi_{\eta,p_i} : P \to Q$ is given by $1 \mapsto \rho + \ell_i$. As $u_{p_i}(1) = -1$, $i = 1, 2$, we see the puncturing ideal $K$ is generated by $\rho + \ell_1, \rho + \ell_2$. Writing $\Bbbk[Q] = \Bbbk[x, y, z]$, with the three variables corresponding to $\rho, \ell_1, \ell_2$ respectively, we see $\operatorname{Spec} \Bbbk[Q]/K = \operatorname{Spec} \Bbbk[x, y, z]/(xy, xz)$, which has two irreducible components of differing dimension.

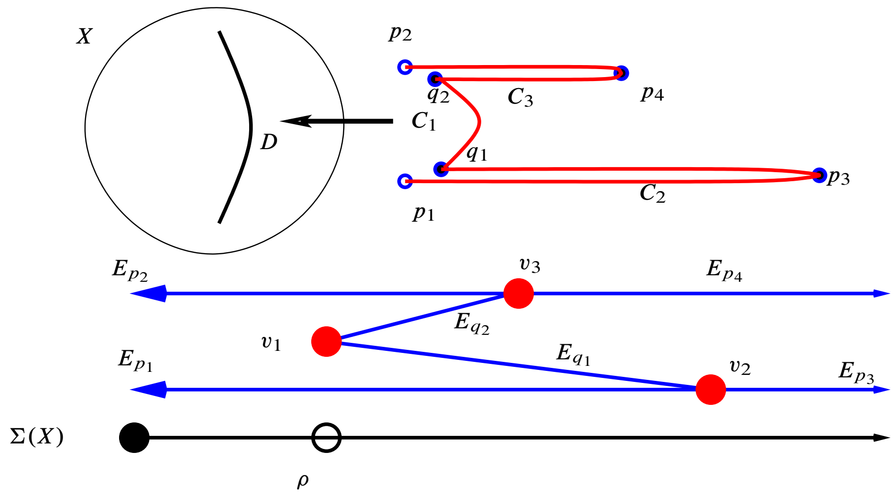


**Figure 2.9.** The algebraic map and its tropical counterpart. Here $\rho = 1$, $\ell_1 = 2$, and $\ell_2 = 1$.

*The participating and excluded cones.* The decomposition $\bigcup_\sigma A_\sigma$ of Lemma 2.56 translates to the statement that the cones *excluded* in this decomposition are the origin, the $\ell_1$-axis, and the $\ell_2$-axis. Indeed, these are the cones where at least one puncture is positioned with its tail at the origin, hence forced to have length 0, which is excluded by Proposition 2.57.

*The components of the algebraic moduli space.* Note that deformation theory provides two deformation classes of the punctured map. The first smooths one or both of the nodes, resulting in a punctured map with at least one pair $p_1, p_3$ or $p_2, p_4$ now being distinct points on the component of the domain mapping surjectively to $D$. Since this component contains a negative contact order point, its image cannot be deformed away from $D$ by Remark 2.20.

The second deformation class keeps the domain of $f$ fixed, but deforms the image of $C_1$ away from $D$, so that it meets $D$ transversally in two points. The remaining components $C_2$ and $C_3$ are then contracted to the points of intersection of $f(C_1)$ with $D$. It is then no longer possible to smooth the nodes.

*The data captured by the ideal.* This local reducibility of moduli space happens despite the obstruction group $H^1(C, f^*\Theta_X)$ for deformations with fixed domain (see Chapter 4) being zero. The point of the puncturing ideal is that it captures these intrinsic singularities of the moduli space. These obstructions really come from obstructions to deforming the punctured domain curve.

The general picture explaining this phenomenon is developed in Section 3.5. In particular, Example 3.32 revisits the present example from the general perspective.

## 2.6 Targets with monodromy

We now drop the assumption that $X$ is simple and discuss what is needed to treat the general case.

### 2.6.1 Tropicalization of punctured maps with non-simple targets

Let $(C^\circ/W, \mathbf{p}, f)$ be a punctured map over a logarithmic point $W = \operatorname{Spec}(Q \to \kappa)$ with $\kappa$ algebraically closed. Then the inclusion of a nodal point $q$ or punctured point $p$ into $C^\circ$ is a geometric point of $\underline{C}$ that we denote by $\bar{q}$ and $\bar{p}$, respectively. For a node $q$ of $\underline{C}$, the generic points $\eta, \eta' \in \operatorname{Spec} \mathcal{O}_{\underline{C},\bar{q}}$ of the two branches of $\underline{C}$ at $\bar{q}$ provide two specialization arrows of geometric points (see Appendix C)

$$\overline{\eta} \to \bar{q}, \quad \overline{\eta}' \to \bar{q},$$

unique up to order and precomposition with an isomorphism in the category of geometric points in $\operatorname{Spec} \mathcal{O}_{\underline{C},\bar{q}}$. The node $q$ is a self-intersection point of $\underline{C}$ iff $\overline{\eta}, \overline{\eta}'$ have the same image in $\underline{C}$, that is, iff they are isomorphic as geometric points of $\underline{C}$. In any case, denoting by $G$ the dual intersection graph of $C^\circ$, each specialization arrow $\overline{\eta} \to \bar{x}$ with $x \in E(G) \cup L(G)$ gives rise to a face inclusion

$$Q^\vee = \overline{\mathcal{M}}^\vee_{C,\overline{\eta}} \to \overline{\mathcal{M}}^\vee_{C,\bar{x}}. \tag{2.24}$$

The equality on the left-hand side is the canonical isomorphism obtained since $C^\circ$ is a log smooth curve over $\operatorname{Spec}(Q \to \kappa)$.

Applying $f$ yields a specialization arrow $f(\overline{\eta}) \to f(\bar{x})$ and a corresponding face embedding

$$\overline{\mathcal{M}}^\vee_{X,f(\overline{\eta})} \to \overline{\mathcal{M}}^\vee_{X,f(\bar{x})} \tag{2.25}$$

Our tropicalization procedure for $f : C^\circ \to W$ requires us to choose, for each $x \in V(G) \cup E(G) \cup L(G)$ with associated geometric point $\bar{x}$ of $\underline{C}$, an isomorphism

$$\operatorname{Hom}\big(\overline{\mathcal{M}}_{X,\underline{f}(\bar{x})}, \mathbb{R}_{\geq 0}\big) \to \boldsymbol{\sigma}(x) \tag{2.26}$$

in $\Sigma(X)$. Composing these isomorphisms or their inverses with the arrow in (2.25) defines an arrow

$$\iota_{x\eta} : \boldsymbol{\sigma}(\eta) \to \boldsymbol{\sigma}(x)$$

in $\Sigma(X)$. If $\Sigma(X)$ is simple there is only one arrow $\boldsymbol{\sigma}(\eta) \to \boldsymbol{\sigma}(x)$ in $\Sigma(X)$. In the general case, the $\iota_{x\eta}$ are part of the data defining the tropicalization, up to the simultaneous action of

$$\mathbb{G} = \prod_{x \in V(G) \cup E(G) \cup L(G)} \operatorname{Aut}_{\Sigma(X)}(\boldsymbol{\sigma}(x)) \tag{2.27}$$

on the choices of isomorphisms (2.26). Note that $\mathbb{G}$ may not act transitively on the set of arrows $\boldsymbol{\sigma}(\eta) \to \boldsymbol{\sigma}(x)$, and then the specialization morphism $\overline{\eta} \to \bar{x}$ in $\underline{C}$ at a node or marked point distinguishes a $\mathbb{G}$-orbit of such arrows.

We emphasize that if $x = q$ is a node there are two such arrows, regardless if $q$ is self-intersecting or not, one for each branch of $\underline{C}$ at $q$. Thus the proper labelling would not be by pairs $(\eta, q)$ but by half-edges of the dual intersection graph $G$ of $C^\circ$. By abuse of notation we nevertheless denote these two half-edges by $(q, \eta)$ and $(q, \eta')$.

Given a node $q$ with adjacent geometric generic point $\overline{\eta}$, we can compose $f^\flat_{\overline{\eta}} : \overline{\mathcal{M}}_{X,\underline{f}(\overline{\eta})} \to \overline{\mathcal{M}}_{C,\overline{\eta}}$ with the identification $\overline{\mathcal{M}}_{C,\overline{\eta}} = Q$ and the isomorphisms (2.26), and dualize to obtain the map of cones

$$V_\eta : Q^\vee \to \boldsymbol{\sigma}(\eta).$$

The defining equation [3, eq. (2.22)] of the contact order $u_q \in \boldsymbol{\sigma}(q)$ at $q$ now takes the form

$$\iota_{q\eta'} \circ V_{\eta'} - \iota_{q\eta} \circ V_\eta = \ell(E_q) \cdot u_q, \tag{2.28}$$

an equality in $\mathrm{Hom}(Q^\vee, \boldsymbol{\sigma}(q))$. Here $\eta'$ is the other geometric generic point of $\mathrm{Spec}\, \mathcal{O}_{C,\bar{q}}$ as above.

The pair $(V_\eta, V_{\eta'})$, or equivalently $(V_\eta, \ell(E_q), u_q)$, determines the tropicalization of $(C^\circ/W, \mathbf{p}, f)$ at $q$. At a marked point $p$, the tropicalization is similarly defined by $V_\eta$ and the contact order $u_p$.

Taken together, we obtain the following description of the tropicalization of $(C^\circ/W, \mathbf{p}, f)$.

**Proposition 2.59.** *The tropicalization of a punctured map $(C^\circ/W, \mathbf{p}, f)$ to $X$ with $W = \mathrm{Spec}(Q \to \kappa)$ an algebraically closed logarithmic point is given by the abstract tropical curve $(G, \mathbf{g}, \ell)$, i.e. the tropicalization of $C^\circ/W$, and the tuple*

$$(V_\eta, u_x, \iota_{x\eta})_{\eta,x},$$

*as discussed. Here $\eta \in V(G)$, $x \in E(G) \cup L(G)$, with $\eta$ adjacent to $x$ for $\iota_{x\eta}$, and the data is subject to (2.28). A self-intersecting node $q$ produces two arrows $\iota_{x\eta}$, as commented on above. The tuple $(V_\eta, u_x, \iota_{x\eta})_{\eta,x}$ is unique up to the obvious action of $\mathbb{G}$ from (2.27) on the set of tuples.*

Conversely, a tropical punctured map over $\omega \in \mathbf{Cones}$ consists of two maps $\Gamma \to \omega$ and $\Gamma \to \Sigma(X)$ of generalized cone complexes. Lifting both maps locally near the strata of $|\Gamma|$ labeled by vertices, edges and legs to maps of cone complexes provides a tuple $(V_\eta, u_x, \iota_{x\eta})_{\eta,x}$ that is again unique up to the action of $\mathbb{G}$. Thus we have a one-to-one correspondence between tropical punctured maps and $\mathbb{G}$-orbits of tuples $(V_\eta, u_x, \iota_{\eta x})_{\eta,x}$. Note in particular that each individual contact order $u_x \in \boldsymbol{\sigma}(x)$,

$x \in E(G) \cup L(G)$, is only defined up to the action of $\mathrm{Aut}_{\Sigma(X)}(\boldsymbol{\sigma}(x))$, but more information is retained when considering contact orders simultaneously and together with the set of face inclusions $\iota_{x\eta}$. Here is a simple example illustrating the effect of monodromy on the procedure.

**Example 2.60.** This is a modification of the Whitney umbrella example in [4, Section 5.4.1]. Let $C$ be the nodal cubic with its log smooth structure over the standard log point $\mathrm{Spec}(\mathbb{N} \to \Bbbk)$. Define $X$ as the quotient of $(\mathbb{A}^1 \setminus \{0\}) \times C$ by the $\mathbb{Z}/2$-action that swaps the two branches of $C$ at the node and acts by multiplication by $-1$ on $\mathbb{A}^1 \setminus \{0\}$. We can view $X$ as a non-trivial, log smooth fibration over $(\mathbb{A}^1 \setminus \{0\}) \times \mathrm{Spec}(\mathbb{N} \to \Bbbk)$ with all fibers $X_s$ isomorphic to the nodal cubic $C$. Thus $X$ is irreducible with two logarithmic strata with closures $\underline{X}$ and $\underline{X}_{\mathrm{sing}}$, respectively. Denoting by $\bar{\eta}_0$, $\bar{\eta}_1$ geometric generic points for these strata, we have $\overline{\mathcal{M}}_{X,\bar{\eta}_0} = \mathbb{N}$, $\overline{\mathcal{M}}_{X,\bar{\eta}_1} = \mathbb{N}^2$. The tropicalization $\Sigma(X)$ has a presentation with two non-zero cones

$$\sigma_0 = \mathbb{R}_{\geq 0}, \quad \sigma_1 = \mathbb{R}^2_{\geq 0},$$

and non-trivial arrows the two face inclusions $\sigma_0 \to \sigma_1$ and the automorphism $\sigma_1 \to \sigma_1$ swapping the two coordinates.

The inclusion $C \to X$ of a closed fiber defines a stable log map with unique generic point $\eta$, one node $q$, and no marked points. We have $\boldsymbol{\sigma}(\eta) = \sigma_0$, $\boldsymbol{\sigma}(q) = \sigma_1$, and a unique arrow (2.26) in $\Sigma(X)$ for $x = \eta$, hence a unique map of cones $V_\eta : Q^\vee = \mathbb{R}_{\geq 0} \to \sigma_0$. There are, however, two choices of isomorphisms

$$\mathrm{Hom}\big(\overline{\mathcal{M}}_{X,\underline{f}(\bar{q})}, \mathbb{R}_{\geq 0}\big) \to \boldsymbol{\sigma}(q) = \sigma_1.$$

Each such choice gives two arrows $\iota_{q\eta}, \iota_{q\eta'} : \sigma_0 \to \sigma_1$ and a contact order $u_q$. If one choice gives

$$(V_\eta, u_q, \iota_{q\eta}, \iota_{q\eta'})$$

for the tuple in Proposition 2.59, the other choice swaps $\iota_{q\eta}$, $\iota_{q\eta'}$ and replaces $u_q$ by $-u_q$. This is indeed the action of $\mathbb{G} = \mathbb{Z}/2$ on the set of tuples as stated in the same proposition.

The relation to the Whitney umbrella $Y = V(x^2z - y^2) \subseteq \mathbb{A}^3$ is as follows. Endow $Y$ with the restriction of the divisorial log structure on $\mathbb{A}^3$ defined by $Y$. We view $Y \setminus V(z)$ as a fibration over $\mathbb{A}^1 \setminus \{0\}$ by one-nodal rational curves via projection to the $z$-coordinate. Then there is an étale map $Y \setminus V(z) \to X$ of degree two of fiber spaces over $\mathbb{A}^1 \setminus \{0\}$ that separates the branches of the fibers of $X \to \mathbb{A}^1 \setminus \{0\}$.

### 2.6.2 Types of punctured maps with non-simple targets

One way to define the type of a punctured map in general is as an equivalence class of tropicalizations which identifies two tropical punctured maps whenever they fit into one family. The action of the automorphism group $\mathbb{G}$ on a face map $\iota_{x\eta}$ in Proposition 2.59 is induced by propagation along appropriate families. Thus in the general case, the type of a punctured map at a geometric point, or of a tropical punctured map, in addition to $(G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u})$ needs to specify these face maps $\iota_{x\eta}$, at least up to the overall action by $\mathbb{G}$. This leads to the following modification of Definition 2.24.

**Definition 2.61.** (1) A *framed type (of a family of tropical punctured maps)* is a tuple $(G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u})$ with $\mathbf{u}(x) \in N_{\boldsymbol{\sigma}(x)}$ for all $x \in E(G) \cup L(G)$ as in Definition 2.24, together with arrows[8] in $\Sigma(X)$,

$$\iota_{xv} : \boldsymbol{\sigma}(v) \to \boldsymbol{\sigma}(x),$$

for all $x \in E(G) \cup L(G)$ and $v \in V(G)$ an adjacent vertex.

(2) The *type (of a family of tropical punctured maps)* is an equivalence class of framed types under the obvious action of $\mathbb{G}$ on the set of framed types, as obtained from Proposition 2.59. The notation for a framed type is $(G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\iota})$ with $\boldsymbol{\iota} = (\iota_{xv})_{x,v}$.

The *type of a punctured map* $(C^\circ/W, \mathbf{p}, f)$ *to* $X$ *at a geometric point* $\bar{w}$ *of* $W$ is the type of the associated tropical map $\Gamma \to \Sigma(X)$ over $\omega = (\bar{\mathcal{M}}^\vee_{W,\bar{w}})_\mathbb{R}$.

Note that $\mathbb{G}$ acts trivially on the domain data $(G, \mathbf{g})$, the strata map $\boldsymbol{\sigma}$ and on global contact orders. So for framed types the action is on the tuple $(\mathbf{u}(x), \iota_{xv})$ with $x$ running through $E(G) \cup V(G)$ and $v$ through vertices adjacent to $x$. In particular, since the group $\mathbb{G}$ acts also trivially on the space $\mathfrak{C}_\sigma$ of global contact orders for $\sigma \in \Sigma(X)$, the definition of global type in Definition 2.44 remains unchanged.

We skip the obvious decorated versions of the notions of types in the general case. These just add the data of curve classes to vertices.

### 2.6.3 Contraction morphisms of types for non-simple targets

The definition of contraction morphism of types

$$\phi : \tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u}) \to \tau' = (G', \mathbf{g}', \boldsymbol{\sigma}', \mathbf{u}')$$

from [3, Definition 2.24] imposes the condition that $\boldsymbol{\sigma}'(\phi(x))$ is a face of $\boldsymbol{\sigma}(x)$ for all $x \in V(G) \cup (E(G) \setminus E_\phi) \cup L(G)$. In the general case, this condition has to be replaced by the choice of an arrow

$$\boldsymbol{\sigma}'(\phi(x)) \to \boldsymbol{\sigma}(x)$$

in $\Sigma(X)$ as part of the data defining $\phi$. We obtain the following definition.

[8] In the case of a self-intersecting node $x = q$ there are two such arrows, which as before we do not distinguish by the notation.

**Definition 2.62.** (1) Let $\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\iota})$, $\tau' = (G', \mathbf{g}', \boldsymbol{\sigma}', \mathbf{u}', \boldsymbol{\iota}')$ be two framed types. A *contraction morphism of framed types* $\tau \to \tau'$ is a contraction morphism $\phi : (G, \mathbf{g}) \to (G', \mathbf{g}')$ of genus-decorated graphs together with arrows

$$\iota_x : \boldsymbol{\sigma}'(\phi(x)) \to \boldsymbol{\sigma}(x)$$

in $\Sigma(X)$ for all $x \in V(G) \cup (E(G) \setminus E_\phi) \cup L(G)$. We require that the $\iota_x$ are compatible with $\boldsymbol{\iota}$, $\boldsymbol{\iota}'$, that is, the diagrams

$$\begin{array}{ccc} \boldsymbol{\sigma}'(\phi(v)) & \xrightarrow{\iota_v} & \boldsymbol{\sigma}(v) \\ \downarrow{\scriptstyle \iota'_{\phi(x)\phi(v)}} & & \downarrow{\scriptstyle \iota_{xv}} \\ \boldsymbol{\sigma}'(\phi(x)) & \xrightarrow{\iota_x} & \boldsymbol{\sigma}(x) \end{array} \tag{2.29}$$

commute, for all $x \in (E(G) \setminus E_\phi) \cup L(G)$ and all $v \in V(G)$ an adjacent vertex.[9]

(2) An equivalence class for the obvious action of the group $\mathbb{G}$ from (2.27) acting on the set of contraction morphisms with domain framed types with given $(G, \mathbf{g}, \boldsymbol{\sigma})$ defines the notion of *contraction morphism of types*.

There is again no change in the definition of contraction morphism of global types compared to the case with simple $X$.

As in the discussion of types in the preceding Section 2.6.2, we have again skipped spelling out the trivial generalization to the decorated versions.

Contraction morphisms arise from specializations in families of punctured maps, as proved in the case of simple $X$ in Proposition 2.25. Here is the version for the general case.

**Proposition 2.63.** *Let $(C^\circ/W, \mathbf{p}, f)$ be a stable punctured map to $X$ over some logarithmic scheme $W$, and let $\bar{w}' \to \bar{w}$ be a specialization arrow of geometric points of $W$. Let $(\tau, \mathbf{A})$ with $\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\iota})$ be the decorated framed type of $(C/W, \mathbf{p}, f)$ at the geometric point $\bar{w}$ of $\underline{W}$ according to Definition 2.61 (2) by a choice of arrows (2.26). Let similarly $(\tau', \mathbf{A}')$ with $\tau' = (G', \mathbf{g}', \boldsymbol{\sigma}', \mathbf{u}')$ be the decorated framed type of $(C^\circ/W, \mathbf{p}, f)$ at $\bar{w}'$, for the induced choice of arrows (2.26).*

*Then the map*

$$(\tau, \mathbf{A}) \to (\tau', \mathbf{A}')$$

*induced by generization is a contraction morphism.*

*Proof.* The proof is again identical to the proof of [3, Lemma 2.30] save keeping track of the choices of arrows in $\Sigma(X)$. ■

[9] Note that $\iota'_{\phi(x)\phi(v)}$ is uniquely determined by the diagram from $\iota_v, \iota_{xv}, \iota_x$.

### 2.6.4 The basic monoid and tropical moduli in general

The definition of basicness (Definition 2.31) makes sense in complete generality by replacing "type" by "a framed type representing the type of $(C^\circ/W, \mathbf{p}, f)$ at the geometric point $\bar{w}$". Indeed, given a framed type, the space of tropical curves of the given framed type is a subspace of the set of tuples $(V_\eta, \ell_q)$ with entries taking values in strongly convex rational polyhedral cones and subject to some integral equalities, hence is parametrized by a strongly convex rational polyhedral cone itself. This cone has been made explicit in Proposition 2.32 in the case of simple $X$. Here is the restatement of this proposition with reference to a framed type.

**Proposition 2.64.** *Let $(\pi : C^\circ/W, \mathbf{p}, f)$ be a basic, pre-stable punctured map over a logarithmic point* $\mathrm{Spec}(Q \to \kappa)$ *with $\kappa$ an algebraically closed field. Denote by $G$ the dual intersection graph of $C^\circ$. For each $x \in V(G) \cup E(G) \cup L(G)$ with associated geometric point $\bar{x}$ of $\underline{C}^\circ$ and smallest stratum $\boldsymbol{\sigma}(x) \in \Sigma(X)$ containing $f(\bar{x})$ choose an isomorphism*

$$\mu_x : \overline{\mathcal{M}}_{X,f(\bar{x})} \to (\boldsymbol{\sigma}(x)_{\mathbb{Z}})^\vee,$$

*dual to an arrow in $\Sigma(X)$ as in (2.26). Denote by $(G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\iota})$ the framed type of $(\pi : C^\circ/W, \mathbf{p}, f)$ defined by this choice according to the discussion leading to Proposition 2.59. Then the map*

$$Q^\vee \to \Big\{((V_\eta)_\eta, (\ell_q)_q) \in \prod_\eta \boldsymbol{\sigma}(\eta)_{\mathbb{Z}} \times \prod_q \mathbb{N} \mid \iota_{q\eta} \circ V_\eta - \iota_{q\eta'} \circ V_{\eta'} = \ell_q \cdot \mathbf{u}(q)\Big\} \tag{2.30}$$

*with $V_\eta$-entry the dual of $\overline{(\pi_\eta^\flat)^{-1} \circ \bar{f}_{\bar{\eta}}^\flat \circ \mu_\eta^{-1}} : \boldsymbol{\sigma}(\eta)_{\mathbb{Z}}^\vee \to Q$ and $\ell_q$-entries given by the dual of the classifiying map $\prod_q \mathbb{N} \to Q$ of the log smooth curve $C/W$, is an isomorphism. Here $\eta$ and $q$ run over the set of generic points and nodes of $\underline{C}$, respectively. The equation in the bracket holds in $N_{\boldsymbol{\sigma}(q)}$ for all nodal points $q$ with adjacent generic points $\eta$, $\eta'$ ordered according to the orientation of $E_q$ (with the usual ambiguity of notation concerning self-intersecting nodes).*

*Proof.* The proof is identical to the proof of Proposition 2.32 once the refined tropicalization procedure of Section 2.6.1 is taken into account. ∎

With this description of the basic monoid in the general case the proof of Proposition 2.34, which proves that basicness is an open condition, generalizes without problems.

The final point we want to discuss concerns the monoid quotient

$$\chi_{\tau\tau'} : Q_{\tau'} \to Q_{\tau\tau'}, \tag{2.31}$$

of basic monoids from (2.21) obtained from a framed type $\tau'$ and contraction morphism $\overline{\tau}' \to \tau$ of the associated global type. The basic monoid $Q_{\tau'}$ depends only on the framed type, as spelled out in (2.30). But note that the group $\mathbb{G}$ from (2.27) generally acts non-trivially on the right-hand side of (2.30), so the basic monoid is *not* intrinsic to the type.

Similarly, the description of $Q_{\tau\tau'}$ in (2.20) requires the knowledge of the image of the arrows $\iota_v : \boldsymbol{\sigma}(\phi(v)) \to \boldsymbol{\sigma}'(v)$, hence works only for a contraction morphism of framed types as follows. Let $(C^\circ/W, \mathbf{p}, f)$ be a basic punctured map and $\bar{w}$ a geometric point of $\underline{W}$. Then a choice of isomorphisms in (2.26), or equivalently of $\boldsymbol{\mu} = (\mu_x)$ in Proposition 2.64, provides a framed type $\tau' = (G', \mathbf{g}', \boldsymbol{\sigma}', \mathbf{u}', \boldsymbol{\iota}')$ and an isomorphism of $\bar{\mathcal{M}}^\vee_{W,\bar{w}}$ with the submonoid $Q^\vee_{\tau'} \subseteq \prod_\eta \boldsymbol{\sigma}'(\eta)^\vee_{\mathbb{Z}} \times \prod_q \mathbb{N}$ on the right-hand side of (2.30). Let $\phi : \overline{\tau}' \to \tau$ be a contraction morphism of the global type $\overline{\tau}'$ associated to $\tau'$ to some other global type $\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \bar{\mathbf{u}})$. Then each choice $\boldsymbol{\iota}_\bullet$ of arrows

$$\iota_v : \boldsymbol{\sigma}(\phi(v)) \to \boldsymbol{\sigma}'(v), \quad v \in V(G)$$

in $\Sigma(X)$ provides a face $Q^\vee_{\tau\tau'}(\boldsymbol{\iota}_\bullet) \subseteq Q_{\tau'}$ as in (2.20), hence a dual localization morphism

$$\chi_{\tau\tau'}(\boldsymbol{\mu}, \boldsymbol{\iota}_\bullet) : \bar{\mathcal{M}}_{W,\bar{w}} \xrightarrow{\simeq} Q_{\tau'} \to Q_{\tau\tau'}(\boldsymbol{\iota}_\bullet)$$

as in (2.31). Thus this quotient of $\bar{\mathcal{M}}_{W,\bar{w}}$ depends on both the choices of $\boldsymbol{\mu}$ and $\boldsymbol{\iota}_\bullet$. Note that $Q_{\tau\tau'} \neq 0$ only if there exists a degeneration of tropical punctured maps of framed type $\tau$ compatible with the restriction on the images of vertices given by $\boldsymbol{\iota}_\bullet$.

The schematic restriction to punctured maps of global type $\tau$ is then locally reflected in the monoid ideal

$$I_{\tau\tau'} = \bigcap_{\boldsymbol{\iota}} (\chi_{\tau\tau'}(\boldsymbol{\mu}, \boldsymbol{\iota}_\bullet))^{-1}(Q_{\tau\tau'}(\boldsymbol{\iota}) \setminus \{0\}) \subseteq \bar{\mathcal{M}}_{W,\bar{w}}. \tag{2.32}$$

Note that unlike in the simple case, $\operatorname{Spec} \Bbbk[Q_{\tau'}]/I_{\tau\tau'}$ may now be a reducible scheme. See Definition 3.4 (3) for the use of this ideal in a moduli context.