Chapter 3

The stack of punctured maps

Throughout this chapter we fix as the target a morphism $X \to B$ locally of finite type between separated, locally noetherian fs logarithmic schemes over $\Bbbk$. We assume further that X is connected and that $X \to B$ fits into a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{A}_X \\ \downarrow & & \downarrow \\ B & \longrightarrow & \mathcal{A}_B \end{array}$$

with strict horizontal arrows, $\mathcal{A}_B$ the Artin fan of B, and $\mathcal{A}_X$ an Artin fan representable over Log or over Log^1. If X has a Zariski log structure and $X \to B$ is log smooth then [3, Proposition 2.8] shows that we can take the Artin fan of X for $\mathcal{A}_X$, which is representable over Log by definition. In general, [5, Corollary 3.3.5] provides the desired diagram with $\mathcal{A}_X$ representable over Log^1.[1] We define

$$\mathcal{X} = B \times_{\mathcal{A}_B} \mathcal{A}_X,$$

which by abuse of notation we refer to as *the relative Artin fan* of $X \to B$.

3.1 Stacks of punctured curves

The purpose of this section is the introduction of stacks of punctured curves as domains for punctured maps.

3.1.1 Stacks of marked pre-stable curves

For a genus-decorated graph $(G, \mathbf{g})$ recall from [3, Section 2.4] the logarithmic stacks $\mathbf{M}(G, \mathbf{g})$ of $(G, \mathbf{g})$-marked pre-stable curves over the ground field $\Bbbk$ with its basic log structure as a nodal curve, and $\mathfrak{M}_B(G, \mathbf{g}) = \mathrm{Log}_{\mathbf{M}(G,\mathbf{g})\times B}$ of $(G, \mathbf{g})$-marked log smooth curves over B with arbitrary fs log structures on the base. For a leg $L \in L(G)$ denote by p_L the associated marked section.

3.1.2 The nodal log-ideal on $\mathbf{M}(G, \mathbf{g})$

Since the basic monoid of an r-nodal curve is $\mathbb{N}^r$, each $(G, \mathbf{g})$-marked nodal curve $C \to W$ comes with a homomorphism $\mathbb{N}^r \to \overline{\mathcal{M}}_W$ with $r = |E(G)|$. The image of

[1] The representability assumption is used in the proof of Lemma 3.11.

$\mathbb{N}^r \setminus \{0\}$ generates a coherent sheaf of ideals $\overline{\mathcal{I}} \subset \overline{\mathcal{M}}_W$ with preimage $\mathcal{I} \subset \mathcal{M}_W$ mapping to 0 under the structure homomorphism $\mathcal{M}_W \to \mathcal{O}_W$. Thus $\mathcal{I}$ endows $\mathbf{M}(G, \mathbf{g})$ with the structure of an idealized log stack.

Definition 3.1. We refer to $\mathcal{I}$ and to any pullback of $\mathcal{I}$ to a stack over $\mathbf{M}(G, \mathbf{g})$ such as $\mathfrak{M}(G, \mathbf{g})$ (and $\breve{\mathfrak{M}}(G, \mathbf{g})$ below) as the *nodal log-ideal*.

The local structure of moduli spaces of nodal curves implies that $\mathbf{M}(G, \mathbf{g})$ with the nodal log-ideal is idealized logarithmically smooth over the trivial log point Spec $\Bbbk$. If $(C/W, \mathbf{p})$ is a $(G, \mathbf{g})$-marked curve, the log ideal is generated at a geometric point $\bar{w}$ of $\underline{W}$ by those standard basis vectors of $\overline{\mathcal{M}}_{W,\bar{w}} \simeq \mathbb{N}^r$ mapping to the smoothing parameters of the nodes labeled by $E(G)$.

3.1.3 Enter stacks of punctured curves

We now define a stack $\breve{\mathfrak{M}}_B(G, \mathbf{g})$ of punctured curves by admitting arbitrary puncturings at these marked sections.

Definition 3.2. Let $(G, \mathbf{g})$ be a genus-decorated graph. A $(G, \mathbf{g})$-*marking* of a punctured curve $(C^\circ/W, \mathbf{p})$ is a $(G, \mathbf{g})$-marking of the underlying marked curve $(\underline{C}/\underline{W}, \mathbf{p})$. The stack $\breve{\mathfrak{M}}_B(G, \mathbf{g})$ is the fibered category over $(\mathbf{Sch}/\underline{B})$ with objects $(G, \mathbf{g})$-marked punctured curves $(C^\circ/W, \mathbf{p})$ over B. Morphisms are given by *strict* fiber diagrams of punctured curves respecting the markings by $(G, \mathbf{g})$.

Note that the morphisms in $\breve{\mathfrak{M}}_B(G, \mathbf{g})$ are pull-backs of punctured curves as defined in Definition 2.13.

The maps associating to a $(G, \mathbf{g})$-marked punctured curve the underlying $(G, \mathbf{g})$-marked nodal curve with its *basic* log structure defines a morphism of logarithmic stacks

$$\breve{\mathfrak{M}}_B(G, \mathbf{g}) \to \mathbf{M}(G, \mathbf{g}). \tag{3.1}$$

3.1.4 The stacks of punctured curves are algebraic

Proposition 3.3. (1) *The stack $\breve{\mathfrak{M}}_B(G, \mathbf{g})$ is a logarithmic algebraic stack.*

(2) *Endowing $\breve{\mathfrak{M}}_B(G, \mathbf{g})$ with the idealized log structure defined by the union of its puncturing log-ideal (Definition 2.49) and its nodal log-ideal (Definition 3.1) and $\mathfrak{M}_B(G, \mathbf{g})$ with its nodal log-ideal, the strict morphism*

$$\breve{\mathfrak{M}}_B(G, \mathbf{g}) \to \mathfrak{M}_B(G, \mathbf{g})$$

forgetting the puncturing, is locally of finite type, quasi-separated, representable, unramified, and idealized logarithmically étale.

Proof. We argue by showing that the morphism $\breve{\mathfrak{M}}_B(G,\mathbf{g}) \to \mathfrak{M}_B(G,\mathbf{g})$ is representable by algebraic spaces, satisfying the adjectives spelled out in (2).[2] This is sufficient as $\mathfrak{M}_B(G,\mathbf{g})$ is a logarithmic algebraic stack.

The stack $\mathfrak{M}_B(G,\mathbf{g})$ is locally noetherian, so it has a covering $\sqcup W_\alpha \to \mathfrak{M}_B(G,\mathbf{g})$ in the strict smooth topology, where W_α are noetherian logarithmic schemes. Letting W be one of these, define

$$\breve{W} = W \times_{\mathfrak{M}_B(G,\mathbf{g})} \breve{\mathfrak{M}}_B(G,\mathbf{g}),$$

viewed as a category fibered in groupoids over $\underline{W}$, or, equivalently, over the category of strict morphisms $T \to W$. It suffices to prove that $\breve{W}$ is an algebraic space satisfying the conditions of (2).

We show this directly by exhibiting $\breve{W}$ as a *sheaf of sets*, with representable diagonal, having an étale covering by a scheme, and satisfying the above conditions.

The morphism $W \to \mathfrak{M}_B(G,\mathbf{g})$ corresponds to a $(G,\mathbf{g})$-marked logarithmic curve $\pi : C \to W$. Spelled out, the formation of $\breve{W}$ means that for any strict morphism $T \to W$, the objects in $\breve{W}(T)$ are punctured curves $(C_T^\circ \to C_T \to T, \mathbf{p}_T)$ with punctures at the markings of C_T. Here $C_T = C \times_W T \to T$ is the pullback of the logarithmic curve $C \to W$. Pull-backs in $\breve{W}$ are defined as pull-backs of punctured curves along strict morphisms over W. The markings by $(G,\mathbf{g})$ are inherited from C/W and do not play any further role.

First, we note that $\breve{W}$ is a sheaf of sets over W. We have to show that any automorphism of the log curve parametrized by W induces at most one automorphism of any corresponding punctured curve above it. Indeed, an isomorphism of punctured curves over the identity of a given logarithmic curve is a pullback diagram as in Diagram (2.3), with $h : T = W \to W$ and $C_T = C \to C$ the identity. Such an isomorphism is an equality of the submonoids of $\mathcal{M} \oplus_{\mathcal{O}^\times} \mathcal{P}^{\mathrm{gp}}$ in the notation of Definition 2.1. In particular, such an isomorphism is unique when it exists.

Second, Isom functors are representable, in fact by open subschemes of the base T. Indeed, the locus on C_T where two logarithmic structures inside $\mathcal{M}_{C_T}^{\mathrm{gp}}$ coincide is open in C_T (as can be deduced from Lemma 2.17), and its complement is a closed subscheme of the markings of C_T, whose image in T is closed. The complement is the desired open subscheme of T. In particular, $\breve{W} \to \breve{W} \times_W \breve{W}$ is an open embedding; once we prove $\breve{W}$ is locally of finite type over W, we will know the diagonal $\breve{W} \to \breve{W} \times_W \breve{W}$ is quasi-compact. This will prove the quasi-separatedness in (2).

Third, it now remains to construct an étale atlas by a scheme, and verify the various adjectives in (2).

We note that the statements of the proposition are both local on W. Further shrinking W, we may assume that the Artin fan $\mathcal{A}_W$ equals $\mathcal{A}_Q$ for an fs and sharp monoid Q.

[2] A simple reduction to known stacks would be welcome.

To prove both statements of the proposition, it suffices to proceed as follows: For any object $C_T^\circ \to C_T \to T$ in $\breve{W}(T)$,

(1) we will construct a locally of finite type, unramified, idealized logarithmically étale, and strict morphism $V \to W$, for V some log scheme,

(2) show that $T \to W$ factors through V,

(3) construct a punctured curve $C_V^\circ \to C_V \to V$, and

(4) show that $C_T^\circ \to C_T \to T$ is the pullback of $C_V^\circ \to C_V \to V$.

(5) Finally, we will show that the tautological morphism $V \to \breve{W}$ defined by the family $C_V^\circ \to C_V \to V$ is étale.

In particular, we obtain an étale cover $\sqcup V \to \breve{W}$ of the sheaf $\breve{W}$ by ordinary schemes, or equivalently, by strict étale morphisms of log schemes.

Since the statements above are étale local on T, we may assume the Artin fan $\mathcal{A}_T$ equals $\mathcal{A}_{Q'}$ for some fs sharp monoid Q'. Since the puncturing ideal $\mathcal{K}_T$ of $(C_T^\circ \to T, \mathbf{p}_T)$ is coherent, further shrinking T we may assume that there is a monoid ideal $K \subset Q'$ such that the corresponding log ideal $\mathcal{K}$ on $\mathcal{A}_{Q'}$ pulls-back to $\mathcal{K}_T$.

The strict morphism $T \to W$ induces a strict open embedding $\mathcal{A}_{Q'} \to \mathcal{A}_Q$. Replacing W by its strict open subscheme $W \times_{\mathcal{A}_Q} \mathcal{A}_{Q'}$, we may assume that $Q = Q'$.

Step 1. Construction of $V \to W$. Fix any point $t \in T$ over the unique closed point of $\mathcal{A}_Q$. Consider the monoid ideal $K = \overline{\mathcal{K}}_T|_t \subset Q$. Let $\mathcal{V} \to \mathcal{A}_Q$ be the strict closed embedding defined by the ideal K, and $\mathcal{K}_{\mathcal{V}}$ be the corresponding log ideal over $\mathcal{V}$. Then $\mathcal{V} \to \mathcal{A}_Q$ is finite type, strict, and idealized logarithmically étale. Thus the projection $V := \mathcal{V} \times_{\mathcal{A}_Q} W \to W$ with the log ideal $\mathcal{K}_V := \mathcal{K}_{\mathcal{V}}|_V$ is a finite type, strict closed embedding and idealized logarithmically étale.

Step 2. $T \to W$ factors through V. Recall that $\mathcal{K}_T$ is the pullback of $\mathcal{K}$. By Proposition 2.52 applied to C_T°/T the image $\alpha_T(\mathcal{K}_T) = (\alpha_{\mathcal{A}_Q}(\mathcal{K}))_T$ is the zero ideal. Hence the morphism $T \to \mathcal{A}_Q$ factors through $\mathcal{V}$. Consequently, $T \to W$ factors through V, as claimed.

For the point t as in Step 1, we denote its image in V by w.

Step 3. Construction of the punctured curves $C_V^\circ \to C_V \to V$. To construct the sheaf of monoids $\overline{\mathcal{M}}_{C_V^\circ}$, first notice that the inclusion $\overline{\mathcal{M}}_{C_V} \subseteq \overline{\mathcal{M}}_{C_V^\circ}$ is an isomorphism away from the points of $\mathbf{p}$. For each puncture $p_w \in \mathbf{p}_w$ over w, we define $\overline{\mathcal{M}}_{C_V^\circ, p_w} := \overline{\mathcal{M}}_{C_T^\circ, p_t}$ using the fiber over t. Let p_T, p_V be the punctured sections corresponding to p_w of $\underline{C}_T/\underline{T}$, $\underline{C}_V/\underline{V}$ respectively. Note that we have

$$\overline{\mathcal{M}}_{C_T^\circ, p_t} \xrightarrow{\cong} \Gamma\big(\underline{T}, p_T^* \overline{\mathcal{M}}_{C_T^\circ}\big), \quad \overline{\mathcal{M}}_{C_T, p_t} = \overline{\mathcal{M}}_{C_V, p_w} = Q \oplus \mathbb{N} \xrightarrow{\cong} \Gamma\big(\underline{V}, p_V^* \overline{\mathcal{M}}_{C_V}\big).$$

Define $\overline{\mathcal{M}}_{C_V^\circ} \subset \overline{\mathcal{M}}_{C_V}^{\mathrm{gp}}$ as the subsheaf of fine monoids generated by the image of $\overline{\mathcal{M}}_{C_V^\circ, p_w} \subset \overline{\mathcal{M}}_{C_V, p_w}^{\mathrm{gp}}$ under this isomorphism.

Consider $\mathcal{M}_{C_V^\circ} := \mathcal{M}_{C_V}^{\mathrm{gp}} \times_{\bar{\mathcal{M}}_{C_V}^{\mathrm{gp}}} \bar{\mathcal{M}}_{C_V^\circ}$. Observe that $\mathcal{M}_{C_V} \subseteq \mathcal{M}_{C_V^\circ}$. We define the structure morphism $\alpha_{C_V^\circ} : \mathcal{M}_{C_V^\circ} \to \mathcal{O}_{C_V}$ as follows. First, we require $\alpha_{C_V^\circ}|_{\mathcal{M}_{C_V}} = \alpha_{C_V}$. Second, for a local section δ of $\mathcal{M}_{C_V^\circ}$ not contained in $\mathcal{M}_{C_V}$, we define $\alpha_{C_V^\circ}(\delta) = 0$.

This defines a monoid homomorphism. Indeed, using the decomposition $\mathcal{M}_{C_V^\circ} \subseteq \mathcal{M} \oplus_{\mathcal{O}_{C_v}^\times} \mathcal{P}^{\mathrm{gp}}$ as in Definition 2.1, write $\delta = (\delta', \delta'')$ with δ' the pullback of a section of $\mathcal{M}_V$. It is sufficient to check that when $\delta \notin \mathcal{M}_{C_V}$ we have $\alpha_V(\delta') = 0$.

In the notation of Section 2.5.2 the assumption $\delta \notin \mathcal{M}_{C_V}$ implies $\delta \in \mathcal{I}_p$. Hence according to Definition 2.49 we have $\delta' \in \mathcal{K}_V$. As V is defined by $\alpha_V(\mathcal{K}_V) = 0$, we have $\alpha_V(\delta') = 0$ as needed.

This defines a logarithmic structure $\mathcal{M}_{C_V^\circ}$ over $\underline{C}_V$. The inclusion of logarithmic structures $\mathcal{M}_{C_V} \subseteq \mathcal{M}_{C_V^\circ}$ is a puncturing, hence defines a punctured curve $C_V^\circ \to C_V \to V$.

Step 4. $C_T^\circ \to C_T \to T$ is the pullback of $C_V^\circ \to C_V \to V$ via $i : T \to V$. Denote by $\underline{j} : \underline{C}_T \to \underline{C}_V$ the pullback of $\underline{i}$. Since $C_T \to T$ is given by base change from $C_V \to V$, it suffices to show that $\underline{j}^*\mathcal{M}_{C_V^\circ} = \mathcal{M}_{C_T^\circ}$ as sub-sheaves of monoids in $\mathcal{M}_{C_T}^{\mathrm{gp}}$. Away from the punctures, the equality clearly holds. Along each puncture $p \in \mathbf{p}_T$, we have the equality $j^*\bar{\mathcal{M}}_{C_V^\circ, p_w} = \bar{\mathcal{M}}_{C_T^\circ, p_t}$ at p_t by the construction in Step 3, which extends along the marking p by generization. This proves the desired equality.

Step 5. Étale covering. Consider a strict, square-zero extension $T \to T'$ over W and a family of punctured curves $C_{T'}^\circ \to C_{T'} \to T'$ such that $C_{T'} = C \times_W T'$, and $C_T^\circ \to C_T \to T$ is the pullback of $C_{T'}^\circ \to C_{T'} \to T'$. Since the strict morphism $T' \to \mathcal{A}_Q$ again factors through $\mathcal{A}_{Q'}$, we may continue to assume $Q = Q'$. Applying Step 2 again, we see that $T' \to W$ factors through V uniquely.

Denote by $t' \in T'$ the image of t via $T \to T'$. The family $C_V^\circ \to C_V \to V$ is constructed using the same geometric fiber over t. Applying Step 4 again, we see that $C_{T'}^\circ \to C_{T'} \to T'$ can be obtained via pulling back $C_V^\circ \to C_V \to V$.

This shows that $V \to \breve{W}$ is formally étale, and we claim it is actually étale, in other words, for any scheme T'' and morphism $T'' \to \breve{W}$, we need to show that $T'' \times_{\breve{W}} V \to T''$ is locally of finite presentation. The question being local, we may assume $T'' \to \breve{W}$ factors through some $V'' \to \breve{W}$ in our covering, and may as well replace T'' by V''. In this case $V \times_{\breve{W}} V'' \to V \times_W V''$, the pullback of the diagonal $\breve{W} \to \breve{W} \times_W \breve{W}$ along $V \times_W V'' \to \breve{W} \times_W \breve{W}$, is an open embedding. As V, V'' and W are noetherian, the map $V \times_{\breve{W}} V'' \to V''$ is of finite presentation.

Moreover, since $\sqcup V \to W$ is locally of finite presentation and $\sqcup V \to \breve{W}$ is étale and surjective, we have that $\breve{W} \to W$ is locally of finite presentation, see [67, Section 06Q1]. As indicated earlier, this implies that the diagonal is quasi-separated, completing the proof. ■

3.2 Stacks of punctured maps marked by tropical types

3.2.1 Weak markings and markings

In analogy with [3, Definition 2.31] we define the following notion.

Definition 3.4. Let $\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \bar{\mathbf{u}})$ be a global type of punctured maps (Definition 2.44). A *weak marking by* τ of a basic punctured map $(C^\circ/W, \mathbf{p}, f)$ to X is a $(G, \mathbf{g})$-marking of the domain curve $(C^\circ/W, \mathbf{p})$ (Definition 3.2) with the following properties:

(1) The restriction of $\underline{f}$ to the closed subscheme $Z \subseteq \underline{C}$ (a subcurve or punctured or nodal section of $\underline{C}$) defined by $x \in V(G) \cup E(G) \cup L(G)$ factors through the closed stratum $X_{\boldsymbol{\sigma}(x)} \subseteq \underline{X}$ (Section 2.2.1).

(2) For each geometric point $\bar{w}$ of $\underline{W}$ with $\tau_{\bar{w}} = (G_{\bar{w}}, \mathbf{g}_{\bar{w}}, \boldsymbol{\sigma}_{\bar{w}}, \mathbf{u}_{\bar{w}})$ the associated type of $(C^\circ/W, \mathbf{p}, f)$ at $\bar{w}$ (Definition 2.24), the contraction morphism $(G_{\bar{w}}, \mathbf{g}_{\bar{w}}) \to (G, \mathbf{g})$ of decorated graphs given by the marking defines a contraction morphism of the associated global types

$$(G_{\bar{w}}, \mathbf{g}_{\bar{w}}, \boldsymbol{\sigma}_{\bar{w}}, \bar{\mathbf{u}}_{\bar{w}}) \to \tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \bar{\mathbf{u}}). \tag{3.2}$$

A weak marking of $(C^\circ/W, \mathbf{p}, f)$ by τ is a *marking* if in addition the following condition holds.

(3) For all geometric points $\bar{w}$ of $\underline{W}$, the ideal in $\mathcal{M}_{W,\bar{w}}$ defined by the monoid ideal $I_{\tau\tau_{\bar{w}}} \subseteq \overline{\mathcal{M}}_{W,\bar{w}}$ in (2.32) maps to 0 under the structure morphism $\mathcal{M}_{W,\bar{w}} \to \mathcal{O}_{W,\bar{w}}$.

A *marking of* $(C^\circ/W, \mathbf{p}, f)$ *by a decorated global type* $\boldsymbol{\tau} = (\tau, \mathbf{A})$ is defined analogously, with the associated types replaced by associated decorated types introduced in (2.13).

In the definition, basicness is not necessary for (1) and (2), but is needed when referring to (2.32) in (3).

Note that a marking of a punctured map by a global type τ does not mean that τ is realizable. It just means that there is a contraction morphism $\overline{\tau}' \to \tau$ from the global type $\overline{\tau}'$ associated to a realizable type τ', the type of the given punctured map.

Remark 3.5. The difference between weak markings and markings is fairly subtle and is related to saturation in the definition of the basic monoid. Recall first the construction of the basic monoid from [30, Construction 1.16]. Let $f : C/W \to X$ be a punctured map defined over a log point. The basic monoid Q associated to this log map was constructed as the saturation of a quotient of the monoid $\prod_{\eta \in \underline{C}} P_\eta \times \prod_{q \in \underline{C}} \mathbb{N}$. Here η runs over generic points of $\underline{C}$ and q runs over the nodes of $\underline{C}$. Denote by Q^{fine} this quotient before saturating, so that Q is the saturation of Q^{fine}, as in [30, eq. (1.14)].

Now suppose that $f : C/W \to X$ is a weakly τ-marked log map with W an arbitrary fs log scheme, but suppose in addition that for every geometric point $\bar{w}$ of W, $C_{\bar{w}} \to X$ is of type τ. Thus $\overline{\mathcal{M}}_W$ is locally constant with stalk Q. The proof of Lemma 3.21 below implies in particular that if s is any section of $\mathcal{M}_W$ whose image $\bar{s}$ in $\overline{\mathcal{M}}_W$ has stalk lying in $Q^{\mathrm{fine}} \setminus \{0\}$ at each geometric point, then $\alpha_W(s) = 0$. However, the condition for being marked requires this vanishing even when $\bar{s}$ lies in $Q \setminus \{0\}$.

For an explicit example where Q^{fine} is not saturated, see [30, Example 1.17 (3)]. There, Q^{fine} is the submonoid of $\mathbb{Z}^2$ generated by $(1, -6)$, $(0, 2)$ and $(0, 3)$. In such a situation, it is not difficult to construct an example of a weakly τ-marked but not τ-marked curve, as follows.

Start with a basic τ-marked log map $f : C/W \to X$ with W a log point, and assume that $Q \neq Q^{\mathrm{fine}}$. Let $W^{\mathrm{fine}} = \mathrm{Spec}(Q^{\mathrm{fine}} \to \Bbbk)$. Since all nodal generators $\rho_E \in Q$ already lie in Q^{fine} by construction, we may find a sub-log structure $\mathcal{M}_{C^{\mathrm{fine}}} \subseteq \mathcal{M}_C$ so that $C^{\mathrm{fine}} \to W^{\mathrm{fine}}$ is a log smooth curve (in the category of fine log schemes) and f induces a morphism $C^{\mathrm{fine}} \to X$. Saturating W^{fine} may yield a non-reduced scheme W^{sat} with reduction W. The composition

$$C^{\mathrm{sat}} := C^{\mathrm{fine}} \times_{W^{\mathrm{fine}}} W^{\mathrm{sat}} \to C^{\mathrm{fine}} \to X$$

yields a stable log map in the category of fs log schemes which is weakly marked, but not marked, by τ.

In the cited example [30, Example 1.17 (3)], Q is the submonoid of $\mathbb{Z}^2$ generated by $(1, -6)$ and $(0, 1)$, and one checks that

$$\underline{W}^{\mathrm{sat}} \cong \mathrm{Spec}\, \Bbbk[Q]/\langle z^{(1,-6)}, z^{(0,2)} \rangle,$$

which is a scheme of length two.

Under the presence of monodromy, the following more refined version of marked punctured maps using framed types rather than global types is sometimes more appropriate, notably in gluing. Note however that framed types work with contact orders living on a single stratum X_σ. Hence this refined notion is inappropriate when studying punctured maps with a contact order propagating into several X_σ not contained in a single stratum.

Definition 3.6. Let $\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u}, \iota)$ be a framed type of a family of tropical punctured maps (Definition 2.24). A *weak marking by* τ of a basic punctured map $(C^\circ/W, \mathbf{p}, f)$ to X is a weak marking by the global type $(G, \mathbf{g}, \boldsymbol{\sigma}, \bar{\mathbf{u}})$ associated to τ, along with, for each $x \in E(G) \cup L(G)$ with associated nodal or punctured locus $Z_x \subseteq \underline{C}$, a homomorphism of sheaves of monoids

$$\mu(x) : (\underline{f}|_{Z_x})^{-1} \overline{\mathcal{M}}_X \to \underline{\boldsymbol{\sigma}(x)^\vee_{\mathbb{Z}}},$$

whose stalkwise duals at all geometric points $\bar{w}$ of W are arrows in $\Sigma(X)$, and which lift the contraction morphism of global types (3.2) to a contraction morphism of framed types (Definition 2.62). Here $\underline{\boldsymbol{\sigma}(x)^\vee_{\mathbb{Z}}}$ is the constant sheaf with stalks the dual of the set of integral points of $\boldsymbol{\sigma}(x)$.

A *marking by a framed type* is then defined by replacing $I_{\tau\tau'}$ in Definition 3.2 (3) by $\chi^{-1}_{\tau\tau'}(Q_{\tau\tau'} \setminus \{0\})$, noting that $\mu(x)$ makes it possible to define $Q_{\tau\tau'}$ and $\chi_{\tau\tau'}$ unambiguously and consistently.

Remark 3.7. We expect that all results that we formulate for (weak) markings by global types hold for (weak) markings by framed types. Since the framed notions have only been included in a late revision of the paper, we nevertheless decided to leave the full development of this modified theory to other occasions. We emphasize that in most applications one is either interested in simple X from the outset or one can reduce to this situation, and in this case the framed perspective does not provide any additional information.

3.2.2 Enter stacks of punctured maps

We continue to assume that $X \to B$ is a morphism of fs log algebraic schemes fulfilling the assumptions stated at the beginning of Chapter 3.

Definition 3.8. Let $\boldsymbol{\tau} = (G, \mathbf{g}, \boldsymbol{\sigma}, \bar{\mathbf{u}}, \mathbf{A}) = (\tau, \mathbf{A})$ be a decorated global type (Definition 2.44). Then

$$\mathcal{M}(X/B, \boldsymbol{\tau}) \quad \text{and} \quad \mathcal{M}(X/B, \tau)$$

are defined as the stacks over $(\mathbf{Sch}/\underline{B})$ with objects basic stable punctured maps to X over B (Definition 2.15) marked by $\boldsymbol{\tau}$ and by τ, respectively (Definition 3.4).

Weakening stability to pre-stability, the analogous stacks to the relative Artin fan $\mathcal{X}$ of X over B, as defined at the beginning of Chapter 3, are denoted[3]

$$\mathfrak{M}(\mathcal{X}/B, \boldsymbol{\tau}) \quad \text{and} \quad \mathfrak{M}(\mathcal{X}/B, \tau).$$

The corresponding stacks with markings replaced by weak markings are denoted by the same symbols adorned with primes:

$$\mathcal{M}'(X/B, \boldsymbol{\tau}), \quad \mathcal{M}'(X/B, \tau), \quad \mathfrak{M}'(\mathcal{X}/B, \boldsymbol{\tau}), \quad \mathfrak{M}'(\mathcal{X}/B, \tau).$$

An important special case is that $\boldsymbol{\tau}$ is the class $\beta = (g, \bar{\mathbf{u}}, A)$ of a punctured map (Definition 2.44). Then G is the graph with only one vertex v of some genus g, stratum $\boldsymbol{\sigma}(v) = 0 \in \Sigma(X)$, and curve class A, no edges, and any number of legs.

[3] Stability being a concept for graphs decorated by genera and curve classes, there does exist a stable version of $\mathfrak{M}(\mathcal{X}/B, \boldsymbol{\tau})$. We omit this variant.

Recalling from Section 2.2.1 that the stratum of X associated to the origin $0 \in \Sigma(X)$ equals $\underline{X}$, the resulting stacks

$$\mathcal{M}'(X/B,\beta) = \mathcal{M}(X/B,\beta), \quad \mathfrak{M}'(\mathcal{X}/B,\beta) = \mathfrak{M}(\mathcal{X}/B,\beta) \tag{3.3}$$

restrict only the total genus and total curve class, as well as the number of punctures and their global contact orders.

Remark 3.9. We will see in Proposition 3.30 that for a realizable global type τ the moduli spaces $\mathfrak{M}(\mathcal{X}/B,\tau)$ of τ-marked punctured maps to $\mathcal{X}/B$ are reduced and pure-dimensional, at least for simple X. For a general global type the reduction of $\mathfrak{M}(\mathcal{X}/B,\tau)$ is stratified by the images of the morphisms $\mathfrak{M}(\mathcal{X}/B,\tau') \to \mathfrak{M}(\mathcal{X}/B,\tau)$ for realizable types τ' dominating τ, see Remark 3.31 below. Thus from the stratified point of view, markings as in Definition 3.4 (3) are the correct notion. This feature explains their appearance in [3, Definition 2.31].

However, the notion of weak marking, as in Definition 3.4 (1)–(2), appears naturally in gluing situations. Notably the commutative square in Theorem 5.8 is only cartesian with weak markings. For applications in Gromov–Witten theory, one works with cycles in the moduli spaces of punctured maps appearing in this diagram and the difference between markings and weak markings disappears, possibly up to computable multiplicities. See for example [71] where this approach is taken.

3.2.3 The stacks are algebraic

Theorem 3.10. *Let $X \to B$ be a morphism of fs logarithmic schemes fulfilling the assumptions stated at the beginning of Chapter 3, and let $\boldsymbol{\tau} = (G, \mathbf{g}, \boldsymbol{\sigma}, \bar{\mathbf{u}}, \mathbf{A}) = (\tau, \mathbf{A})$ be a decorated global type of punctured maps to X. Then the stacks*

$$\mathcal{M}(X/B,\boldsymbol{\tau}), \quad \mathcal{M}(X/B,\tau), \quad \mathfrak{M}(\mathcal{X}/B,\boldsymbol{\tau}), \quad \mathfrak{M}(\mathcal{X}/B,\tau)$$

are logarithmic algebraic stacks locally of finite type over B. Moreover, $\mathcal{M}(X/B,\boldsymbol{\tau})$ and $\mathcal{M}(X/B,\tau)$ are Deligne–Mumford, and the forgetful morphisms to the stack $\mathcal{M}(\underline{X}/\underline{B})$ of ordinary stable maps are representable.

Analogous results hold for the weakly marked versions $\mathcal{M}'(X/B,\boldsymbol{\tau})$, $\mathcal{M}'(\mathcal{X}/B,\tau)$, $\mathfrak{M}'(\mathcal{X}/B,\boldsymbol{\tau})$, $\mathfrak{M}'(\mathcal{X}/B,\tau)$.

Proof. We first restrict to $\mathcal{M}'(X/B,\boldsymbol{\tau})$ and then comment on the minor changes for the other cases.

Step 1: An algebraic stack of prestable maps. Denote by

$$\breve{\mathfrak{C}} = \breve{\mathfrak{C}}(G,\mathbf{g}) \to \breve{\mathfrak{M}} = \breve{\mathfrak{M}}(G,\mathbf{g})$$

the universal curve over the logarithmic algebraic stack $\breve{\mathfrak{M}}(G,\mathbf{g})$ of $(G,\mathbf{g})$-marked punctured curves from Definition 3.2 and Proposition 3.3. This morphism is proper, flat, integral, of finite type and has geometrically reduced fibers. Hence [70, Corollary 1.1.1] applies to show that

$$\mathrm{Hom}_{\breve{\mathfrak{M}}}\big(\breve{\mathfrak{C}}, \breve{\mathfrak{M}} \times^{\mathrm{f}}_{B} X\big)$$

is representable by a logarithmic algebraic stack, locally of finite type.[4]

The rest of the proof is analogous to [3, Proposition 2.34].

Step 2: Carving out weakly marked basic stable maps. Condition (1) in Definition 3.4 of marking by $\boldsymbol{\tau}$ defines a closed substack of $\mathrm{Hom}_{\breve{\mathfrak{M}}}(\breve{\mathfrak{C}}, \breve{\mathfrak{M}} \times^{\mathrm{f}}_{B} X)$, while all the remaining conditions in Definition 3.4 (2) are open, see Proposition 2.63. Note here we are using that curve classes are locally constant in flat families. The condition on a map being basic is open by Proposition 2.34; stability is open since it is open on the underlying stable maps. Thus the morphism

$$\mathcal{M}'(X/B, \boldsymbol{\tau}) \to \mathrm{Hom}_{\breve{\mathfrak{M}}}\big(\breve{\mathfrak{C}}, \breve{\mathfrak{M}} \times^{\mathrm{f}}_{B} X\big)$$

forgetting all parts of the marking except the $(G, \mathbf{g})$-marking of the domain curve identifies the stack $\mathcal{M}'(X/B, \boldsymbol{\tau})$ with an open substack of a strict closed substack of $\mathrm{Hom}_{\breve{\mathfrak{M}}}(\breve{\mathfrak{C}}, \breve{\mathfrak{M}} \times^{\mathrm{f}}_{B} X)$.

Step 3: Verifying properties. By Proposition 2.37, logarithmic automorphisms of basic stable maps acting trivially on underlying maps are trivial. Hence $\mathcal{M}'(X/B, \boldsymbol{\tau}) \to \mathcal{M}(\underline{X}/\underline{B})$ is representable. Since $\mathcal{M}(\underline{X}/\underline{B})$ is a Deligne–Mumford stack, so is $\mathcal{M}'(X/B, \boldsymbol{\tau})$. Ignoring curve classes yields the statement for $\mathcal{M}'(X/B, \tau)$.

Step 4: Weakly marked maps to $\mathcal{X}$. The morphism $\mathcal{X} \to B$ from the relative Artin fan is well behaved.

Lemma 3.11. *The morphism $\mathcal{X} \to B$ is quasi-separated, locally of finite type, and has affine stabilizers.*

Proof. It suffices to verify these properties for the morphism $\mathcal{A}_X \to \mathcal{A}_B$. This is shown in [6, Lemma 2.5.5] in case $X \to B$ is logarithmically smooth, and we indicate here why the argument applies here. Since the properties claimed are local in B (or $\mathcal{A}_B$), we may assume $\mathcal{A}_B$ is an Artin cone $\mathcal{A}_\tau$. Since $\mathcal{A}_X$ has a cover by étale maps from Artin cones $\mathcal{A}_{\sigma\to\tau}$, we have that $\mathcal{A}_X$ is locally of finite type.

Quasiseparation follows in the same way as in [6, Lemma 2.3.8 (ii)], applied to $\mathcal{A}_X \to \mathcal{A}_X \times_{\mathcal{A}_B} \mathcal{A}_X$ instead of $\mathcal{A}_X \to \mathcal{A}_X \times \mathcal{A}_X$ and using representability over Log^1 instead of Log: one needs to show, for two charts $\mathcal{A}_{\sigma_1\to\tau}$ and $\mathcal{A}_{\sigma_2\to\tau}$ of $\mathcal{A}_X$, that $\mathcal{A}_{\sigma_1\to\tau} \times_{\mathcal{A}_X} \mathcal{A}_{\sigma_2\to\tau}$ is quasicompact. By [6, Lemma 2.3.8 (i)] and representability

[4] This last property is not explicitly stated in [70], but follows by inspection of the proof.

it suffices to show that the stack $\mathcal{A}_{\sigma_1\to\tau} \times_{\mathrm{Log}^1} \mathcal{A}_{\sigma_2\to\tau}$ has finitely many points. The argument of [6, Lemma 2.3.8 (ii)] then applies as stated.

The claim about stabilizers follows as in [6, Lemma 2.5.5]. ■

It follows that [70, Corollary 1.1.2] still applies. The rest of the proof for the stacks $\mathfrak{M}'(\mathcal{X}/B,\boldsymbol{\tau})$ and $\mathfrak{M}'(\mathcal{X}/B,\tau)$ is the same, except we can not conclude the Deligne–Mumford property due to the absence of stability.

Step 5: Marked maps. Stacks of marked maps are closed substacks of stacks of weakly marked maps, locally defined by the log-ideal $I_{\tau\tau_{\bar{w}}}$ in Definition 3.4 (3).[5] Hence the result also holds for these cases. ■

3.3 Boundedness

For ordinary stable logarithmic maps, boundedness of $\mathcal{M}(X/B,\beta)$ is established in [2, 30] for projective $X \to B$ under the technical assumption that $\overline{\mathcal{M}}_X$ is globally generated. Reference [5] removed the technical assumption by showing that there is a logarithmic blowing up $Y \to X$ with $\overline{\mathcal{M}}_Y$ globally generated and then using birational invariance of the moduli spaces $\mathfrak{M}(\mathcal{X}/B,\beta)$ under this process. Since this birational invariance seems to be rather more subtle in the punctured case, we content ourselves with a statement assuming global generatedness, which suffices for most practical applications. Throughout this and the next subsections we assume that the log structure on X is Zariski as in [30], which we follow. We believe this assumption could be removed by minor adaptations of the proof.

Theorem 3.12. *Suppose the underlying family $X \to B$ is projective, and the sheaf $\overline{\mathcal{M}}_X^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is generated by its global sections.*[6] *Then the projection $\mathcal{M}(X/B,\beta) \to B$ is of finite type.*

Proof. We split the proof into several steps. The theorem follows from Propositions 3.16 and 3.17 below. ■

Global generatedness of $\overline{\mathcal{M}}_X^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ can be easily read off from the cone complex $\Sigma(X)$ as follows.

Proposition 3.13. *The sheaf $\overline{\mathcal{M}}_X^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is generated by global sections if and only if there exists a continuous map*

$$|\Sigma(X)| \to \mathbb{R}^r$$

with restriction to each $\sigma \in \Sigma(X)$ an injective homomorphism of additive monoids.

[5] For a much more detailed discussion of this point, in terms of the idealized structure defined by markings, see Section 3.5 below, and notably Theorem 3.25.

[6] Samuel Johnston in [39] has meanwhile removed the global generatedness assumptions along the same line as [5].

Proof. A map $|\Sigma(X)| \to \mathbb{R}^r$ which is injective when restricted to any $\sigma \in \Sigma(X)$ is dual to a system of surjective homomorphisms

$$\varphi_\sigma : \mathbb{R}^r \to \mathrm{Hom}(\sigma, \mathbb{R}),$$

compatible with the dual of the face maps defining $\Sigma(X)$. But such a compatible system $(\varphi_\sigma)_{\sigma\in\Sigma(X)}$ of surjections is equivalent to a linear map

$$\mathbb{R}^r \to \Gamma\big(\underline{X}, \overline{\mathcal{M}}_X^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{R}\big) = \Gamma\big(\underline{X}, \overline{\mathcal{M}}_X^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}\big) \otimes_{\mathbb{Q}} \mathbb{R}$$

with composition to $\overline{\mathcal{M}}_{X,x}^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{R}$ surjective for each $x \in X$. The claim follows. ■

Remark 3.14. We remark that if $\overline{\mathcal{M}}_X^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is generated by global sections, then all global contact orders of X are monodromy free, which we see as follows. The map of Proposition 3.13 gives a well-defined map $\mathfrak{C}_\sigma(X) \to \mathbb{Z}^r \subseteq \mathbb{R}^r$. Indeed, if $\sigma' \in \Sigma_\sigma(X)$ and $u \in N_{\sigma'}$, we may view u as an integral tangent vector (i.e., an element of $N_{\sigma'}$) to $\sigma' \in \Sigma(X)$ and take its image under the map $|\Sigma(X)| \to \mathbb{R}^r$. Since u is compatible with inclusion of faces, this provides a point v of $\mathbb{Z}^r \subseteq \mathbb{R}^r$ only depending on $\iota_{\sigma\sigma'}(u)$ (see Definition 2.40 for notation). Since $|\Sigma(X)| \to \mathbb{R}^r$ is injective on cones, v arises, for each σ', as the image of at most one $u \in N_{\sigma'}$. Hence all global contact orders are monodromy free.

3.3.1 Boundedness of $\mathcal{M}(X/B, \beta)$

Definition 3.15. A class β of a punctured map (Definition 2.44) is called *combinatorially finite* if the set of types (Definition 2.24) of stable punctured maps with associated class β is finite.

Proposition 3.16. *Suppose β is combinatorially finite. Then the forgetful map*

$$\mathcal{M}(X/B, \beta) \to \mathcal{M}(\underline{X}/\underline{B}, \underline{\beta}) \tag{3.4}$$

is of finite type.

Proof. The strategy of the proof is similar to those in [30, Section 3.2] and [15, Section 5.4] by showing that each stratum with constant combinatorial structure is bounded. The proof is largely the same, with extra care needed only in the proof of [30, Proposition 3.17].

By Theorem 3.10, $\mathcal{M}(X/B, \beta) \to B$ is locally of finite type, and hence so is the morphism $\mathcal{M}(X/B, \beta) \to \mathcal{M}(\underline{X}/\underline{B}, \underline{\beta})$. Thus it is sufficient to prove the latter morphism is quasi-compact. We thus need to show that $\underline{W} \times_{\mathcal{M}(\underline{X}/\underline{B},\underline{\beta})} \mathcal{M}(X/B, \beta)$ is quasi-compact for any quasi-compact scheme $\underline{W}$ and morphism $\underline{W} \to \mathcal{M}(\underline{X}/\underline{B}, \underline{\beta})$. Using [30, Lemma 3.14], it is enough to find a weak cover in the sense of [30, Definition 3.13] of $\underline{W} \to \mathcal{M}(\underline{X}/\underline{B}, \underline{\beta})$ by finitely many quasi-compact subsets. We may weakly cover $\underline{W}$ by a finite number of locally closed strata on which the corresponding ordinary stable map is combinatorially constant (in the sense of [30, Definition 3.15]), and replace $\underline{W}$ with one of these locally closed strata. Thus we may assume given $\mathfrak{f} = (\underline{C}/\underline{W}, \mathbf{p}, \underline{f})$ a combinatorially constant ordinary stable map over an integral, quasi-compact scheme $\underline{W}$. Then $\underline{W} \times_{\mathcal{M}(\underline{X}/\underline{B},\beta)} \mathcal{M}(X/B,\beta)$ classifies punctured enhancements of the ordinary stable maps parametrized by $\underline{W}$, and we need to show this fiber product is quasi-compact.

As the combinatorial type of a log curve with constant dual intersection graph is locally constant, we have a decomposition

$$\underline{W} \times_{\mathcal{M}(\underline{X}/\underline{B},\underline{\beta})} \mathcal{M}(X/B,\beta) = \coprod_{\mathbf{u}} \mathcal{M}(X,\mathfrak{f},\mathbf{u})$$

into disjoint open substacks according to the type $\mathbf{u}$. As β is assumed combinatorially finite, this is a finite union. Hence it is sufficient to show that each $\mathcal{M}(X,\mathfrak{f},\mathbf{u})$ is quasi-compact. As in the proof of [30, Proposition 3.17], it is sufficient to construct a quasi-compact stack Z with a morphism $Z \to \mathcal{M}(X,\mathfrak{f},\mathbf{u})$ which is surjective on geometric points.

To do so, set $Q_1 := \mathbb{N}^k$, where k is the number of nodes of any fiber of $\underline{C} \to \underline{W}$. By Proposition 2.32 and the fact we have fixed the type $\mathbf{u}$, the basic monoid Q is constant on $\mathcal{M}(X,\mathfrak{f},\mathbf{u})$, and there is a canonical morphism $Q_1 \to Q$. The latter induces a morphism of Artin cones $\mathcal{A}_{Q^\vee} \to \mathcal{A}_{Q_1^\vee}$. We equip $\underline{W}$ with the canonical log structure coming from the family of pre-stable curves $\underline{C} \to \underline{W}$, and consider $Z_1 = \mathcal{A}_{Q^\vee} \times_{\mathcal{A}_{Q_1^\vee}} W$. Pulling back the universal family from W, we obtain a family of log curves $C_1 \to Z_1$ and an ordinary stable map $\underline{f} : \underline{C}_1 \to \underline{X}/\underline{B}$. Observe that there is a global chart $Q \to \overline{\mathcal{M}}_{Z_1}$. To check Z_1 is quasi-compact we can, and do, replace Z_1 with its underlying reduced substack.

The type $\mathbf{u}$ prescribes, for each marked section $p \in \mathbf{p}$, an ideal sheaf

$$\overline{\mathcal{I}}_p \subseteq \overline{\mathcal{M}}_W \oplus \mathbb{Z} \subseteq p^*\overline{\mathcal{M}}_C^{\mathrm{gp}}$$

generated by $u_p^{-1}(\mathbb{Z}_{<0})$, which, we note, is constant along Z_1. These ideals produce an ideal $\overline{\mathcal{K}} \subseteq Q$ as in Definition 2.49 by taking into account all punctures in $\mathbf{p}$. Denote by $\mathcal{K} = \overline{\mathcal{K}} \times_{\overline{\mathcal{M}}_{Z_1}} \mathcal{M}_{Z_1}$ the resulting log ideal, where the arrow on the left is given by the composition $\overline{\mathcal{K}} \to Q \to \overline{\mathcal{M}}_{Z_1}$ with the last arrow the global chart.

To obtain a family of punctured stable maps of type $\mathbf{u}$ over Z_1 then requires that $\alpha_{Z_1}(\mathcal{K}) = 0$ by Proposition 2.52. Thus in particular if $0 \in \overline{\mathcal{K}}$, then there are no punctured maps of type $\mathbf{u}$ and we can ignore such a $\mathbf{u}$; otherwise, as Z_1 is reduced and $\overline{\mathcal{M}}_{Z_1}$ is locally constant with stalk Q, necessarily $\alpha_{Z_1}(\mathcal{K}) = 0$. Indeed, any local section m of $\mathcal{K}$ maps to a nowhere zero section of $\overline{\mathcal{M}}_{Z_1}$, and hence $\alpha_{Z_1}(m)$ is nowhere invertible, thus zero, since Z_1 is reduced.

We now construct a punctured family of curves $C_1^\circ \to Z_1$. First, the ghost sheaf $\overline{\mathcal{M}}_{C_1^\circ}$ is identical to $\overline{\mathcal{M}}_{C_1}$ away from the punctures. Along each puncture $p \in \mathbf{p}$, we take $\overline{\mathcal{M}}_{C_1^\circ,p} \subset \overline{\mathcal{M}}^{\mathrm{gp}}_{C_1,p}$ to be the smallest fine submonoid generated by $\overline{\mathcal{M}}_{C_1,p}$ and the image of $f^{-1}\overline{\mathcal{M}}_X \to \overline{\mathcal{M}}^{\mathrm{gp}}_{C_1,p}$ determined by the type $\mathbf{u}$. As all the ghost sheaves and morphisms between them are constant along Z_1, this yields a well-defined sheaf of monoids $\overline{\mathcal{M}}_{C_1^\circ}$, hence $\mathcal{M}_{C_1^\circ} := \overline{\mathcal{M}}_{C_1^\circ} \times_{\overline{\mathcal{M}}^{\mathrm{gp}}_{C_1}} \mathcal{M}^{\mathrm{gp}}_{C_1}$ over $\underline{C}_1$.

We define the structure homomorphism $\alpha_{C_1^\circ} : \mathcal{M}_{C_1^\circ} \to \mathcal{O}_{C_1}$ by $\alpha_{C_1^\circ}|_{\mathcal{M}_{C_1}} = \alpha_{C_1}$ and $\alpha_{C_1^\circ}|_{\mathcal{M}_{C_1^\circ}\setminus\mathcal{M}_{C_1}} = 0$. The same argument as in the proof of Proposition 3.3, Step 3, shows that this defines a logarithmic structure $\mathcal{M}_{C_1^\circ}$, hence the desired punctured curve $C_1^\circ \to Z_1$.

The remainder of the proof is now identical to that of [30, Proposition 3.17]. ∎

3.3.2 Finiteness of the combinatorial data

In order to complete the proof that $\mathscr{M}(X/B, \beta)$ is finite type, it remains to bound the combinatorial data.

Proposition 3.17. *Suppose $\overline{\mathcal{M}}^{\mathrm{gp}}_X \otimes_{\mathbb{Z}} \mathbb{Q}$ is generated by its global sections. Then any class of punctured map β is combinatorially finite.*

Proof. Arguing stratawise as in [30, Section 3.2], it is sufficient to show that for any combinatorially constant family of ordinary stable maps $(\underline{C}/\underline{W}, \mathbf{p}, \underline{f})$ in the sense of [30, Definition 3.15], there are only finitely many combinatorial types of liftings of such a family to a punctured log curve of type β. Since types are constant along a combinatorially constant family, we may further assume that $\underline{W}$ is the spectrum of a field. Finiteness of the number of types of a logarithmic stable map with a given underlying stable map over a field with fixed contact orders u_p is proved in [30, Theorem 3.9].

One small difference in our setup concerns the definition of contact orders. In [30] these were given by a sheaf homomorphism $\overline{\mathcal{M}}_Z \to \mathbb{N}$, hence were fixed at $\underline{f}(p)$ by the underlying ordinary stable map and the contact orders u_p. In contrast, a global contact order may give an infinite set of maps $\overline{\mathcal{M}}_{X,\underline{f}(p)} \to \mathbb{Z}$. The argument is saved under the assumption that $\overline{\mathcal{M}}^{\mathrm{gp}}_X \otimes_{\mathbb{Z}} \mathbb{Q}$ is generated by its global sections: The injectivity statement in Proposition 3.13 implies that there is at most one local representative of u_p. ∎

3.4 Valuative criterion

We now show stable reduction for basic stable punctured maps, which allows us to conclude properness of the moduli spaces of such maps. Recall that for a given class

$\beta = (g, \bar{\mathbf{u}}, A)$ of stable punctured maps to $X \to B$, we have the class $\underline{\beta} = (g, k, A)$ for ordinary stable maps to $\underline{X} \to \underline{B}$ by removing contact orders. We will show that

Theorem 3.18. *Assume that the log structure on X is defined in the Zariski topology. Then the tautological morphism removing all logarithmic structures*

$$\mathcal{M}(X/B, \beta) \to \mathcal{M}(\underline{X}/\underline{B}, \underline{\beta})$$

satisfies the valuative criterion for properness.

Proof. In what follows, we assume given R a discrete valuation ring over $\underline{B}$ with maximal ideal $\mathfrak{m}$, residue field $\kappa = R/\mathfrak{m}$, and fraction field K. Suppose we have a commutative square of solid arrows of the underlying stacks:

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & \mathcal{M}(X/B, \beta) \\ \downarrow & \nearrow & \downarrow \\ \operatorname{Spec} R & \longrightarrow & \mathcal{M}(\underline{X}/\underline{B}, \underline{\beta}). \end{array}$$

We want to show that there is a dashed arrow making the above diagram commutative, which is unique up to a unique isomorphism.

The top arrow of the above diagram yields a stable punctured map

$$(\pi_K : C_K^\circ \to \operatorname{Spec}(Q_K \to K), \mathbf{p}_K, f_K)$$

over the logarithmic point $\operatorname{Spec}(Q_K \to K)$. The bottom arrow of the above diagram yields an ordinary stable map $(\underline{C}/\operatorname{Spec} R, \mathbf{p}, \underline{f})$ with its generic fiber given by the underlying stable map of f_K. To construct the dashed arrow, it suffices to extend the stable punctured map f_K across the closed point $0 \in \operatorname{Spec} R$ with the given underlying stable map $\underline{f}$. The task is to then extend the logarithmic structures and morphisms thereof. The proof is almost identical to that of [30, Theorem 4.1]. Since that proof is quite long, we only note the salient differences.

Reference [30, Section 4.1] accomplishes this extension at the level of ghost sheaves; in particular, [30, Proposition 4.3], which states that the type of the central fiber is uniquely determined by the stable log map on the generic fiber, carries through with u_p for a puncture p determined as for marked points. Indeed, if p is a punctured point on $\underline{C}_0$ in the closure of the punctured point p_K on $\underline{C}_K$, then u_p must be the composition

$$P_p \to P_{p_K} \xrightarrow{u_{p_K}} \mathbb{Z}, \tag{3.5}$$

where the first map is the generization map $(\underline{f}^* \bar{\mathcal{M}}_X)_p \to (\underline{f}^* \bar{\mathcal{M}}_X)_{p_K}$. In particular, the contact orders u_p and u_{p_K} both have global contact order as specified in β.

By Proposition 2.32, the type of the central fiber then determines the extension $\overline{\mathcal{M}}_{C^\circ}$ of $\overline{\mathcal{M}}_{C_K^\circ}$ and a map $\bar{f}^\flat : \underline{f}^*\overline{\mathcal{M}}_X \to \overline{\mathcal{M}}_{C^\circ}$ extending the corresponding map on the generic fiber. Here $\overline{\mathcal{M}}_{C^\circ}$ is defined at punctures by pre-stability via Corollary 2.7.

Next, [30, Section 4.2][7] shows that the logarithmic structure on the base Spec R is uniquely defined. In this argument, marked points play no role, and the argument remains unchanged in the punctured case. In particular, this produces a unique choice of logarithmic structure $\mathcal{M}_R$ on Spec R, which in addition comes with a morphism of logarithmic structures $\mathcal{M}_R^0 \to \mathcal{M}_R$ where $\mathcal{M}_R^0$ is the basic logarithmic structure (pulled back from the moduli space of pre-stable curves $\mathbf{M}$ with its basic logarithmic structure, see [30, Appendix A]) associated to the family $\underline{C} \to \operatorname{Spec} R$. In particular, one obtains a logarithmic structure $(\underline{C}, \mathcal{M}'_C) = (\operatorname{Spec} R, \mathcal{M}_R) \times_{(\operatorname{Spec} R, \mathcal{M}_R^0)} (\underline{C}, \mathcal{M}_C^0)$, where $\mathcal{M}_C^0$ is the logarithmic structure pulled back from the basic logarithmic structure of the universal curve over $\mathcal{M}(\underline{X}/\underline{B}, \beta)$. The logarithmic structure $\mathcal{M}'_C$ then has logarithmic marked points along the punctures p, but there is a sub-logarithmic structure $\mathcal{M}_C \subset \mathcal{M}'_C$ which only differs in that we remove the marked points, that is, we make $(\underline{C}, \mathcal{M}_C) \to (\operatorname{Spec} R, \mathcal{M}_R)$ strict away from the nodes.

By Corollary 2.7, there is a natural inclusion $\overline{\mathcal{M}}_{C^\circ} \subset (\overline{\mathcal{M}}'_C)^{\mathrm{gp}}$. We form $\mathcal{M}_{C^\circ} := \overline{\mathcal{M}}_{C^\circ} \times_{(\overline{\mathcal{M}}'_C)^{\mathrm{gp}}} (\mathcal{M}'_C)^{\mathrm{gp}}$ and define a structure homomorphism $\alpha_{C^\circ} : \mathcal{M}_{C^\circ} \to \mathcal{O}_C$ by $\alpha_{C^\circ}|_{\mathcal{M}_{C'}} = \alpha_{C'}$ and $\alpha_{C^\circ}(\mathcal{M}_{C^\circ} \setminus \mathcal{M}'_C) = 0$, as in Proposition 3.3, Step 3. To show that this is a homomorphism, it is enough to show that if $s \in \mathcal{M}_{C^\circ,p} \setminus \mathcal{M}'_{C,p}$, writing $s = (s_1, s_2)$ as a stalk of $\mathcal{M}_C \oplus_{\mathcal{O}_C^\times} \mathcal{P}^{\mathrm{gp}}$, then $\alpha_C(s_1) = 0$. But necessarily $(\bar{s}_1, \bar{s}_2) = \bar{f}^\flat(m) + (\bar{s}'_1, \bar{s}'_2)$ for some $m \in P_p$ with $u_p(m) < 0$ and $(\bar{s}'_1, \bar{s}'_2) \in \overline{\mathcal{M}}_{C,p} \oplus \mathbb{N}$. Write for points $x, x' \in \underline{C}$ with x in the closure of x' the generization map $\chi_{x',x} : P_x \to P_{x'}$. Then $u_{p_K}(\chi_{p_K,p}(m)) = u_p(m)$ by (3.5). Thus $u_{p_K}(\chi_{p_K,p}(m)) < 0$ and necessarily $\alpha_{C_K}(s_1|_{C_K}) = 0$. But since C is reduced and C_K is dense in C, this implies $\alpha_C(s_1) = 0$, as desired. Thus we have a punctured log scheme C°.

We can now extend $f_K^\flat : f_K^*\mathcal{M}_X \to \mathcal{M}_{C_K^\circ}$ to $f^\flat : f^*\mathcal{M}_X \to \mathcal{M}_{C^\circ}$ as in [30, Section 4.3]. ∎

Corollary 3.19. *Let $\boldsymbol{\tau} = (G, \mathbf{g}, \boldsymbol{\sigma}, \bar{\mathbf{u}}, \mathbf{A})$ be a decorated global type of punctured maps (Definition 2.44) and assume $X \to B$ is projective, the log structure on X is Zariski, and $\overline{\mathcal{M}}_X^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is globally generated.*[8] *Then $\mathcal{M}(X/B, \boldsymbol{\tau}) \to B$ is proper. In particular, $\mathcal{M}(X/B, \beta)$ is proper for any $\beta = (g, \bar{\mathbf{u}}, A)$.*

---

[7] We take the opportunity to correct an error, pointed out by the referee of the current paper, in the first paragraph of [30, Section 4.2]. Two descriptions of a set $U(\eta)$ are given. The first description, as the set of generizations of points in A, the set of non-special points in $\underline{C}_0$, is not correct (it is not necessarily an open set). Thus the reader should rely only on the second description of the set $U(\eta)$.

[8] Again, the latter assumption has been removed by [39].

Proof. Theorem 3.12 shows that $\mathcal{M}(X/B,\beta) \to B$ is of finite type. Properness for $\boldsymbol{\tau} = \beta$ now follows from the valuative criterion verified in Theorem 3.18.

For general $\boldsymbol{\tau}$, the proof of [3, Proposition 2.34] generalizes to the present punctured setup to exhibit $\mathcal{M}(X/B,\boldsymbol{\tau})$ as a closed substack of the base change of the stack $\mathcal{M}(X/B,\beta)$ by the finite map $\mathbf{M}(G,\mathbf{g}) \to \mathbf{M}$. ∎

3.5 Idealized smoothness of $\mathfrak{M}(\mathcal{X}/B,\tau) \to B$

For simplicity of presentation, we restrict to X simple throughout this section. Thus for any $\sigma,\tau \in \Sigma(X)$ there is at most one arrow $\sigma \to \tau$ in $\Sigma(X)$.

3.5.1 Marking log-ideals

Let $\tau = (G,\mathbf{g},\boldsymbol{\sigma},\bar{\mathbf{u}})$ be a global type of punctured maps. Recall from the discussion after Definition 3.1 that the moduli stack $\mathbf{M}(G,\mathbf{g})$ of $(G,\mathbf{g})$-marked pre-stable curves with its nodal log ideal sheaf is idealized logarithmically smooth over the trivial log point $\operatorname{Spec}\Bbbk$. A similar result holds for our moduli spaces $\mathfrak{M}(\mathcal{X}/B,\tau)$. To introduce the idealized structure let $(\pi : C^\circ \to W,\mathbf{p},f)$ be a τ-marked basic punctured map and let $\bar{w}$ of $\underline{W}$ be a geometric point. Let $\tau_{\bar{w}} = (G_{\bar{w}},\mathbf{g}_{\bar{w}},\boldsymbol{\sigma}_{\bar{w}},\mathbf{u}_{\bar{w}})$ be the type of the punctured map over $\bar{w}$, equipped with its marking contraction morphism $\phi : \tau_{\bar{w}} \to \tau$ (Definitions 2.24 and 3.4 (2)), with set of contracted edges E_ϕ. For the sake of Definition 3.20 below, we introduce the following notation. For $x \in V(G_{\bar{w}}) \cup L(G_{\bar{w}}) \cup (E(G_{\bar{w}}) \setminus E_\phi)$ the face inclusion $\boldsymbol{\sigma}(\phi(x)) \to \boldsymbol{\sigma}_{\bar{w}}(x)$ is dual to a localization map

$$\chi_x : P_x \to P_{\phi(x)}$$

of stalks of $\overline{\mathcal{M}}_X$. We also have homomorphisms

$$\varphi_x : P_x \to \overline{\mathcal{M}}_{C^\circ,x}, \quad u_x : P_x \to \mathbb{Z}$$

defined by $\bar{f}^\flat_{\bar{w}}$ and by the contact order $\mathbf{u}_{\bar{w}}$. For uniformity of notation we define $u_x = 0$ for $x \in V(G_{\bar{w}})$. Moreover, by Definition 2.18 of contact order, $\varphi_x(u_x^{-1}(0)) \subseteq \overline{\mathcal{M}}_{C^\circ,x}$ is contained in the image of $\bar{\pi}^\flat_x : \overline{\mathcal{M}}_{W,\bar{w}} \to \overline{\mathcal{M}}_{C^\circ,x}$. For the following definition recall also the homomorphism $\chi_{\tau\tau_{\bar{w}}} : Q_{\tau_{\bar{w}}} \to Q_{\tau\tau_{\bar{w}}}$ from (2.21).

Definition 3.20. The τ*-marking ideal* $\overline{\mathcal{I}}^\tau_W$ of the τ-marked basic punctured map $(\pi : C \to W,\mathbf{p},f)$ is the sheaf of ideals in $\overline{\mathcal{M}}_W$ with stalk at the geometric point $\bar{w}$ of $\underline{W}$ generated by the following subsets:

(i) (Target stratum generators) the preimage under $\bar{\pi}^\flat_x$ of $\varphi_x(P_x \setminus \chi_x^{-1}(0))$ for $x \in V(G_{\bar{w}}) \cup L(G_{\bar{w}}) \cup (E(G_{\bar{w}}) \setminus E_\phi)$;

(ii) (Nodal generators) the nodal generators $\rho_E \in \overline{\mathcal{M}}_{W,\bar{w}} = Q_{\tau_{\bar{w}}}$ for $E \in E(G_{\bar{w}}) \setminus E_\phi$;

(iii) (Basic monoid generators) $\chi^{-1}_{\tau\tau_{\bar{w}}}(Q_{\tau\tau_{\bar{w}}} \setminus \{0\})$.

The collection of stalks $\overline{\mathcal{I}}^{\tau}_{W,\bar{w}} \subset \overline{\mathcal{M}}_{W,\bar{w}}$ in Definition 3.20 form a coherent ideal $\overline{\mathcal{I}}^{\tau}_{W} \subset \overline{\mathcal{M}}_{W}$. Indeed, we obtain a sheaf by the method of Remark 2.36, and, as W is fine and saturated, we may apply Lemma 2.47, noting that all generating sets are compatible with generization. As usual, we also refer to the preimage $\mathcal{I}_W \subset \mathcal{M}_W$ of $\overline{\mathcal{I}}_W$ under $\mathcal{M}_W \to \overline{\mathcal{M}}_W$ as the *τ-marking ideal*. Without the generators specified in (iii) we speak of the *weak τ-marking ideal*.

3.5.2 The base of a punctured map is idealized by the marking log-ideal

The τ-marking ideal defines an idealized log structure on base spaces of τ-marked punctured maps as follows.

Lemma 3.21. *Let $(C^\circ/W, \mathbf{p}, f)$ be a τ-marked basic punctured map. Then the τ-marking ideal $\mathcal{I}_W \subset \mathcal{M}_W$ maps to* 0 *under the structure homomorphism $\mathcal{M}_W \to \mathcal{O}_W$.*

Proof. It is enough to show that any lift $s \in \mathcal{M}_{W,\bar{w}}$ of an element of one of the generating sets satisfies $\alpha_W(s) = 0$. This holds for elements described in (iii) of Definition 3.20 by Definition 3.4 (3).

Similarly, Definition 3.4 (1) guarantees the required vanishing for elements described in (i) of Definition 3.20. Indeed, consider first the case of $x = v \in V(G_{\bar{w}})$, where we defined $u_v = 0$. Then

$$u_v^{-1}(0) \setminus \chi_v^{-1}(0) = P_v \setminus \chi_v^{-1}(0),$$

and $\alpha_X(P_v \setminus \chi_v^{-1}(0)) \subset \mathcal{O}_X$ locally generates the ideal $\mathcal{I}_{X_{\boldsymbol{\sigma}(\phi(v))}} \subset \mathcal{O}_X$ of the stratum $X_{\boldsymbol{\sigma}(\phi(v))}$ in X. Thus, the condition that the restriction of $\underline{f}$ to the closed subscheme of $\underline{C}$ corresponding to $\phi(v)$ factors through $\underline{X}_{\boldsymbol{\sigma}(\phi(v))}$ implies the desired vanishing in this case. A similar argument works for legs and edges.

Finally, the lift to $\mathcal{M}_{W,\bar{w}}$ of a nodal generator $\rho_E \in \overline{\mathcal{M}}_{W,\bar{w}}$ lies in the nodal log-ideal (Definition 3.1) of the $(G, \mathbf{g})$-marked curve C/W, which maps to zero in $\mathcal{O}_W$ by Proposition 3.3 (2). ∎

Remark 3.22. Omitting the last set (iii) of generators in Definition 3.20 leads to the idealized structure for moduli spaces of *weakly* marked punctured maps (Definition 3.4).

As shown in Proposition 2.52, the base W is also idealized by the puncturing log ideal $\mathcal{K}$. It is therefore natural to combine the two.

Definition 3.23. We call the union $\mathcal{I}^\tau \cup \mathcal{K}$ of the τ-marking and the puncturing log ideals the *canonical idealized structure* on our τ-marked moduli spaces such as $\mathfrak{M}(\mathcal{X}/B, \tau)$.

3.5.3 The realizable case

While the definition of the τ-marking ideal may seem complicated, in fact in the case we most frequently need it, namely the realizable case, the canonical idealized structure has a simpler description: By Lemma 2.45 there is a unique lift to a type, and the associated basic monoid already knows about marked strata, non-deforming nodes and punctures.

Proposition 3.24. *If τ is a realizable global type, then $\overline{\mathcal{I}}^{\tau}_{W,\bar w}+\overline{\mathcal{K}}_{W,\bar w}$ with $\overline{\mathcal{K}}_W$ the puncturing log ideal (Definition 2.55) is given by the set* (iii) *in Definition 3.20.*

Proof. Denote by $\chi: Q_{\tau_{\bar w}}\to Q_{\tau\tau_{\bar w}}$ the localization homomorphism from (2.21) defined by the τ-marking of $(C^\circ/W,\mathbf{p},f)$. By Lemma 2.45 there is a unique type of punctured map with associated global type τ. Hence in particular $Q_{\tau\tau_{\bar w}}$ agrees with the basic monoid for a tropical punctured map of this type and does not depend on $\bar w$. We write this basic monoid as Q_τ. Denote by $R\subset Q_{\tau_{\bar w}}$ the ideal $\chi^{-1}(Q_\tau\setminus\{0\})$.

We need to show that R contains the elements listed in (i) and (ii) of Definition 3.20 as well as generators of the puncturing log ideal stated in Definition 2.49. Adopting the notation given in Definition 3.20, for $v\in V(G_{\bar w})$ we have a commutative diagram

$$\begin{array}{ccc} P_v & \xrightarrow{\varphi_v} & Q_{\tau_{\bar w}} \\ {\scriptstyle\chi_v}\downarrow & & \downarrow{\scriptstyle\chi} \\ P_{\phi(v)} & \xrightarrow[\varphi_{\phi(v)}]{} & Q_\tau \end{array}$$

The fact that τ is realizable implies that $\varphi_{\phi(v)}$ is a local homomorphism, i.e. $\varphi^{-1}_{\phi(v)}(0)=\{0\}$. Indeed, dually, the map $Q_\tau^\vee\to P^\vee_{\phi(v)}$ is given by evaluation of the tropical map at the vertex v, and realizability implies the image of this map intersects the interior of $P^\vee_{\phi(v)}$. This is equivalent to the local homomorphism statement. But this implies that $\varphi_v(P_v\setminus\chi_v^{-1}(0))\subseteq\chi^{-1}(Q_\tau\setminus\{0\})=R$.

In the case of a leg L, we similarly have a diagram

$$\begin{array}{ccccc} P_L & \xrightarrow{\varphi_L} & Q_{\tau_{\bar w}}\oplus\mathbb{Z} & \xrightarrow{\mathrm{pr}_1} & Q_{\tau_{\bar w}} \\ {\scriptstyle\chi_L}\downarrow & & \downarrow{\scriptstyle\chi\oplus\mathrm{id}} & & \downarrow{\scriptstyle\chi} \\ P_{\phi(L)} & \xrightarrow[\varphi_{\phi(L)}]{} & Q_\tau\oplus\mathbb{Z} & \xrightarrow[\mathrm{pr}_1]{} & Q_\tau \end{array}$$

Again, $\varphi_{\phi(L)}$ is necessarily local by realizability. Note that, with $\iota: Q_{\tau_{\bar w}}\to Q_{\tau_{\bar w}}\oplus\mathbb{Z}$ given by $m\mapsto(m,0)$,

$$\iota^{-1}\big(\varphi_L\big(P_L\setminus\chi_L^{-1}(0)\big)\big)=\iota^{-1}\big(\varphi_L\big(u_L^{-1}(0)\setminus\chi_L^{-1}(0)\big)\big)=\mathrm{pr}_1\circ\varphi_L\big(u_L^{-1}(0)\setminus\chi_L^{-1}(0)\big).$$

Thus $\mathrm{pr}_1 \circ \varphi_L(u_L^{-1}(0) \setminus \chi_L^{-1}(0)) \subseteq \chi^{-1}(Q_\tau \setminus \{0\}) = R$, as desired. In fact we obtain more from this. If instead $p \in P_L$ with $u_L(p) < 0$, then $\mathrm{pr}_1(\varphi_L(p))$ is a generator of $\bar{\mathcal{K}}_{W,\bar{w}}$, and $\chi(\mathrm{pr}_1(\varphi_L(p)))$ is a generator of the puncturing ideal for the type τ. But as the type is realizable, this ideal does not contain 0. Thus $\mathrm{pr}_1(\varphi_L(p)) \in R$, so $\bar{\mathcal{K}}_{W,\bar{w}} \subseteq R$.

For an edge $E \in V(G)$, the argument that $\phi_E(u_E^{-1}(0) \setminus \chi_E^{-1}(0)) \subseteq R$ is similar and we leave the details to the reader. Finally, for the corresponding nodal generator $\rho_E \in Q_{\tau_{\bar{w}}}$ from Definition 3.20 (ii), observe that $\chi(\rho_E)$ is the edge length function of the edge E. Again, since τ is realizable, $\chi(\rho_E) \neq 0$ and $\rho_E \in R$. ■

3.5.4 The stacks are idealized log smooth

Theorem 3.25. *Assume that X is simple. Then the forgetful morphisms*

$$\mathfrak{M}(\mathcal{X}/B, \tau) \to \mathbf{M}(G, \mathbf{g}) \times B$$

remembering only the domain curve as a family of marked curves over B, is idealized logarithmically étale for the canonical idealized structures. An analogous result holds for τ replaced by a decorated global type $\boldsymbol{\tau} = (\tau, \mathbf{A})$ of a punctured map, and for weak markings.

Proof. Step 1. Lifting to the stack of punctured curves. We first note that the morphism in question is in fact idealized. Indeed, the generators of the nodal log-ideal (Definition 3.1) on $\mathbf{M}(G, \mathbf{g}) \times B$ are pulled back to the nodal generator ρ_E of Definition 3.20 (ii) for $E \in E(G)$. The morphism then factors over the idealized logarithmically étale morphism

$$\breve{\mathfrak{M}}_B(G, \mathbf{g}) \to \mathfrak{M}_B(G, \mathbf{g}) = \mathrm{Log}_{\mathbf{M}(G,\mathbf{g})\times B}$$

from Proposition 3.3 (2). Moreover, by [53, Theorem 4.6 (iii)], the morphism

$$\mathrm{Log}_{\mathbf{M}(G,\mathbf{g})\times B} \to \mathbf{M}(G, \mathbf{g}) \times B$$

is also logarithmically étale. It thus suffices to prove the statement with $\mathbf{M}(G, \mathbf{g}) \times B$ replaced by the stack $\breve{\mathfrak{M}}_B(G, \mathbf{g})$ of $(G, \mathbf{g})$-marked punctured curves. Note that the morphism $\mathfrak{M}(\mathcal{X}/B, \tau) \to \breve{\mathfrak{M}}_B(G, \mathbf{g})$ is strict, but not in general idealized strict: the nodal log-ideal of $\breve{\mathfrak{M}}_B(G, \mathbf{g})$ from Definition 3.1 involves only the nodes of the domain curves, whereas the τ-marking ideal of $\mathfrak{M}(\mathcal{X}/B, \tau)$ from Definition 3.20, in particular part (i), also records target data.

Step 2. Lifting to the prestable map. According to the definition of idealized log étale, it is sufficient to consider a diagram of solid arrows in the category of idealized log spaces

$$\begin{array}{ccc} T_0 & \xrightarrow{g_0} & \mathfrak{M}(\mathcal{X}/B,\tau) \\ \downarrow & \nearrow & \downarrow \\ T & \xrightarrow[g]{} & \breve{\mathfrak{M}}_B(G,\mathbf{g}) \end{array} \tag{3.6}$$

where $T_0 \hookrightarrow T$ is an idealized strict closed embedding defined by a square-zero ideal. Denote by $\mathcal{K}_{T_0}$ and $\mathcal{K}_T$ the log-ideals of T_0 and T respectively. We wish to show that there is a unique dashed arrow making the above diagram commutative.

Denote by $f_{T_0} : C^\circ_{T_0} \to \mathcal{X}$ the punctured map over T_0 corresponding to the morphism g_0, and by $C^\circ_{T_0} \hookrightarrow C^\circ_T$ the extension given by g. Write also $\pi_{T_0} : C^\circ_{T_0} \to T_0$, $\pi_T : C^\circ_T \to T$. Thus the lifting problem (3.6) reduces to the following:

$$\begin{array}{ccc} C^\circ_{T_0} & \xrightarrow{f_{T_0}} & \mathcal{X} \\ \downarrow & \overset{f_T}{\nearrow} & \downarrow \\ C^\circ_T & \longrightarrow & B \end{array}$$

Since $\mathcal{X}\to B$ is log étale, by the infinitesimal lifting property of log étale morphisms in the category of idealized log schemes [52, p. 399], such f_T exists and is unique.

It remains to check that f_T is also a τ-marked curve. Item (2) of Definition 3.4 is automatic as T_0 and T have the same geometric points. As a preparation for establishing (1) and (3), we first check the vanishing of the τ-marking ideal.

Step 3. The marking ideal vanishes. Fix a geometric point $\bar{t}$ of T_0. Let $I^\tau_0 \subseteq \mathcal{M}_{T_0,\bar{t}}$ be the stalk of the log-ideal $g_0^\bullet \mathcal{I}^\tau_{\mathfrak{M}(\mathcal{X}/B,\tau)}$ at $\bar{t}$, and write $\overline{I}^\tau_0 \subseteq \overline{\mathcal{M}}_{T_0,\bar{t}}$ for its image. As $\overline{\mathcal{M}}_{T_0,\bar{t}} = \overline{\mathcal{M}}_{T,\bar{t}}$, we also obtain an ideal $I^\tau \subseteq \mathcal{M}_{T,\bar{t}}$ as the inverse image of $\overline{I}^\tau_0$ under the map $\mathcal{M}_{T,\bar{t}} \to \overline{\mathcal{M}}_{T,\bar{t}}$. As g_0 is idealized, necessarily $I^\tau_0 \subseteq \mathcal{K}_{T_0,\bar{t}}$. Since $T_0 \to T$ is idealized strict, we thus have $I^\tau \subseteq \mathcal{K}_{T,\bar{t}}$ and hence $\alpha_T(I^\tau) = 0$. This finishes Step 3.

Now let $x \in V(G)\cup E(G)\cup L(G)$, and let $Z \subseteq C_T$ be the corresponding closed subscheme. To verify condition (1) of Definition 3.4, we need to show that $f_T|_Z$ factors through $\mathcal{X}_{\boldsymbol{\sigma}(x)}$. Let $\bar{w} = g_0(\bar{t})$, with corresponding type of tropical curve $\tau_{\bar{w}}$, equipped with a contraction morphism $\phi : \tau_{\bar{w}} \to \tau$. We now check the needed factorization for each kind of x in the following steps.

Step 4. The marking lifts at a vertex. First consider the case that x is a vertex. In this case Z is a sub-curve of $\underline{C}_T$, flat over $\underline{T}$. Let $U \subseteq Z$ be the open subset of non-special points; it is then sufficient to show that $\underline{f}_T|_U$ factors through the closed substack $\mathcal{X}_{\boldsymbol{\sigma}(x)}$. So let $\bar{u}$ be a geometric point of U lying over $\bar{t}$, contained in an irreducible component of $Z_{\bar{t}}$ indexed by a vertex $v \in V(G_{\tau_{\bar{w}}})$. Note then that $\phi(v) = x$. It is enough to show that $f^\sharp_T : \mathcal{O}_{\mathcal{X},\underline{f}_T(\bar{u})} \to \mathcal{O}_{C_T,\bar{u}}$ takes the stalk $\mathcal{J}_{\underline{f}_T(\bar{u})}$ of the ideal $\mathcal{J}$

of $\mathcal{X}_{\boldsymbol{\sigma}(x)}$ in $\mathcal{X}$ to 0. Using the notation of Definition 3.20, we have $\overline{\mathcal{M}}_{\mathcal{X},\underline{f}_T(\bar{u})} = P_v$ and a generization map $\chi_v : P_v \to P_{\phi(v)}$. If $p \in P_v$, write $s_p \in \mathcal{M}_{\mathcal{X},\underline{f}_T(\bar{u})}$ for a lift of p. We next observe that since B is a log point or is log smooth over $\operatorname{Spec} \Bbbk$ and X is simple, the ideal $\mathcal{J}_{\underline{f}_T(\bar{u})}$ is generated by the set $\{\alpha_{\mathcal{X}}(s_p) \mid p \in P_v \setminus \chi_v^{-1}(0)\}$. Indeed, this is the idealized smoothness statement of the strata in Proposition 2.48, applied on a smooth chart of $\mathcal{X}$, together with the stalkwise characterization (2.22) of the log ideal $\mathcal{K}$ in the proof of that proposition. Note that due to simplicity, the only face map is $\chi_v^t : \boldsymbol{\sigma}(x) \to P_{\mathbb{R}}^{\vee}$ in the present case, and hence $\overline{\mathcal{K}}_{\underline{f}_T(\bar{u})} = P_v \setminus \chi_v^{-1}(0)$.

Now by Definition 3.20 (i) and strictness of π_T at $\bar{u}$, for each $p \in P_v$ there exists $s'_p \in I^\tau \subseteq \mathcal{M}_{T,\bar{t}}$ with $f_T^\flat(s_p) = h \cdot \pi_T^\flat(s'_p)$ for some $h \in \mathcal{O}_{C_T,\bar{u}}^\times$. Thus

$$f_T^\sharp(\alpha_{\mathcal{X}}(s_p)) = \alpha_{C_T}(f_T^\flat(s_p)) = \alpha_{C_T}(h \cdot \pi_T^\flat(s'_p)) = h \cdot \pi_T^\sharp(\alpha_T(s'_p)) = 0.$$

This shows that $\underline{f}_T|_U$ factors through $\mathcal{X}_{\boldsymbol{\sigma}(x)}$.

Step 5. The marking lifts at a leg. Second consider the case that $x = L \in L(G)$. In this case Z is the image of a section of π_T, with $\underline{Z} \cong \underline{T}$. Let $\bar{u}$ be the unique geometric point of $\underline{Z}$ over $\bar{t}$. We now have a generization map $\chi_L : P_L = \overline{\mathcal{M}}_{\mathcal{X},\underline{f}_T(\bar{u})} \to P_{\phi(L)}$. Following the same notation as in the previous paragraph, it is then sufficient to show that for each $p \in P_L \setminus \chi_L^{-1}(0)$, we have $0 = \alpha_{C_T}(f_T^\flat(s_p))|_Z \in \mathcal{O}_{Z,\bar{u}}$. As in the previous paragraph, this is forced by the generators of the puncturing ideal in Definition 3.20 (i) in case $u_L(p) = 0$. If $u_L(p) > 0$, then $\alpha_{C_T}(f_T^\flat(s_p))$ contains a positive power of the defining equation of Z as a subscheme of C_T, and hence vanishes along Z. If $u_L(p) < 0$, then we achieve vanishing by Definition 2.1 (2). Thus we obtain the desired vanishing.

Step 6. The marking lifts at an edge. The third case is $x = E \in E(G)$. The argument is similar to the second case, and we leave the details to the reader. This verifies that f_T satisfies condition (1) of τ-marked curve.

Step 7. Base marking, decoration and weak marking. Finally, condition (3) holds. Indeed, the generators in Definition 3.20 (iii) guarantee the desired vanishing.

This completes the proof for markings by τ. The proof for τ replaced by $\boldsymbol{\tau}$ is identical. The weakly marked case is obtained by the same proof omitting (iii) in Definition 3.20. ∎

Remark 3.26. The proof in the weakly marked case uses simplicity only when arguing that the ideal defining $\mathcal{X}_{\boldsymbol{\sigma}(x)}$ locally is generated by expressions $\alpha_{\mathcal{X}}(s_p)$, $p \in P_v \setminus \chi_v^{-1}(0)$ for the unique generization map $\chi_v : P_v \to P_{\phi(v)}$, $P_v = \overline{\mathcal{M}}_{\underline{f}_T(\bar{u})}$. In general there is still always a log ideal $K \subseteq \overline{\mathcal{M}}_{\underline{f}_T(\bar{u})}$ with this property, as we saw in the proof of Proposition 2.48. This larger log ideal can be accounted for by modifying Definition 3.20 (i) accordingly. In the marked case, we also need to refine $Q_{\tau\boldsymbol{\tau}_{\bar{w}}}$ in Definition 3.20 (iii) to the version stated in (2.32) in Section 2.6.4.

Thus we expect the statement of Theorem 3.25 to hold true in the non-simple case with these adjustments. Details are left to the interested reader.

Remark 3.27 (Local structure of stacks of prestable maps). Theorem 3.25 gives the following local description of $\mathfrak{M}(\mathcal{X}/B,\tau)$. Let $(C^\circ/W,\mathbf{p},f)$ be a basic stable punctured map over a log point $W=\operatorname{Spec}(Q\to\kappa)$ over B marked by τ. Denote by s the number of edges of the graph G given by $\tau=(G,\mathbf{g},\boldsymbol{\sigma},\bar{\mathbf{u}})$ and assume that $\underline{C}$ has $s+r$ nodes. Thus r nodes of $\underline{C}$ can be smoothed while keeping a marking by $(G,\mathbf{g})$.

The underlying object $(\underline{C}/\underline{W},\mathbf{p})$, viewed as a pre-stable curve with its basic log structure, is a point $\operatorname{Spec}\kappa\to\mathbf{M}(G,\mathbf{g})\times B$.

By the deformation theory of nodal curves, there exists a strict smooth neighborhood of this point étale locally isomorphic to

$$\mathbb{A}^r\times U\times B. \tag{3.7}$$

Here $\mathbb{A}^r$ is endowed with the idealized log structure obtained by restricting the toric log structure of $\mathbb{A}^{s+r}$ to an intersection of s coordinate hyperplanes, and corresponds to deforming the r smoothable nodes; U is smooth with trivial log structure corresponding to equisingular deformations of $\underline{C}$; and the étale local isomorphism is a product of an étale local isomorphism of $\mathbb{A}^r\times U$ with an open substack of $\mathbf{M}(G,\mathbf{g})$ and id_B.

Note that the image of $(C^\circ/W,\mathbf{p})$ in $\mathbf{M}(G,\mathbf{g})$ is defined by the underlying marked nodal curve $(\underline{C}/\underline{W},\mathbf{p})$ endowed with its basic log structure of marked nodal curves.

Consider the point $W\to\mathfrak{M}(\mathcal{X}/B,\tau)$ corresponding to the object $(C^\circ/W,\mathbf{p},f)$. Pulling back the neighborhood (3.7) along $\mathfrak{M}(\mathcal{X}/B,\tau)\to\mathbf{M}(G,\mathbf{g})\times B$ gives a smooth neighborhood V of $W\to\mathfrak{M}(X/B,\tau)$ equipped with a morphism $\phi:V\to\mathbb{A}^r\times U\times B$. We may now apply Proposition B.4 to describe this neighborhood explicitly étale locally, as follows. We use the notation $\mathcal{A}_P$ and $\mathcal{A}_{P,I}$ defined in (B.1), for P a monoid and $I\subseteq P$ a monoid ideal.

The log-ideal $\mathcal{I}^\tau_{\mathfrak{M}(\mathcal{X}/B,\tau)}\cup\mathcal{K}_{\mathfrak{M}(\mathcal{X}/B,\tau)}$ induces a monoid ideal $I\subseteq Q$, as constructed in Definition 3.20, with associated idealized Artin fan $\mathcal{A}_{Q,I}$. Let Q_B be the stalk of $\overline{\mathcal{M}}_B$ at the image point of the composition $W\to\mathfrak{M}(\mathcal{X}/B,\tau)\to B$. We may first replace B with an étale neighborhood of this image point and so assume given a map $Q_B\to\overline{\mathcal{M}}_B$, or equivalently a strict morphism $B\to\mathcal{A}_{Q_B}$. Then by Proposition B.4, possibly after passing to an étale neighborhood of V, there is a diagram

$$\begin{array}{ccccc}
V & & \xrightarrow{\quad\psi\quad} & & \\
& \searrow^{\theta} & & & \\
& & V' & \longrightarrow & \mathcal{A}_{Q,I} \\
& {\scriptstyle\phi}\swarrow & \downarrow & & \downarrow{\scriptstyle\iota} \\
& & \mathbb{A}^r\times U\times B & \longrightarrow & \mathcal{A}_{\mathbb{N}^{s+r},J}\times\mathcal{A}_{Q_B}
\end{array} \tag{3.8}$$

with the square Cartesian in the log, fine and fs categories, ψ and both horizontal arrows strict and idealized strict, and θ étale and strict. Further, ι is induced by the map on stalks of ghost sheaves $\mathbb{N}^{s+r} \oplus Q_B \to Q$ given by the morphism $\mathfrak{M}(\mathcal{X}/B, \tau) \to \mathbf{M}(G, \mathbf{g}) \times B$. Finally, $J \subseteq \mathbb{N}^{s+r}$ is the ideal generated by the first s generators of $\mathbb{N}^{s+r}$, so that the morphism $\mathbb{A}^r \to \mathcal{A}_{\mathbb{N}^{s+r},J}$ is strict and idealized strict.

In conclusion, we see that V is étale locally isomorphic to

$$V' \cong U \times \big((\mathbb{A}^r \times B) \times_{\mathcal{A}_{\mathbb{N}^{s+r},J} \times \mathcal{A}_{Q_B}} \mathcal{A}_{Q,I}\big). \tag{3.9}$$

Thus the local models of $\mathfrak{M}(\mathcal{X}/B, \tau)$ and their idealized structures are explicitly described from the types of tropical punctured maps admitting a contraction morphism to τ.

3.5.5 Dimension formulas

Example 2.58 exhibits a case where $\mathfrak{M}(\mathcal{X}/B, \tau)$ is not pure-dimensional. Before revisiting this example, we give a useful condition which implies $\mathfrak{M}(\mathcal{X}/B, \tau)$ is pure-dimensional, of the expected dimension. The statement involves a refinement of the notion of realizability of global types from Definition 2.44 (2) relative to B.

Definition 3.28. Let τ be a global type of punctured map to X. We say that τ *is realizable over* B if there exists a geometric point $\bar{w}$ of $\mathfrak{M}(\mathcal{X}/B, \tau)$ such that the corresponding punctured map has global type τ.

Proposition 3.29. *Suppose the Artin fan $\mathcal{A}_X$ of X is Zariski (Definition A.7). Then a global type $\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \bar{\mathbf{u}})$ is realizable over B if and only if the following conditions hold:*

(1) *τ is realizable, hence there is a universal family $h : \Gamma = \Gamma(G, \ell) \to \Sigma(X)$ of type τ, parametrized by $\omega_\tau := Q^\vee_{\tau,\mathbb{R}}$, where Q_τ is the basic monoid for tropical maps of type τ.*

(2) *The universal family of tropical maps of type τ is defined over $\Sigma(B)$, i.e., there is a map $\omega_\tau \to \Sigma(B)$ making the diagram*

$$\begin{array}{ccc} \Gamma & \xrightarrow{h} & \Sigma(X) \\ \downarrow & & \downarrow \\ \omega_\tau & \longrightarrow & \Sigma(B) \end{array}$$

commute.

(3) *Let $\sigma \in \Sigma(B)$ be the minimal cone containing the image of ω_τ. Then there exists a point $b \in B_\sigma$ such that $\sigma = \mathrm{Hom}(\overline{\mathcal{M}}_{B,b}, \mathbb{R}_{\geq 0})$.*

Proof. That conditions (1)–(3) are necessary is clear. Conversely, suppose (1)–(3) hold. Let $\underline{C}/\operatorname{Spec}\Bbbk$ be a pre-stable curve with dual intersection graph G. Pull-back the basic log structure on $\underline{C}/\operatorname{Spec}\Bbbk$ by the canonical morphism $\mathbb{N}^{|E(G)|} \to Q_\tau$ from the nodal parameters to the basic monoid for τ to define a log smooth curve C/W over the log point $W = \operatorname{Spec}(Q_\tau \to \Bbbk)$. We may then construct a morphism $W \to B$ with image a point $b \in B_\sigma$ given by item (3) in the statement of the proposition. Note we may take b to be a closed point, so that $b = \operatorname{Spec}\Bbbk$. At the logarithmic level, this morphism can be taken so its induced tropicalization is the given map $\omega_\tau \to \sigma$.

Next apply the correspondence [3, Proposition 2.10] (it is here we need the hypothesis that $\mathcal{A}_X$ is Zariski) between morphisms from a logarithmic space to an Artin fan and their tropicalizations to first construct a saturated puncturing $\widetilde{C}^\circ \to C$ and then a logarithmic map $\widetilde{C}^\circ \to \mathcal{A}_X$ with tropicalization of type τ. Prestabilizing then leads to a basic pre-stable punctured map $(C^\circ/W, \mathbf{p}, f)$ to $\mathcal{A}_X$ of type τ. Note that C° is not necessarily saturated. On the other hand, we have a composed morphism $C^\circ \to W \to B$, with $W \to B$ constructed in the previous paragraph. The compositions $C^\circ \to \mathcal{A}_X \to \mathcal{A}_B$ and $C^\circ \to B \to \mathcal{A}_B$ agree by item (2) of the proposition, and hence we obtain a punctured map $C^\circ \to \mathcal{X} = \mathcal{A}_X \times_{\mathcal{A}_B} B$ defined over B with the necessary properties. ∎

Proposition 3.30. *Let $\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \bar{\mathbf{u}})$ be a global type (Definition 2.44) and assume X is simple and B is either log smooth over* $\operatorname{Spec}\Bbbk$ *or $B = \operatorname{Spec}\Bbbk^\dagger$, the standard log point. Assume further τ is realizable over B. Then $\mathfrak{M}(\mathcal{X}/B,\tau)$ is non-empty, reduced and pure-dimensional. If B is log smooth over* $\operatorname{Spec}\Bbbk$*, then*

$$\dim \mathfrak{M}(\mathcal{X}/B,\tau) = 3|\mathbf{g}| - 3 + |L(G)| - \operatorname{rk} Q_\tau^{\mathrm{gp}} + \dim B,$$

while if $B = \operatorname{Spec}\Bbbk^\dagger$, then

$$\dim \mathfrak{M}(\mathcal{X}/B,\tau) = 3|\mathbf{g}| - 3 + |L(G)| - \operatorname{rk} Q_\tau^{\mathrm{gp}} + 1.$$

Proof. By Proposition 3.24, as τ is a realizable type, the τ-marked ideal at a point $\bar{w}'$ of $\mathfrak{M}(\mathcal{X}/B,\tau)$ takes the form $\chi^{-1}_{\tau\tau_{\bar{w}'}}(Q_\tau \setminus \{0\})$. Thus, in the description of a smooth neighborhood V of $\bar{w}'$ as given in (3.8), $\mathcal{A}_{Q,I}$ is reduced, and if B is log smooth over $\operatorname{Spec}\Bbbk$, the bottom horizontal arrow is smooth, and hence V' is also reduced as the square is Cartesian. This shows that $\mathfrak{M}(\mathcal{X}/B,\tau)$ is reduced in this case.

If on the other hand $B = \operatorname{Spec}\Bbbk^\dagger$, we may take $Q_B = \mathbb{N}$ in (3.8). Since $\mathfrak{M}(\mathcal{X}/B,\tau)$ is defined over B, the induced morphism of stalks of ghost sheaves $\mathbb{N} \to Q_\tau$ is local and hence $\mathbb{N} \setminus \{0\}$ maps into $Q_\tau \setminus \{0\}$, and thus more generally $\mathbb{N} \to Q_{\tau_{\bar{w}'}}$ maps $\mathbb{N} \setminus \{0\}$ into $\chi^{-1}_{\tau\tau_{\bar{w}'}}(Q_\tau \setminus \{0\})$ by compatibility of these maps with generization. Hence we may replace $\mathcal{A}_{Q_B}$ with the closed substack $\mathcal{A}_{\mathbb{N},\mathbb{N}\setminus\{0\}}$ in (3.8) without affecting this diagram in any other way. In particular, the bottom horizontal arrow is now still smooth. So V' is again reduced.

Let $\bar{w}$ be a point as in Definition 3.28. We may now calculate dimensions by looking at the description of (3.8) for a neighborhood of $\bar{w}$ in $\mathfrak{M}(\mathcal{X}/B,\tau)$. Since the corresponding curve $C^\circ/\bar{w}$ now has no smoothable nodes, we may take $r=0$ and $s=|E(G)|$ in (3.8). Further, since $I=Q_\tau\setminus\{0\}$, necessarily $\dim \mathcal{A}_{Q_\tau,I}=-\operatorname{rank} Q_\tau^{\mathrm{gp}}$. Thus we may calculate, with the cases being for B log smooth and $B=\operatorname{Spec}\Bbbk^\dagger$ respectively,

$$\begin{aligned}\dim\mathfrak{M}(\mathcal{X}/B)-\dim\mathbf{M}(G,\mathbf{g})\times B &= \dim V'-\dim U\times B\\ &= \begin{cases}\dim\mathcal{A}_{Q_\tau,I}-\dim\mathcal{A}_{\mathbb{N}^s,J}\times\mathcal{A}_{Q_B}\\ \dim\mathcal{A}_{Q_\tau,I}-\dim\mathcal{A}_{\mathbb{N}^s,J}\times\mathcal{A}_{\mathbb{N},\mathbb{N}\setminus\{0\}}\end{cases}\\ &= \begin{cases}-\operatorname{rank} Q_\tau^{\mathrm{gp}}-(-s)\\ -\operatorname{rank} Q_\tau^{\mathrm{gp}}-(-s-1)\end{cases}\end{aligned}$$

As $\dim\mathbf{M}(G,\mathbf{g})=3|\mathbf{g}|-3+|L(G)|-|E(G)|$, and $s=|E(G)|$, we then obtain the desired dimension formulas in the two cases. ■

Remark 3.31 (*Stratified structure of* $\mathfrak{M}(\mathcal{X}/B,\tau)$). If $\tau'\to\tau$ is a morphism of global types (Definition 2.44), a marking by τ' induces a marking by τ by composition of the marking morphism with $\tau'\to\tau$. The same arguments as for ordinary logarithmic maps [3, Proposition 2.34] shows that the corresponding morphism of stacks

$$j_{\tau\tau'}:\mathfrak{M}(\mathcal{X}/B,\tau')\to\mathfrak{M}(\mathcal{X}/B,\tau)$$

is finite and unramified. If τ' is realizable over B, then Proposition 3.30, under the assumptions on B stated there, further shows that $\operatorname{im}(j_{\tau\tau'})$ defines a pure-dimensional substack of $\mathfrak{M}(\mathcal{X}/B,\tau)$. Conversely, if there is no τ'' which is realizable over B mapping to τ' then $\mathfrak{M}(\mathcal{X},\tau')=\emptyset$. Thus the images of $j_{\tau\tau'}$ for morphisms of global types $\tau'\to\tau$ with τ' realizable over B define a stratification of $\mathfrak{M}(\mathcal{X}/B,\tau)$ into pure-dimensional strata.

In particular, the closure of a maximal stratum is the image of $\mathfrak{M}(\mathcal{X}/B,\tau')$ for τ' a *minimal* global type realizable over B dominating τ. Minimality here means that the morphism $\tau'\to\tau$ does not factor over any other global type realizable over B.

Note, however, that $\mathfrak{M}(\mathcal{X}/B,\tau')$ is not in general irreducible even for realizable τ', due to saturation phenomena already present in ordinary stable logarithmic maps. In the logarithmic enhancement question for transverse stable logarithmic maps of [3, Theorem 4.13], this reducibility is reflected in various choices of roots of unity.

Example 3.32 (Example 2.58 revisited, see Figure 3.1). Let τ be the global type with G having just one vertex of genus 0, no edges, and four legs, all image cones equal to $0\in\Sigma(X)=\{0,\mathbb{R}_{\geq 0}\}$ and global contact orders $-1,-1,2,2$. This global type is not realizable because there can be no positive length legs for the two punctures, but there are several minimal realizable global types marked by τ. Here are two of

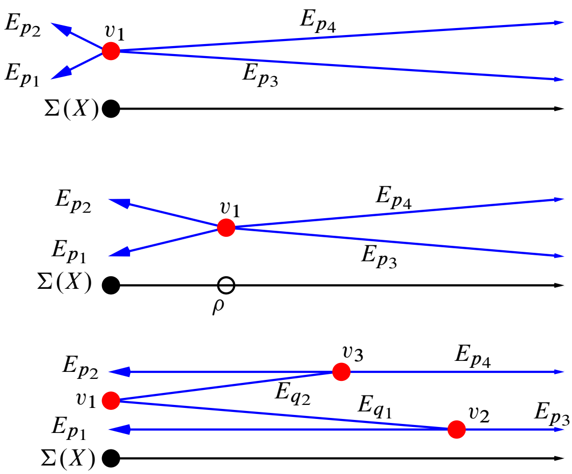

Figure 3.1. The top combinatorial map is not tropically realizable since E_{p_1}, E_{p_2} have nowhere to stretch. The first realizable type has no nodes, with $\ell_1 = \ell_2 = 0$, but with v_1 positioned at $\rho > 0$. The second has v_1 positioned at $\rho = 0$ but then $\ell_1, \ell_2 > 0$.

them. The first, τ_1, has the same $(G, \mathbf{g})$ as τ, but all image cones are $\mathbb{R}_{\geq 0}$. In the notation of Example 2.58, the tropical punctured map realizing this type has $\rho > 0$ and $\ell_1 = \ell_2 = 0$. The other minimal realizable type, τ_2, has G with three vertices v_1, v_2, v_3 with $\boldsymbol{\sigma}(v_1) = \{0\}$, $\boldsymbol{\sigma}(v_2) = \boldsymbol{\sigma}(v_3) = \mathbb{R}_{\geq 0}$ and two edges, connecting v_1 to v_2 and v_3, respectively, and one positive and one negative leg attached to each of v_2 and v_3. This global type is realizable by tropical punctured maps with $\rho = 0$ and $\ell_1, \ell_2 > 0$. Note that by Proposition 3.30, $\dim \mathfrak{M}(\mathcal{X}/B, \tau_1) = 0$ but $\dim \mathfrak{M}(\mathcal{X}/B, \tau_2) = -1$, showing non-pure-dimensionality of $\mathfrak{M}(\mathcal{X}/B, \tau)$.

3.5.6 Comparing marked and weakly marked stacks

We end this section by showing that the marked and weakly marked moduli spaces have the same reduction.

Proposition 3.33. *Let $\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u})$ be a global type of punctured maps and assume X is simple. Then the canonical morphism*

$$\mathfrak{M}(\mathcal{X}/B, \tau) \to \mathfrak{M}'(\mathcal{X}/B, \tau)$$

is a closed embedding defined by a nilpotent ideal. Analogous statements hold for moduli spaces of punctured maps to X/B and for decorated global types.

Proof. By the idealized description in Theorem 3.25 of the moduli spaces in question, the statement amounts to showing that the τ-marked ideal from Definition 3.20 is contained in the radical of the weakly τ-marked ideal defined in Remark 3.22.

Let $(C/W, \mathbf{p}, f)$ be a punctured map weakly marked by τ and $\bar{w}$ of $\underline{W}$ a geometric point. We adopt the notation from Definition 3.20 and in particular write $\phi : \tau_{\bar{w}} \to \tau$ for the contraction morphism given by the marking and

$$\chi_{\tau\tau_{\bar{w}}} : Q_{\tau_{\bar{w}}} \to Q_{\tau\tau_{\bar{w}}}$$

for the localization morphism of basic monoids. We have to show that for each $q \in Q_{\tau_{\bar{w}}}$ with $\chi_{\tau\tau_{\bar{w}}}(q) \neq 0$ a multiple kq lies in the monoid ideal generated by the elements listed in Definition 3.20 (i) and (ii). The description of the dual basic monoids in Proposition 2.32 provides the following commutative diagram with horizontal arrows surjective up to saturation, with as usual P_v denoting the monoid dual to the cone $\boldsymbol{\sigma}(v)$:

$$\begin{array}{ccc} \prod_{v \in V(G_{\bar{w}})} P_v \times \prod_{E \in E(G_{\bar{w}})} \mathbb{N} & \longrightarrow & Q_{\tau_{\bar{w}}} \\ \big\downarrow & & \big\downarrow \chi_{\tau\tau_{\bar{w}}} \\ \prod_{v \in V(G)} P_v \times \prod_{E \in E(G)} \mathbb{N} & \longrightarrow & Q_{\tau\tau_{\bar{w}}} \end{array}$$

The left vertical homomorphism is as follows:

$$\big((p_v)_{v \in V(G_{\bar{w}})}, (\ell_E)_{E \in E(G_{\bar{w}})}\big) \mapsto \Big(\Big(\sum_{\phi(v')=v} \chi_{v'}(p_{v'})\Big)_v, (\ell_{\phi^{-1}(E)})_E\Big).$$

As the top arrow is surjective up to saturation, there exists $k \geq 0$ such that $kq \in Q_{\tau_{\bar{w}}}$ lifts to an element (p_v, ℓ_E) in the left upper corner. Since $\chi_{\tau\tau_{\bar{w}}}(q) \neq 0$, the image of this lift in the lower left corner is non-zero. We conclude that there exists (1) $v \in V(G_{\bar{w}})$ with $\chi_v(p_v) \neq 0$ or (2) $E \in E(G_{\bar{w}}) \setminus E_\phi$ with $\ell_E \neq 0$. In the first case kq lies in the ideal generated by $\varphi_v(P_v \setminus \chi_v^{-1}(0))$, part of Definition 3.20 (i), while in the second case kq lies in the ideal generated by the nodal generator q_E from Definition 3.20 (ii). ∎