Chapter 4

The perfect obstruction theory

Throughout this chapter, we fix a log smooth morphism $X \to B$ of fs logarithmic schemes fulfilling the assumptions stated at the beginning of Chapter 3 and $n \in \mathbb{N}$. Crucial for the following discussion is the factorization of $X \to B$ over the relative Artin fan $\mathcal{X} \to B$.

Denote by $\mathcal{M}_n(X/B)$ (resp. $\mathfrak{M}_n(\mathcal{X}/B)$) the stack of marked or weakly marked punctured maps to $X \to B$ (resp. $\mathcal{X} \to B$), with n the number of punctured or nodal sections, fixing and suppressing all other decorations in the notation. In Sections 4.1 and 4.2, we construct two perfect relative obstruction theories, in the sense of [13, Definition 4.4], one for $\mathcal{M}_n(X/B) \to \mathfrak{M}_n(\mathcal{X}/B)$ and one for a related morphism $\mathcal{M}_n(X/B) \to \mathfrak{M}_n^{\mathrm{ev}}(\mathcal{X}/B)$; the latter space incorporates data of maps to X at a set of special points on the domain curve, see (4.13). Working over $\mathfrak{M}_n^{\mathrm{ev}}(\mathcal{X}/B)$ is crucial for understanding gluing at a virtual level in Section 5.3.

We will avail ourselves of the dualizing complex of various Gorenstein morphisms π. To avoid adjusting for shifts of dimension in the formulas, we denote by ω_π the relative dualizing complex, usually denoted $\omega_\pi^\bullet$, of a relatively Gorenstein morphism π, that is, the complex with the invertible relative dualizing sheaf defined in [36, Example III.9.7] (see also [20, p. 157]) shifted to the left by the relative dimension.

4.1 Obstruction theories for logarithmic maps from pairs

All cases of interest fit into the following general setup. For this subsection we do not enforce the assumptions on B from the Conventions, Section 1.6.

4.1.1 Source family

Let S be a log stack over B and assume we are given a proper and representable morphism of fine log stacks

$$Y \to S,$$

with underlying map of ordinary stacks $\underline{Y} \to \underline{S}$ flat and relatively Gorenstein. The fibers of this morphism serve as domains for a space of logarithmic maps.

In the application, Y is either the universal curve over $S = \mathfrak{M}_n(\mathcal{X}/B)$ or over $S = \mathfrak{M}_n^{\mathrm{ev}}(\mathcal{X}/B)$, or a union of sections of the universal curve with induced log structure.

4.1.2 Target family

As a target, we take a composition of morphisms of fine log stacks

$$V \to W \to B,$$

with $V \to W$ log smooth. In applications this will be the sequence[1] $X \to \mathcal{X} \to B$. We assume further given a B-morphism $Y \to W$ defining a commutative square

$$\begin{array}{ccc} Y & \longrightarrow & W \\ \big\downarrow & & \big\downarrow \\ S & \longrightarrow & B; \end{array}$$

In our applications this is the universal family of maps to the Artin fan, either prestable maps of curves or the corresponding maps of the union of sections, as the case may be.

4.1.3 Moduli of lifted maps

Let M be an open algebraic substack of the following algebraic stack over S. An object over an affine S-scheme T, considered as a log scheme by pulling back the log structure from S, consists of a commutative diagram

$$\begin{array}{ccccc} Y_T & & \longrightarrow & & V \\ \big\downarrow & \searrow & & & \big\downarrow \\ T & & Y & \longrightarrow & W \\ & \searrow & \big\downarrow & & \big\downarrow \\ & & S & \longrightarrow & B \end{array} \tag{4.1}$$

where the square formed by Y_T, T, S and Y is cartesian. Thus we are interested in lifting the map $Y \to W$ to V fiberwise relative to S. We endow M with the log structure making the morphism $M \to S$ strict. The pullback of Y to M defines the universal domain $\pi : Y_M \to M$. We have the following 2-commutative diagram of stacks

$$\begin{array}{ccccc} Y_M & & \xrightarrow{\quad f \quad} & & V \\ {\scriptstyle\pi}\big\downarrow & \searrow & & & \big\downarrow \\ M & & Y & \longrightarrow & W \\ & \searrow & \big\downarrow & & \big\downarrow \\ & & S & \longrightarrow & B \end{array} \tag{4.2}$$

[1] In this case, $V \to W$ is strict and we could indeed work with ordinary cotangent complexes throughout, but for possible other applications we do not make this assumption.

In the main application, with $Y \to S$ the family of prestable curves, M is an open substack of the stack of punctured maps of interest; thus our deformation theory fixes both the domain of the punctured map to X and the map to the relative Artin fan $\mathcal{X}$. In the secondary application, with $Y \to S$ the family of sections with logarithmic structures, the stack M parametrizes liftings of the sections from $\mathcal{X}$ to X.

4.1.4 An obstruction theory

Functoriality of log cotangent complexes [54, Property 1.1 (iv)] yields the morphism

$$f^*\Omega_{V/W} = Lf^*\mathbb{L}_{V/W} \to \mathbb{L}_{Y_M/Y} = \pi^*\mathbb{L}_{M/S}. \tag{4.3}$$

The equality on the left holds by [54, Property 1.1 (iii)] since $V \to W$ is log smooth, while the equality on the right follows since

$$\mathbb{L}_{M/S} = \mathbb{L}_{\underline{M}/\underline{S}} \quad \text{and} \quad \mathbb{L}_{Y_M/Y} = \mathbb{L}_{\underline{Y}_M/\underline{Y}}$$

by strictness of $M \to S$ [54, Property 1.1 (ii)] and then using compatibility of the ordinary cotangent complexes with flat pullback by $\underline{\pi}$.

Since $\underline{Y} \to \underline{S}$ is relatively Gorenstein by assumption, so is $\underline{Y}_M \to \underline{M}$ and we have a natural isomorphism of exact functors $\pi^! = \pi^* \otimes \omega_\pi$. Thus (4.3) is equivalent to a morphism $f^*\Omega_{V/W} \otimes \omega_\pi \to \pi^!\mathbb{L}_{M/S}$, which by adjunction is equivalent to a morphism

$$\Phi : \mathbb{E} \to \mathbb{L}_{M/S} \tag{4.4}$$

with

$$\mathbb{E} = R\pi_*(f^*\Omega_{V/W} \otimes \omega_\pi).$$

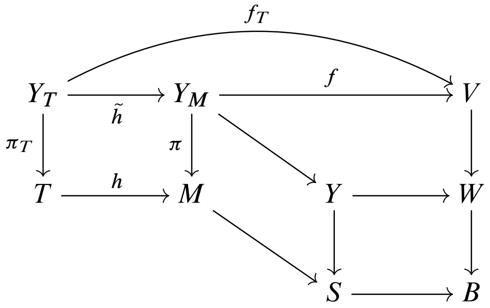
4.1.5 Functoriality

We will show in Proposition 4.2 that Φ is a perfect obstruction theory for M over S. A most transparent proof that Φ is a perfect obstruction theory for M over S relies on the fact that the construction of Φ is functorial. For lack of reference we provide a proof for this well-known property in the following lemma. If $T \to M$ is any map, denote by

$$\Phi_T : \mathbb{E}_T \to \mathbb{L}_{T/S}$$

the morphism in (4.4) constructed from (4.1) instead of (4.2).

Lemma 4.1. *The construction of Φ in (4.4) is functorial in the following sense: Let $\underline{T} \to \underline{M}$ be a morphism of stacks. Denoting $T \to M$ the associated strict morphism* *of log stacks, we obtain the commutative diagram*



with the two squares of domains (i.e., the left-most square and the parallelogram) cartesian. Then we have a commutative square

$$\begin{array}{ccc} Lh^*\mathbb{E} & \xrightarrow{Lh^*\Phi} & Lh^*\mathbb{L}_{M/S} \\ {\scriptstyle\beta}\downarrow & & \downarrow \\ \mathbb{E}_T & \xrightarrow{\Phi_T} & \mathbb{L}_{T/S}, \end{array}$$

with left-hand vertical arrow a natural isomorphism and the right-hand vertical arrow defined by functoriality of cotangent complexes.

Proof. Naturality of the base change map [67, Remark 07A7] applied to $f^*\Omega_{V/W} \otimes \omega_\pi \to \mathbb{L}_{Y_M/Y} \otimes \omega_\pi$ together with $f \circ \tilde{h} = f_T$ and $\tilde{h}^*\omega_\pi = \omega_{\pi_T}$ [20, Theorem 3.6.1], leads to the commutative square

$$\begin{array}{ccc} Lh^*\mathbb{E} = Lh^*R\pi_*(f^*\Omega_{V/W} \otimes \omega_\pi) & \longrightarrow & Lh^*R\pi_*(\mathbb{L}_{Y_M/Y} \otimes \omega_\pi) \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle b} \\ \mathbb{E}_T = R\pi_{T*}(f_T^*\Omega_{V/W} \otimes \omega_{\pi_T}) & \longrightarrow & R\pi_{T*}\big(L\tilde{h}^*\mathbb{L}_{Y_M/Y} \otimes \omega_{\pi_T}\big). \end{array} \tag{4.5}$$

Now $\mathbb{L}_{Y_M/Y} \simeq \pi^*\mathbb{L}_{M/S}$, as remarked after (4.3), and hence the adjunction counit $R\pi_*\pi^! \to \mathrm{id}$ applied in the construction of Φ in (4.4) is given by the projection formula followed by the trace morphism,

$$R\pi_*(\pi^*\mathbb{L}_{M/S} \otimes \omega_\pi) \xrightarrow{\simeq} \mathbb{L}_{M/S} \otimes R\pi_*(\omega_\pi) \xrightarrow{\mathrm{Tr}_{\omega_\pi}} \mathbb{L}_{M/S}.$$

Thus the upper horizontal map of (4.5) composed with Lh^* of this adjunction counit isomorphism yields $Lh^*\Phi$.

Similarly, extending the lower horizontal arrow by the map induced by functoriality of cotangent complexes,

$$L\tilde{h}^*\mathbb{L}_{Y_M/Y} \to \mathbb{L}_{Y_T/Y} = \pi_T^*\mathbb{L}_{T/S},$$

composed with the adjunction counit morphism

$$R\pi_{T*}(\pi_T^*\mathbb{L}_{T/S} \otimes \omega_{\pi_T}) \to \mathbb{L}_{T/S}$$

for π_T retrieves the definition of Φ_T.

Moreover, by compatibility of both the projection formula [67, Lemma 0B6B] and the trace morphism [67, Lemma 0E6C] with base change, the following diagram continuing (4.5) on the right is commutative:

$$\begin{array}{ccccc}
Lh^*R\pi_*(\pi^*\mathbb{L}_{M/S} \otimes \omega_\pi) & \xrightarrow{\simeq} & Lh^*\mathbb{L}_{M/S} \otimes Lh^*R\pi_*\omega_\pi & & \\
\downarrow b & & \downarrow & \searrow^{\mathrm{tr}} & \\
R\pi_{T*}(\pi_T^*Lh^*\mathbb{L}_{M/S} \otimes \omega_{\pi_T}) & \xrightarrow{\simeq} & Lh^*\mathbb{L}_{M/S} \otimes R\pi_{T*}\omega_{\pi_T} & \longrightarrow & Lh^*\mathbb{L}_{M/S} \\
\downarrow & & \downarrow & & \downarrow \\
R\pi_{T*}(\pi_T^*\mathbb{L}_{T/S} \otimes \omega_{\pi_T}) & \xrightarrow{\simeq} & \mathbb{L}_{T/S} \otimes R\pi_{T*}\omega_{\pi_T} & \longrightarrow & \mathbb{L}_{T/S}.
\end{array}$$

The three left horizontal isomorphisms are defined by projection formulas, the diagonal and the two horizontal morphisms on the right induced by trace homomorphisms, the two upper vertical arrows defined by base change, and the three lower vertical arrows defined by functoriality of cotangent complexes. For the identification of the upper left vertical arrow with the right vertical arrow labeled b in (4.5) note that

$$L\tilde{h}^*\mathbb{L}_{Y_M/Y} \simeq L\tilde{h}^*\pi^*\mathbb{L}_{M/S} \simeq \pi_T^*Lh^*\mathbb{L}_{M/S}.$$

This establishes the claimed commutative diagram.

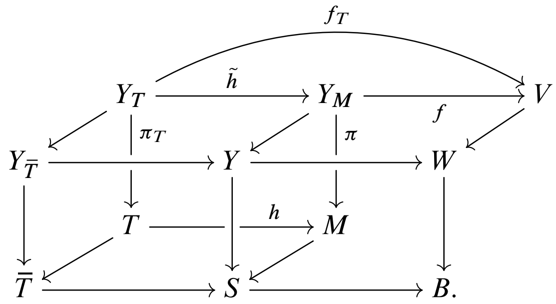
It remains to show that β is a natural isomorphism. This follows from the general base change statement [67, Lemma 0A1K] applied to $\pi : Y_M \to M$, with $f^*\Omega_{V/W}$ for the object in $D_{\mathrm{QCoh}}(\mathcal{O}_{Y_M})$ and with ω_π as complex of π-flat quasi-coherent sheaves. ∎

Proposition 4.2 (Φ is a perfect obstruction theory). *The morphism $\Phi : \mathbb{E} \to \mathbb{L}_{M/S}$ constructed in* (4.4) *is an obstruction theory for $M \to S$ in the sense of* [13, Definition 4.4].

Proof. We check the obstruction-theoretic criterion [13, Theorem 4.5.3], applied in the setting relative to S, similarly to ordinary logarithmic maps carried out in [30, Proposition 5.1].

Assume given a morphism $h : T \to M$, a square-zero extension $T \to \overline{T}$ with ideal sheaf $\mathcal{J}$ and a morphism $\overline{T} \to S$, with log structures turning all three morphisms strict.

This situation leads to the following commutative diagram:



All sides of the cube on the left are cartesian, but not in general the bottom and top faces.

The obstruction class $\omega(h) \in \mathrm{Ext}^1(Lh^*\mathbb{L}_{\underline{M}/\underline{S}}, \mathcal{J})$ for extending h to an S-morphism $\overline{T} \to M$ is the composition

$$Lh^*\mathbb{L}_{\underline{M}/\underline{S}} \to \mathbb{L}_{\underline{T}/\underline{\overline{T}}} \to \tau_{\geq -1}\mathbb{L}_{\underline{T}/\underline{\overline{T}}} = \mathcal{J}[1],$$

the first arrow defined by functoriality of cotangent complexes, see [38, Proposition 2.2.4] with $X_0 = T$, $X = \overline{T}$, $Y_0 = Y = M$ and $Z_0 = Z = S$. Because $T \to \overline{T}$ and $M \to S$ are strict we can replace the ordinary cotangent complex with the log cotangent complex in this construction [54, Property 1.1 (ii)].

Now $\Phi^*\omega(h)$ is the composition of this morphism with $Lh^*\Phi : Lh^*\mathbb{E} \to Lh^*\mathbb{L}_{M/S}$. By functoriality of our obstruction theory (Lemma 4.1), this composition also has the factorization

$$\mathbb{E}_T = R\pi_{T*}(f_T^*\Omega_{V/W} \otimes \omega_{\pi_T}) \xrightarrow{\Phi_T} \mathbb{L}_{T/S} \to \tau_{\geq -1}\mathbb{L}_{T/\overline{T}} = \mathcal{J}[1],$$

which by adjunction is equivalent to the composition

$$f_T^*\Omega_{V/W} \otimes \omega_{\pi_T} \to \mathbb{L}_{Y_T/Y} \otimes \omega_{\pi_T} \to \tau_{\geq -1}\pi_T^!\mathbb{L}_{T/\overline{T}} = \pi_T^*\mathcal{J}[1] \otimes \omega_{\pi_T}.$$

Up to tensoring with ω_{π_T} this is the obstruction class for extending $f_T : Y_T \to V$ to $Y_{\overline{T}}$, as a morphism over W. By our assumption on the objects of M, this extension exists if and only if $T \to M$ extends to $\overline{T}$. This shows the part of the criterion concerning the obstruction.

A similar argument shows that once $\omega(h) = 0$, the space of extensions form a torsor under $\mathrm{Ext}^0(Lh^*\mathbb{L}_{\underline{M}/\underline{S}}, \mathcal{J})$, showing the second part of the criterion. ∎

4.1.6 The dualizing complex of the embedding of markings

After this recapitulation of obstruction theories for logarithmic maps with proper and relatively Gorenstein domains, we are now in position to bring in point conditions.

Abstractly we consider a composition of proper, representable morphisms of fine log stacks

$$Z \xrightarrow{\iota} Y \to S, \tag{4.6}$$

with maps of algebraic stacks underlying $Z \to S$ and $Y \to S$ flat and relatively Gorenstein as before. Note that while ι may not be flat and hence cannot be considered relatively Gorenstein following the usual convention, one can still define a relative dualizing sheaf

$$\omega_\iota = \omega_{Z/S} \otimes \iota^* \omega_{Y/S}^\vee \tag{4.7}$$

fulfilling relative duality, hence defining a right-adjoint functor $\iota^!$ to $R\iota_*$. This works as in the case of smooth morphisms discussed e.g. in [37, Section 3.4].

4.1.7 Obstruction for markings

We now have another algebraic stack N, an open substack of the stack over S with objects given by diagrams as in (4.1) with Y replaced by Z. We assume the open substack N is chosen large enough so that composition with $\iota : Z \to Y$ defines a morphism of stacks

$$\varepsilon : M \to N. \tag{4.8}$$

As in (4.4) we now obtain two obstruction theories, one for $M \to S$, the other for $N \to S$,

$$\Phi : \mathbb{E} \to \mathbb{L}_{M/S}, \quad \Psi : \mathbb{F} \to \mathbb{L}_{N/S}. \tag{4.9}$$

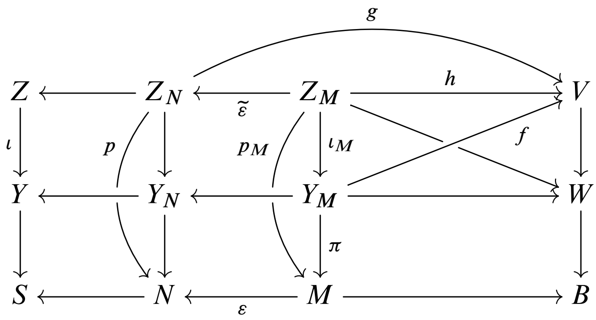
In our application, $Y \to S$ is some universal curve and $Z \to Y$ a strict closed embedding with morphism to S scheme-theoretically étale. In this case, Ψ is simply the obstruction theory for a number of points in V/W, i.e., a trivial obstruction theory in the sense that there are no obstructions. In particular, étale locally $\mathbb{F}$ can be taken as the direct sum of the pullback of $\Omega_{V/W}$ by scheme-theoretic maps from $\underline{N}$ to $\underline{V}$.

Proposition 4.3 (Compatibility of obstruction theories). *The two obstruction theories Φ and Ψ in (4.9) fit into a commutative square*

$$\begin{array}{ccc} L\varepsilon^*\mathbb{F} & \xrightarrow{L\varepsilon^*\Psi} & L\varepsilon^*\mathbb{L}_{N/S} \\ \downarrow & & \downarrow \\ \mathbb{E} & \xrightarrow{\Phi} & \mathbb{L}_{M/S}, \end{array}$$

with the right-hand vertical morphism given by functoriality of the cotangent complex.

Proof. Consider the following commutative diagram with the left four squares cartesian.



The left column is the given morphism (4.6) of domains, the lower horizontal row contains the restriction morphism ε from (4.8) and the morphisms to B and S, while $f : Y_M \to V$ and $g : Z_N \to V$ are the respective universal morphisms defined on the universal domains $Y_M \to M$ and $Z_N \to N$.

The obstruction theory Φ in (4.9) was defined by applying $R\pi_*(\cdot \otimes \omega_\pi)$ to

$$f^*\Omega_{V/W} \to \mathbb{L}_{Y_M/Y} = \pi^*\mathbb{L}_{M/S}$$

followed by the adjunction counit $R\pi_*\pi^! \to \mathrm{id}$, using $\pi^! = \pi^* \otimes \omega_\pi$. For Ψ one analogously takes $Rp_*(\cdot \otimes \omega_p)$ of $g^*\Omega_{V/W} \to \mathbb{L}_{Z_N/Z} = p^*\mathbb{L}_{N/S}$ followed by $Rp_*p^! \to \mathrm{id}$. By functoriality of obstruction theories (Lemma 4.1), the pullback $L\varepsilon^*\Psi$ is similarly obtained by $Rp_{M*}(\cdot \otimes \omega_{p_M})$ of

$$h^*\Omega_{V/W} \to L\widetilde{\varepsilon}^*\mathbb{L}_{Z_N/Z} = L\widetilde{\varepsilon}^* p^*\mathbb{L}_{N/S} = p_M^* L\varepsilon^*\mathbb{L}_{N/S}, \qquad (4.10)$$

followed by $Rp_{M*}p_M^! \to \mathrm{id}$.

From $h = f \circ \iota_M = g \circ \widetilde{\varepsilon}$ we can extend (4.10) to the commutative diagram

$$\begin{array}{ccccc}
\widetilde{\varepsilon}^* g^*\Omega_{V/W} & \longrightarrow & L\widetilde{\varepsilon}^*\mathbb{L}_{Z_N/W} & \longrightarrow & L\widetilde{\varepsilon}^*\mathbb{L}_{Z_N/Z} = p_M^* L\varepsilon^*\mathbb{L}_{N/S} \\
\| & & \downarrow & & \downarrow \\
h^*\Omega_{V/W} & \longrightarrow & \mathbb{L}_{Z_M/W} & \longrightarrow & \mathbb{L}_{Z_M/Z} = p_M^*\mathbb{L}_{M/S} \\
\| & & \downarrow & & \downarrow \simeq \\
\iota_M^* f^*\Omega_{V/W} & \longrightarrow & L\iota_M^*\mathbb{L}_{Y_M/W} & \longrightarrow & L\iota_M^*\mathbb{L}_{Y_M/Y} = p_M^*\mathbb{L}_{M/S}
\end{array}$$

The last row in this diagram is $L\iota_M^*$ of the morphism $f^*\Omega_{V/W} \to \pi^*\mathbb{L}_{M/S}$ that gives rise to the obstruction theory Φ for M. The essential part of this diagram is the square

$$\begin{array}{ccc}
h^*\Omega_{V/W} & \longrightarrow & p_M^* L\varepsilon^*\mathbb{L}_{N/S} \\
\| & & \downarrow \\
\iota_M^* f^*\Omega_{V/W} & \longrightarrow & p_M^*\mathbb{L}_{M/S}
\end{array} \qquad (4.11)$$

Next observe that $\omega_{p_M} = \iota_M^* \omega_\pi \otimes \omega_{\iota_M}$, $h = f \circ \iota_M$, and $\iota_M^! = \iota_M^* \otimes \omega_{\iota_M}$ show that

$$Rp_{M*}(h^*\Omega_{V/W} \otimes \omega_{p_M}) = R\pi_* R\iota_{M*}\iota_M^!(f^*\Omega_{V/W} \otimes \omega_\pi).$$

Thus $Rp_{M*}(\cdot \otimes \omega_{p_M})$ applied to (4.11) yields the upper left square of the following commutative diagram:

$$\begin{array}{ccccc}
L\varepsilon^*\mathbb{F} = Rp_{M*}(h^*\Omega_{V/W} \otimes \omega_{p_M}) & \longrightarrow & Rp_{M*}p_M^! L\varepsilon^*\mathbb{L}_{N/S} & \longrightarrow & L\varepsilon^*\mathbb{L}_{N/S} \\
\| & & \downarrow a & & \downarrow \\
R\pi_* R\iota_{M*}\iota_M^!(f^*\Omega_{V/W} \otimes \omega_\pi) & \xrightarrow[b]{} & Rp_{M*}p_M^!\mathbb{L}_{M/S} & \longrightarrow & \mathbb{L}_{M/S} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{E} = R\pi_*(f^*\Omega_{V/W} \otimes \omega_\pi) & \longrightarrow & R\pi_*\pi^!\mathbb{L}_{M/S} & \longrightarrow & \mathbb{L}_{M/S}.
\end{array} \tag{4.12}$$

The upper right square is from functoriality of adjunction $Rp_{M*}p_M^! \to \mathrm{id}$ applied to the arrow marked a, the lower left one similarly from $R\iota_{M*}\iota_M^! \to \mathrm{id}$ applied to the arrow marked b. The lower right square is from the natural isomorphism of the adjunction counit $Rp_{M*}p_M^! \to \mathrm{id}$ with the composition

$$R\pi_* R\iota_{M*}\iota_M^! \pi^! \to R\pi_*\pi^! \to \mathrm{id},$$

see [36, Proposition VII.3.4 (b)], [20, Lemma 3.4.3].

The outer square of (4.12) provides the claimed commutative diagram. ∎

4.2 Obstruction theories for punctured maps with point conditions

We are now in position to define obstruction theories for moduli spaces of punctured maps with prescribed point conditions. Recall the log smooth morphism $X \to B$ and its factorization over the relative Artin fan $\mathcal{X} \to B$ from the beginning of this chapter. We want to work relative to a stack S of stable punctured maps to $\mathcal{X}/B$. Adopting the notation used elsewhere in the paper, we now write $\mathfrak{M}$ instead of S for the algebraic stack of domains together with the tuple of points at which to impose point conditions. For example, $\mathfrak{M}$ could be $\mathfrak{M}(\mathcal{X}/B, \tau)$ from Definition 3.8. Then $Y \to S = \mathfrak{M}$ is the universal curve, $Z \to Y$ the strict closed embedding of a union of sections, one for each point condition to be imposed, assumed ordered, and we have a universal diagram

$$\begin{array}{ccc}
Y & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow \\
\mathfrak{M} & \longrightarrow & B.
\end{array}$$

As our target we now take the composition

$$X \to \mathcal{X} \to B.$$

Note that $\mathcal{X} \to B$ is log étale and $X \to \mathcal{X}$ is strict and log smooth. Hence $\underline{X} \to \underline{\mathcal{X}}$ is smooth as a morphism of stacks and we have a sequence of canonical isomorphisms

$$\mathbb{L}_{X/B} = \Omega_{X/B} = \Omega_{X/\mathcal{X}} = \Omega_{\underline{X}/\underline{\mathcal{X}}} = \mathbb{L}_{\underline{X}/\underline{\mathcal{X}}}.$$

For easier reference later on we also write $\mathcal{M}$ instead of M for the algebraic stack of punctured maps to X to be considered.

For the moduli space N of point conditions we take the space of factorizations of the composition $Z \to Y \to \mathcal{X}$ via $X \to \mathcal{X}$. Note that since $X \to \mathcal{X}$ is strict, it is enough to provide the lift for $\underline{X} \to \underline{\mathcal{X}}$, that is, ignoring the log structure. Thinking of these factorizations as providing evaluation maps $\mathfrak{M} \to \underline{X}$ at the marked points given by the sections Z of $Y \to S$, we denote the stack of such factorizations by $\mathfrak{M}^{\mathrm{ev}}$. This stack is algebraic by the fiber product description

$$\mathfrak{M}^{\mathrm{ev}} = \mathfrak{M} \times_{\underline{\mathcal{X}} \times_{\underline{B}} \cdots \times_{\underline{B}} \underline{\mathcal{X}}} (\underline{X} \times_{\underline{B}} \cdots \times_{\underline{B}} \underline{X}). \tag{4.13}$$

Here the map $\mathfrak{M} \to \underline{\mathcal{X}} \times_{\underline{B}} \cdots \times_{\underline{B}} \underline{\mathcal{X}}$ is defined by composing the sections $\mathfrak{M} \to \underline{\mathfrak{M}} \to \underline{Z}$ with the composition $\underline{Z} \to \underline{Y} \to \underline{\mathcal{X}}$ in the given order of the sections.

With this notation, the composition $M \to N \to S$ considered in the proof of Proposition 4.3 reads

$$\mathcal{M} \xrightarrow{\varepsilon} \mathfrak{M}^{\mathrm{ev}} \to \mathfrak{M}. \tag{4.14}$$

In Section 4.1 we recalled the construction of obstruction theories for $\mathcal{M}/\mathfrak{M}$ and for $\mathfrak{M}^{\mathrm{ev}}/\mathfrak{M}$, which in the situation at hand are perfect of amplitude contained in $[-1, 0]$, and showed their compatibility (Proposition 4.3). As in [50, Construction 3.13], this situation provides perfect obstruction theories for $\mathcal{M}/\mathfrak{M}^{\mathrm{ev}}$ by completing the compatibility diagram in Proposition 4.3 to a morphism of distinguished triangles:

$$\begin{array}{ccccccc} L\varepsilon^*\mathbb{F} & \longrightarrow & \mathbb{E} & \longrightarrow & \mathbb{G} & \longrightarrow & L\varepsilon^*\mathbb{F}[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ L\varepsilon^*\mathbb{L}_{\mathfrak{M}^{\mathrm{ev}}/\mathfrak{M}} & \longrightarrow & \mathbb{L}_{\mathcal{M}/\mathfrak{M}} & \longrightarrow & \mathbb{L}_{\mathcal{M}/\mathfrak{M}^{\mathrm{ev}}} & \longrightarrow & L\varepsilon^*\mathbb{L}_{\mathfrak{M}^{\mathrm{ev}}/\mathfrak{M}}[1] \end{array} \tag{4.15}$$

Remark 4.4. Note that while the isomorphism class of $\mathbb{G}$ is unique, the dashed arrow is not, so this recipe potentially provides several different obstruction theories for $\mathcal{M}/\mathfrak{M}^{\mathrm{ev}}$. On the other hand, any two dashed arrows differ by an element of the image of

$$\mathrm{Hom}(\mathbb{G}, \mathbb{L}_{\mathcal{M}/\mathfrak{M}}) \to \mathrm{Hom}(\mathbb{G}, \mathbb{L}_{\mathcal{M}/\mathfrak{M}^{\mathrm{ev}}}).$$

Thus the space of obstruction theories $\mathbb{G} \to \mathbb{L}_{\mathcal{M}/\mathfrak{M}^{\mathrm{ev}}}$ constructed as dashed arrow in (4.15) is parametrized by an affine space. This shows that the virtual classes constructed from any two such obstruction theories agree.[2]

For the sake of being explicit and for later use we now work out $\mathbb{G}$. For simplicity of notation write $\pi : C \to \mathcal{M}$ for the pullback $Y_{\mathcal{M}}$ of the universal curve $Y \to \mathfrak{M}$ to $\mathcal{M}$, and, in disagreement with our usual conventions, write $\iota : Z \to C$ for the strict closed substack of special points rather than $Z_{\mathcal{M}}$. We assume that $Z = Z' \amalg Z''$ with Z' disjoint from the critical locus of $\underline{C} \to \underline{\mathcal{M}}$ and Z'' the images of a set of nodal sections, as reviewed in Definition 5.1 below. Denote by $\kappa : \widetilde{C} \to C$ the partial normalization of $\widetilde{C}$ along the nodal sections exhibiting $\underline{C}$ as the fibered sum

$$\underline{C} = \underline{Z}'' \amalg_{\underline{\widetilde{Z}}''} \underline{\widetilde{C}}$$

with $\widetilde{Z}'' = \kappa^{-1}(Z'') \to Z''$ the two-fold unbranched cover induced by κ. Write $\widetilde{\pi} = \pi \circ \kappa : \widetilde{C} \to \mathcal{M}$, $\widetilde{f} = f \circ \kappa : \widetilde{C} \to X$ and $\widetilde{Z} = \kappa^{-1}(Z)$, with the log structures making $\widetilde{C} \to C$ and $\widetilde{Z} \to \widetilde{C}$ strict.[3]

For simplicity of the following statement *we now assume the two-fold covering* $\widetilde{Z}'' \to Z''$ *is trivial*, that is, that there is an isomorphism

$$\widetilde{Z}'' \simeq Z'' \amalg Z''$$

over Z''. This is sufficient for all applications we can currently think of. The general case can be treated by going over to an orientation covering or by twisting with an orientation sheaf.

Proposition 4.5. *For the tangent-obstruction bundle in* (4.15) *it holds*

$$\begin{aligned}\mathbb{G} &\simeq R\pi_*(f^*\Omega_{X/B} \otimes \kappa_*(\omega_{\widetilde{\pi}}(\widetilde{Z}))) \\ &\simeq R\widetilde{\pi}_*(\widetilde{f}^*\Omega_{X/B} \otimes \omega_{\widetilde{\pi}}(\widetilde{Z})) \\ &\simeq (R\widetilde{\pi}_* \widetilde{f}^*\Theta_{X/B}(-\widetilde{Z}))^\vee.\end{aligned}$$

Moreover, $\mathbb{G}$ *is perfect of amplitude* $[-1, 0]$.

Proof. The second isomorphism follows by the projection formula, the third isomorphism by relative duality.

For the first isomorphism we first claim there exists the following exact sequence of complexes, all concentrated in degree -1:

$$0 \to \omega_\pi \to \kappa_*(\omega_{\widetilde{\pi}}(\widetilde{Z})) \to \iota_*\mathcal{O}_Z[1] \to 0. \tag{4.16}$$

[2]We learnt this argument from Tom Graber.

[3]The log structures on $\widetilde{C}$ and $\widetilde{Z}$ are irrelevant for the following discussion and are merely chosen for the sake of uncluttering the notation.

On the complement of the nodal locus Z'', this sequence is defined by

$$0 \to \omega_\pi \to \omega_\pi(Z') \to \omega_\pi \otimes_{\mathcal{O}_C} \iota_*\mathcal{O}_{Z'}(Z') \to 0$$

by means of the canonical isomorphism

$$\omega_\pi \otimes_{\mathcal{O}_C} \iota_*\mathcal{O}_{Z'}(Z') = \iota_*\big(\iota^*\omega_\pi \otimes_{\mathcal{O}_{Z'}} \omega_\iota\big) \simeq \iota_*\mathcal{O}_{Z'}[1]$$

coming from the definition of $\omega_\iota = \iota^*\omega_\pi^\vee \simeq \mathcal{O}_{Z'}(Z')$ in (4.7). Explicitly, the homomorphism $\omega_\pi(Z') \to \iota^*\mathcal{O}_{Z'}[1]$ takes the residue along Z'.

Near the nodal locus, (4.16) is defined by

$$0 \to \omega_\pi \xrightarrow{\kappa^*} \kappa_*(\omega_{\tilde\pi}(\tilde Z)) \to \iota_*\mathcal{O}_{Z''}[1] \to 0.$$

To obtain this sequence, recall that étale locally $\omega_\pi = \Omega_{C/\mathcal{M}}[1]$ with $\Omega_{C/\mathcal{M}}$ the sheaf of relative logarithmic differentials for $C/\mathcal{M}$, while $\omega_{\tilde\pi} = \Omega_{\underline{\tilde C}/\underline{\mathcal{M}}}[1]$ with $\Omega_{\underline{\tilde C}/\underline{\mathcal{M}}}$ the sheaf of relative ordinary differentials for $\underline{\tilde C}/\underline{\mathcal{M}}$. In fiberwise coordinates z, w for the two branches of C along Z'' on an étale neighborhood, $\Omega_{C/\mathcal{M}}$ is locally generated by $z^{-1}dz = -w^{-1}dw$, hence pulls back to ordinary differentials with simple poles along $\kappa^{-1}(Z'') \subseteq \tilde Z$. The map to $\mathcal{O}_Z$ takes the difference of the residues of such rational differential forms on $\tilde C$ along the two preimages of the nodal locus. Note that this map depends on an order of the two branches along each connected component of Z'', hence relies on the assumption $\tilde Z'' = Z'' \amalg Z''$. This establishes sequence (4.16).

Next note that $\omega_{p_\mathcal{M}} \simeq \mathcal{O}_Z$ since $p_\mathcal{M} : Z \to \mathcal{M}$ is étale. Using the projection formula we can thus rewrite

$$L\varepsilon^*\mathbb{F} = Rp_{\mathcal{M}*}(h^*\Omega_{X/B} \otimes \omega_{p_\mathcal{M}}) = R\pi_*\iota_*\iota^* f^*\Omega_{X/B} = R\pi_*(f^*\Omega_{X/B} \otimes \iota_*\mathcal{O}_Z).$$

Finally, apply $R\pi_*$ to (4.16) tensored with $f^*\Omega_{X/B}$ to produce the upper triangle of (4.15) with the claimed middle term $\mathbb{G} = R\pi_*(f^*\Omega_{X/B} \otimes \kappa_*(\omega_{\tilde\pi}(\tilde Z)))$:

$$\begin{array}{ccccc} \mathbb{E} & & \mathbb{G} & & L\varepsilon^*\mathbb{F}[1] \\ \| & & \| & & \| \\ R\pi_*(f^*\Omega_{X/B} \otimes \omega_\pi) & \longrightarrow & R\pi_*(f^*\Omega_{X/B} \otimes \kappa_*(\omega_{\tilde\pi}(\tilde Z))) & \longrightarrow & p_{\mathcal{M}*}(h^*\Omega_{X/B})[1] \end{array} \tag{4.17}$$

Taking cohomologies, this diagram also shows the statement about the amplitude of $\mathbb{G}$. ∎

4.3 Punctured Gromov–Witten invariants

Using properness of $\mathcal{M}(X/B, \beta)$ over B (Corollary 3.19) and the obstruction theory, we can now define punctured Gromov–Witten invariants. To be explicit, we assume the ground field $\Bbbk$ to be a subfield of $\mathbb{C}$ and take $H_2(X)$ to be singular homology of the base change to $\mathbb{C}$. Since $\mathfrak{M}(\mathcal{X}/B,\beta)$ is typically non-equidimensional due to the puncturing ideal, the general definition demands a stratum-by-stratum treatment. Sometimes one can show independence of certain choices, e.g. in the setting of [33], but presently our understanding of the intersection theory of $\mathfrak{M}(\mathcal{X}/B)$ and in logarithmic geometry is too limited to make general statements. Some steps in this direction have been taken in [10, 71].

Let $X \to B$ be projective and log smooth, with Zariski logarithmic structure on X. Let $\boldsymbol{\tau} = (G, \mathbf{g}, \boldsymbol{\sigma}, \bar{\mathbf{u}}, \mathbf{A})$ be a decorated global type (Definition 2.44). Denote by g the total genus and $k = |L(G)|$. We assume $\bar{\mathcal{M}}_X^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ to be generated by global sections to apply Corollary 3.19, or otherwise $\mathcal{M}(X/B,\boldsymbol{\tau}) \to B$ to be proper. Denote by $Z_L = X_{\boldsymbol{\sigma}(L)} \subseteq X$ the evaluation stratum for $L \in L(G)$.

Considering for simplicity evaluations at all punctures rather than at a subset of punctures, we then have an evaluation map

$$\underline{\mathrm{ev}} : \underline{\mathcal{M}}(X/B,\boldsymbol{\tau}) \to \prod_{L\in L(G)} \underline{Z}_L,$$

and, by Section 4.2 and notably (4.15), a perfect relative obstruction theory $\mathbb{G}$ for

$$\varepsilon : \mathcal{M}(X/B,\boldsymbol{\tau}) \to \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\boldsymbol{\tau}).$$

The relative virtual dimension is given by the Riemann–Roch formula applied to the virtual bundle in Proposition 4.5 as

$$d(g,k,A,n) = c_1(\Theta_{X/B}) \cdot A + n \cdot (1-g-k). \tag{4.18}$$

Here $A = |\mathbf{A}|$ and $g = |\mathbf{g}|$ are the total curve class and total genus of $\boldsymbol{\tau}$, $k = |L(G)|$ the number of point conditions imposed and $n = \dim X - \dim B$ the relative dimension of X over B. Denote by $\varepsilon_{\mathbb{G}}^{!}$ the associated virtual pullback from [50], an operational Chow class for ε.

Definition 4.6. The *punctured Gromov–Witten correspondence* defined by the global decorated type $\boldsymbol{\tau}$ is the homomorphism

$$(\underline{\mathrm{ev}} \times p)_* \varepsilon_{\mathbb{G}}^{!} : A_*(\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\boldsymbol{\tau})) \to A_{*+d(g,k,A,n)}\Big(\prod_L \underline{Z}_L \times \mathcal{M}_{g,k}\Big)$$

of rational Chow groups.

Here $\prod_L$ denotes the cartesian product of spaces over B. As usual, pairing with cohomology classes in $\prod_L \underline{Z}_L \times \mathcal{M}_{g,k}$ and taking degrees then produces Gromov–Witten invariants. Note also that Proposition 3.30 defines pure-dimensional cycles in $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\boldsymbol{\tau})$ as the images of the fundamental classes of $\mathfrak{M}(\mathcal{X}/B,\boldsymbol{\tau}')$ for $\boldsymbol{\tau}' \to \boldsymbol{\tau}$ a contraction morphism from a realizable global type.