Chapter 5

Splitting and gluing

As discussed in the introduction, one crucial motivation for the introduction of the notion of punctured maps is the desire to treat logarithmic Gromov–Witten invariants by splitting the domain curves along nodal sections, in situations where such sections occur uniformly in the moduli space.

After briefly formalizing this splitting operation, we present the second series of main results of this paper, the reverse procedure of gluing a pair of punctured sections, followed by its treatment in punctured Gromov–Witten theory. We end this chapter with an application to the degeneration situation of [3].

Throughout this chapter, $X \to B$ denotes a morphism of fs logarithmic schemes fulfilling the assumptions stated at the beginning of Chapter 3.

5.1 Splitting punctured maps

We first discuss the operation of splitting of punctured curves along nodal sections.

Definition 5.1. A *nodal section* of a family of nodal curves $\underline{\pi} : \underline{C} \to \underline{W}$ is a section $s : \underline{W} \to \underline{C}$ of $\underline{\pi}$ that étale locally in $\underline{W}$ factors over the closed embedding defined by the ideal (x, y) in the domain of an étale map

$$\mathbf{Spec}\, \mathcal{O}_W[x, y]/(xy) \to \underline{C}.$$

The *partial normalization of* $\underline{C}/\underline{W}$ *along* s is the map

$$\underline{\kappa} : \underline{\widetilde{C}} \to \underline{C} \tag{5.1}$$

that étale locally is given by base change from the normalization of the plane nodal curve $\operatorname{Spec} \Bbbk[x, y]/(xy)$. We say s is *of splitting type* if the two-fold unbranched cover $\underline{\kappa}^{-1}(\operatorname{im}(s)) \to \operatorname{im}(s)$ is trivial.

A nodal section of a punctured curve $(C^\circ/W, \mathbf{p})$ or punctured map $(C^\circ/W, \mathbf{p}, f)$ is a nodal section of the underlying curve $\underline{C}/\underline{W}$.

Note that a nodal section s of a nodal curve $\underline{C}/\underline{W}$ with partial normalization $\underline{\kappa} : \underline{\widetilde{C}} \to \underline{C}$ and nodal locus $\underline{Z} = \operatorname{im}(s)$ exhibits $\underline{C}$ as the fibered sum

$$\underline{Z} \amalg_{\underline{\kappa}^{-1}(\underline{Z})} \underline{\widetilde{C}} \xrightarrow{\simeq} \underline{C}. \tag{5.2}$$

A punctured curve can be split along a nodal section of splitting type.

Proposition 5.2. *Let $\underline{\kappa} : \underline{\tilde{C}} \to \underline{C}$ be the partial normalization of a punctured curve $(\pi : C^\circ \to W, \mathbf{p})$ defined by the splitting at a nodal section s of splitting type. Let $p_1, p_2 : \underline{W} \to \underline{\tilde{C}}$ be two sections of $\underline{\kappa}^{-1}(\mathrm{im}(s)) \to \mathrm{im}(s)$ with disjoint images.*

Then

$$(\tilde{C}^\circ, \tilde{\mathbf{p}}) = \big(\tilde{\pi} : (\underline{\tilde{C}}, \underline{\kappa}^* \mathcal{M}_{C^\circ}) \xrightarrow{\kappa} C^\circ \to W, \{\hat{\mathbf{p}}, p_1, p_2\}\big)$$

with $\hat{\mathbf{p}} : \underline{W} \to \underline{\tilde{C}}$ the unique set of sections with $\mathbf{p} = \underline{\kappa} \circ \hat{\mathbf{p}}$, is a (possibly disconnected) punctured curve.

Proof. Since $\kappa : \tilde{C}^\circ \to C^\circ$ is an isomorphism away from $\mathrm{im}(p_1) \cup \mathrm{im}(p_2)$, it suffices to consider a neighborhood of a geometric point $\bar{p} \to \underline{\tilde{C}}$ of one of $\mathrm{im}(p_i)$, say $i = 1$. Denote by $\bar{q} = \underline{\kappa} \circ \bar{p}$ the corresponding geometric point of $\underline{C}$, thus a geometric point of the image of the nodal section. By the structure of log smooth curves, $\mathcal{M}_{C^\circ, \bar{q}}$ is generated by $(\pi^* \mathcal{M}_W)_{\bar{q}}$, s_x and s_y, where $s_x, s_y \in \mathcal{M}_{C^\circ, q}$ are induced by the coordinates x, y in Definition 5.1. These are subject to the relation $s_x s_y = s_\rho$ for some $s_\rho \in (\pi^* \mathcal{M}_W)_{\bar{q}}$. Hence $(\pi^* \mathcal{M}_W)_{\bar{q}}$ and s_y locally generate $\mathcal{M}^{\mathrm{gp}}_{C^\circ}$ as a group, with $s_x = s_\rho s_y^{-1}$. Pulling back to $\underline{\tilde{C}}$, along the branch $x = 0$, hence with $y = 0$ giving $\mathrm{im}(p_1)$, we see that $(\kappa^* \mathcal{M}_{C^\circ})^{\mathrm{gp}}$ is locally generated by $(\tilde{\pi}^* \mathcal{M}_W)_{\bar{p}}$ and $\kappa^\flat s_y$. Further, $\kappa^\flat s_y$ is also a section of $\mathcal{P}$, the divisorial log structure given by p_1, and the image of $\kappa^\flat s_y$ in $\overline{\mathcal{P}}$ generates $\overline{\mathcal{P}}$ as a monoid. Thus locally near $\bar{p}$,

$$\tilde{\pi}^* \mathcal{M}_W \oplus_{\mathcal{O}^\times_{\tilde{C}}} \mathcal{P} \subseteq \kappa^* \mathcal{M}_{C^\circ} \subset \tilde{\pi}^* \mathcal{M}_W \oplus_{\mathcal{O}^\times_{\tilde{C}}} \mathcal{P}^{\mathrm{gp}}.$$

Further, any local section of $\kappa^* \mathcal{M}_{C^\circ}$ not contained in $\tilde{\pi}^* \mathcal{M}_W \oplus_{\mathcal{O}^\times_{\tilde{C}}} \mathcal{P}$ can be written in the form $s_x^a s_y^b s_W$ with $a > 0$, $b \geq 0$ and s_W a local section of $\tilde{\pi}^* \mathcal{M}_W$. Since $\alpha(s_x) = 0$ when $x = 0$, we see that α applied to any such element is zero. Thus $(\tilde{C}^\circ / W, \tilde{\mathbf{p}})$ is a punctured curve near $\bar{p}$. ■

For the application to moduli spaces of punctured maps we formalize the splitting procedure as an operation on graphs, hence on (global) types of punctured maps.

Definition 5.3. Let G be a connected graph and $\mathbf{E} \subseteq E(G)$ a subset of edges. Replacing each $E \in \mathbf{E}$ by a pair of legs L_E, L'_E leads to a graph $\widehat{G}$ with

$$V(\widehat{G}) = V(G), \quad E(\widehat{G}) = E(G) \setminus \mathbf{E}, \quad L(\widehat{G}) = L(G) \cup \{L_E, L'_E\}_{E \in \mathbf{E}}.$$

We call the collection of connected subgraphs $G_1, \ldots, G_r$ of $\widehat{G}$ *the graphs obtained from G by splitting along* $\mathbf{E}$.

There is an obvious induced notion of splitting of a genus-decorated graph $(G, \mathbf{g})$, of a (global) type τ, or of a (global) decorated type $\boldsymbol{\tau}$ of a punctured map along a subset of edges of the corresponding graphs.

Proposition 5.4. *Let $X \to B$ be a morphism of fs logarithmic schemes over $\Bbbk$ fulfilling the assumptions stated at the beginning of Chapter 3. Let $\tau_1, \ldots, \tau_r$ be obtained*

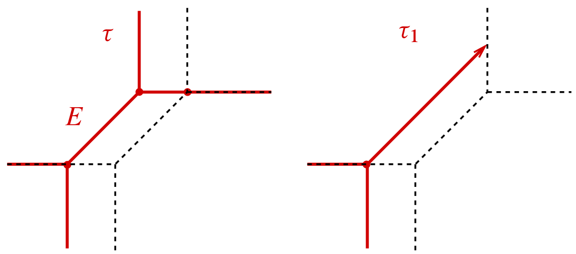


Figure 5.1. Tropical splitting.

from splitting a global type $\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \bar{\mathbf{u}})$ *of a punctured map to* X/B *along a subset of edges* $\mathbf{E} \subseteq E(G)$. *Then the splitting morphism from Proposition 5.2 followed by pre-stabilization (Proposition 2.5) defines morphisms of stacks*

$$\mathfrak{M}(\mathcal{X}/B, \tau) \to \prod_i \mathfrak{M}(\mathcal{X}/B, \tau_i), \quad \mathcal{M}(X/B, \tau) \to \prod_i \mathcal{M}(X/B, \tau_i),$$

with the products understood as fiber products over B.

Analogous results hold for decorated types and for moduli spaces of weakly marked punctured maps.

Proof. The statement is immediate from Propositions 5.2 and 2.5. ■

Example 5.5. As an illustration of the splitting procedure consider the degeneration of $\mathbb{P}^1 \times \mathbb{P}^1$ to two copies of $\mathbb{P}^2$ constructed as follows. Take the polyhedral decomposition $\mathcal{P}$ of $\mathbb{R}^2$ with two vertices at $(0,0)$, $(1,1)$ and four maximal cells given by the dashed part of Figure 5.1. Embed $\mathbb{R}^2$ as affine hyperplane $\mathbb{R}^2 \times \{1\}$ in $\mathbb{R}^3$ and take the closures of the cones over cells of $\mathcal{P}$ to define a fan Σ in $\mathbb{R}^3$ with support $|\Sigma| = \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$. The corresponding toric threefold X comes with a flat morphism

$$\pi : X \to \mathbb{A}^1$$

induced by the projection $|\Sigma| \to \mathbb{R}_{\geq 0}$ to the last coordinate. It is not hard to show that $\pi^{-1}(\mathbb{A}^1 \setminus \{0\}) = (\mathbb{P}^1 \times \mathbb{P}^1) \times (\mathbb{A}^1 \setminus \{0\})$, a trivial family, and $\pi^{-1}(0) = \mathbb{P}^2 \amalg_{\mathbb{P}^1} \mathbb{P}^2$, a gluing of two copies of $\mathbb{P}^2$ along a pair of toric divisors.

Figure 5.1 on the left shows the tropicalization of a family of curves of bidegree $(1,1)$ giving a type τ. The figure shows the intersection with the affine hyperplane $\mathbb{R}^2 \times \{1\}$. Splitting along the edge E yields the two types τ_1, τ_2 whose general members are depicted on the right. Note also that the leg in τ_2 obtained from splitting τ at E extends to the boundary of the cell, while this is not true for τ_1. This illustrates the necessity of pre-stabilization in the splitting procedure.

The opposite process of shrinking legs to an edge of a tropical domain curve appears in gluing, see Remark 5.12.

5.2 Gluing punctured maps to $\mathcal{X}/B$

5.2.1 Notation for splitting edges

In this section we work in categories of spaces over $\underline{B}$ or B. In particular, products are to be understood as fiber products over $\underline{B}$ or B, as appropriate.

Let $\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \bar{\mathbf{u}})$ be a global type of punctured tropical maps and $\tau_i = (G_i, \mathbf{g}_i, \boldsymbol{\sigma}_i, \bar{\mathbf{u}}_i)$, $i = 1, \ldots, r$, the global types obtained by splitting τ at a subset $\mathbf{E} \subseteq E(G)$ of edges (Definition 5.3). We choose an orientation on each edge $E \in \mathbf{E}$ and refer to the two legs obtained by splitting the edge E with vertices v, v' by the corresponding half-edges (E, v), (E, v'), with E oriented from v to v'.[1] Denote by $\mathbf{L} \subseteq \bigcup_i L(G_i)$ the subset of all legs obtained from splitting edges, and by $i(v) \in \{1, \ldots, r\}$ for $v \in V(G)$ the index i with $v \in V(G_i)$.

5.2.2 The stack $\widetilde{\mathfrak{M}}'(\mathcal{X}/B, \tau)$ and its evaluation morphism

Evaluation at the nodal sections for $\mathbf{E}$ defines the morphism

$$\underline{\mathrm{ev}_{\mathbf{E}}} : \underline{\mathfrak{M}'}(\mathcal{X}/B, \tau) \to \prod_{E \in \mathbf{E}} \underline{\mathcal{X}}.$$

For each $E \in \mathbf{E}$ denote by $\mathfrak{M}'_E(\mathcal{X}/B, \tau)$ the image of the corresponding nodal section $s_E : \underline{\mathfrak{M}'}(\mathcal{X}/B, \tau) \to \underline{\mathfrak{C}'}^{\circ}(\mathcal{X}/B, \tau)$ with the restriction of the log structure on the universal domain $\mathfrak{C}'^{\circ}(\mathcal{X}/B, \tau)$. Denote further by $\widetilde{\mathfrak{M}}'(\mathcal{X}/B, \tau)$ the fs fiber product

$$\widetilde{\mathfrak{M}}'(\mathcal{X}/B, \tau) = \mathfrak{M}'_{E_1}(\mathcal{X}/B, \tau) \times^{\mathrm{fs}}_{\mathfrak{M}'(\mathcal{X}/B,\tau)} \cdots \times^{\mathrm{fs}}_{\mathfrak{M}'(\mathcal{X}/B,\tau)} \mathfrak{M}'_{E_r}(\mathcal{X}/B, \tau), \quad (5.3)$$

where $E_1, \ldots, E_r \in E(G)$ are the edges in $\mathbf{E}$.[2] With this enlarged log structure, the pullback $\widetilde{\mathfrak{C}}'^{\circ}(\mathcal{X}/B, \tau) \to \widetilde{\mathfrak{M}}'(\mathcal{X}/B, \tau)$ of the universal domain has sections $\tilde{s}_E$, $E \in \mathbf{E}$, in the category of log stacks. Moreover, $\underline{\mathrm{ev}_{\mathbf{E}}}$ lifts to a logarithmic evaluation morphism

$$\mathrm{ev}_{\mathbf{E}} : \widetilde{\mathfrak{M}}'(\mathcal{X}/B, \tau) \to \prod_{E \in \mathbf{E}} \mathcal{X}, \quad (5.4)$$

with E-component equal to

$$\tilde{f} \circ \tilde{s}_E \quad \text{for } \tilde{f} : \widetilde{\mathfrak{C}}'^{\circ}(\mathcal{X}/B, \tau) \to \mathcal{X}$$

the universal punctured morphism.

[1] We use this notation as it is easy to parse, but note that (E, v) is ambiguous if E is a loop. It will always be clear from the context how to fix this ambiguity with a heavier notation.

[2] Note that we have suppressed the dependence of the stack on $\mathbf{E}$ from the notation.

5.2.3 The stacks $\widetilde{\mathfrak{M}}'(\mathcal{X}/B, \tau_i)$, evaluation and splitting morphisms

Similarly, for each of the global types $\tau_i = (G_i, \mathbf{g}_i, \boldsymbol{\sigma}_i, \bar{\mathbf{u}}_i)$ obtained by splitting and $L \in L(G_i)$, denote by $\mathfrak{M}'_L(\mathcal{X}/B, \tau_i)$ the image of the punctured section s_L : $\underline{\mathfrak{M}}'(\mathcal{X}/B, \tau_i) \to \underline{\mathfrak{C}}'^\circ(\mathcal{X}/B, \tau_i)$ defined by L, again endowed with the pullback of the log structure on $\mathfrak{C}'^\circ(\mathcal{X}/B, \tau_i)$. With $L_1, \ldots, L_s$ the legs of G_i obtained from splitting, define the stack

$$\widetilde{\mathfrak{M}}'(\mathcal{X}/B, \tau_i) = \big(\mathfrak{M}'_{L_1}(\mathcal{X}/B, \tau_i) \times^{\mathrm{f}}_{\mathfrak{M}'(\mathcal{X}/B,\tau_i)} \cdots \times^{\mathrm{f}}_{\mathfrak{M}'(\mathcal{X}/B,\tau_i)} \mathfrak{M}'_{L_s}(\mathcal{X}/B, \tau_i)\big)^{\mathrm{sat}},$$

where sat denotes saturation, bearing in mind that the log structures on the stacks $\mathfrak{M}'_{L_j}(\mathcal{X}/B, \tau_i)$ are not saturated.

This stack differs from $\mathfrak{M}'(\mathcal{X}/B, \tau_i)$ by adding the pullback of the log structure of each puncture obtained from splitting, so that the pullback $\widetilde{\mathfrak{C}}'^\circ(\mathcal{X}/B, \tau_i) \to \widetilde{\mathfrak{M}}'(\mathcal{X}/B, \tau_i)$ of the universal curve now has punctured sections in the category of log stacks. We define the evaluation morphism

$$\mathrm{ev}_{\mathbf{L}} : \prod_{i=1}^{r} \widetilde{\mathfrak{M}}'(\mathcal{X}/B, \tau_i) \to \prod_{E \in \mathbf{E}} \mathcal{X} \times \mathcal{X}, \tag{5.5}$$

by taking as E-component the evaluation at the corresponding two sections $s_{E,v}$, $s_{E,v'}$, observing the chosen orientation of E.

Lemma 5.6. *The splitting morphism $\mathfrak{M}(\mathcal{X}/B, \tau) \to \prod_i \mathfrak{M}(\mathcal{X}/B, \tau_i)$ in Proposition 5.4 lifts to a morphism*

$$\widetilde{\mathfrak{M}}(\mathcal{X}/B, \tau) \to \prod_{i=1}^{r} \widetilde{\mathfrak{M}}(\mathcal{X}/B, \tau_i). \tag{5.6}$$

Analogous statements hold for weak markings and for the moduli spaces of stable maps to X rather than $\mathcal{X}$.

Proof. We only treat the case of marked moduli spaces of punctured maps to $\mathcal{X}$, the other cases being completely analogous.

It suffices to produce a morphism

$$\mathfrak{M}_E(\mathcal{X}/B, \tau) \to \mathfrak{M}_L(\mathcal{X}/B, \tau_i)$$

lifting $\mathfrak{M}(\mathcal{X}/B, \tau) \to \mathfrak{M}(\mathcal{X}/B, \tau_i)$ whenever $L = (E, v) \in L(G_i)$ is one of the two legs obtained from splitting E. Indeed, this then provides a morphism of fibered products, which lifts to the saturation by functoriality of saturation.

To construct this lifting let $\mathfrak{C}^\circ \to \mathfrak{M} := \mathfrak{M}(\mathcal{X}/B, \tau)$ be the universal curve, and $\widetilde{\mathfrak{C}}^\circ \to \mathfrak{C}^\circ$ the splitting of all nodes labeled by an element of $\mathbf{E}$, strict as a morphism of log stacks. The graph G_i given by τ_i selects a connected component $\widetilde{\mathfrak{C}}^\circ_i \subset \widetilde{\mathfrak{C}}^\circ$, and the nodal section s_E lifts to a punctured section $\tilde{s}_i : \underline{\mathfrak{M}} \to \underline{\tilde{\mathfrak{C}}_i^\circ}$. Let similarly $\mathfrak{C}_L^\circ \to \mathfrak{M}_i := \mathfrak{M}(\mathcal{X}/B, \tau_i)$ and $s_L : \underline{\mathfrak{M}}_i \to \underline{\mathfrak{C}}_L$ the corresponding universal curve and punctured section over $\mathfrak{M}_i$. Then $\mathfrak{M}_E(\mathcal{X}/B, \tau) = \underline{\mathfrak{M}} \times_{\underline{\tilde{\mathfrak{C}}_i^\circ}} \tilde{\mathfrak{C}}_i^\circ$ since $\tilde{\mathfrak{C}}_i^\circ \to \mathfrak{C}^\circ$ is strict, and similarly $\mathfrak{M}_L(\mathcal{X}/B, \tau_i) = \underline{\mathfrak{M}}_i \times_{\underline{\mathfrak{C}}_L^\circ} \mathfrak{C}_L^\circ$. Now there is a canonical morphism $\tilde{\mathfrak{C}}_i^\circ \to \mathfrak{C}_L^\circ$ lifting $\mathfrak{M}(\mathcal{X}/B, \tau) \to \mathfrak{M}(\mathcal{X}/B, \tau_i)$—the prestabilization morphism as a punctured map. Pulling back we obtain the desired morphism $\mathfrak{M}_E(\mathcal{X}/B, \tau) \to \mathfrak{M}_L(\mathcal{X}/B, \tau_i)$. ∎

We next show that enlarging the log structures for the punctures may change the structure of the underlying stacks, but only by nilpotents in the structure sheaf.

For $\mathbf{S} \subseteq E(G) \cup L(G)$ we unify the notation, denoting by $\widetilde{\mathfrak{M}}'(\mathcal{X}/B, \tau) \to \mathfrak{M}'(\mathcal{X}/B, \tau)$ the corresponding fiber product over both nodal and punctured sections. In this generality we have:

Proposition 5.7. *Let $\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \bar{\mathbf{u}})$ be a global type of punctured maps, $\mathbf{S} \subseteq E(G) \cup L(G)$ and $\widetilde{\mathfrak{M}}'(\mathcal{X}/B, \tau)$ the corresponding stack of weakly τ-marked punctured maps to $\mathcal{X}/B$ with sections. Then the canonical map*

$$\widetilde{\mathfrak{M}}'(\mathcal{X}/B, \tau) \to \mathfrak{M}'(\mathcal{X}/B, \tau)$$

induces an isomorphism on the reductions of their underlying stacks. If moreover $\mathbf{S} \subseteq E(G)$, the canonical map is an isomorphism on underlying stacks.

Analogous results hold for the marked and decorated versions.

Proof. Going inductively, it suffices to treat the case that $\mathbf{S}$ has only one element. The case $\mathbf{S} = \{E\}$ is an edge leads to the problem of going over from a monoid Q to the saturation of a monoid of the form $Q \oplus_{\mathbb{N}} \mathbb{N}^2$ with $1 \in \mathbb{N}$ mapping to $(1, 1) \in \mathbb{N}^2$. Since the morphism $\mathbb{N} \to \mathbb{N}^2$ is saturated and integral, $Q \oplus_{\mathbb{N}} \mathbb{N}^2$ is saturated and integral as well by [52, Propositions I.4.8.5, I.4.6.3]. In particular, the fs fiber product in (5.3) agrees with the ordinary fiber product, and only changes the log structure.

For $\mathbf{S} = \{L\}$ a leg, we need to take the saturation of the strict subspace given by a punctured section. Let $(C^\circ/W, \mathbf{p}, f)$ be a punctured map to $\mathcal{X}/B$ with $W = \operatorname{Spec}(Q \to A)$. Let $\underline{W} \to \operatorname{Spec} \Bbbk[Q^\circ]$ be a chart for the log structure induced by the punctured section corresponding to the leg L, with $Q^\circ \subset Q \oplus \mathbb{Z}$. Then necessarily the induced map $Q^\circ \to A$ takes $Q^\circ \setminus (Q \oplus 0)$ to zero.

The saturation of $\operatorname{Spec}(Q^\circ \to A)$ equals $W' = \operatorname{Spec}(Q' \to A')$ with Q' the saturation of Q° and $A' = A \otimes_{k[Q^\circ]} k[Q']$. Necessarily, if $m \in Q' \setminus Q^\circ$ then $m \in Q \oplus \mathbb{Z}_{<0}$, and so its image $z^m \in A'$ is nilpotent (following the notation of Section 1.6). It is then immediate that $A \to A'_{\text{red}}$ is surjective. This map factors through $A_{\text{red}} \to A'_{\text{red}}$, so the latter is surjective. Thus W'_{red} is a closed subscheme of W_{red}. On the other hand, by [52, Proposition III.2.1.5] saturation is always a surjective morphism, and hence $W'_{\text{red}} \to W_{\text{red}}$ is an isomorphism. ∎

By Proposition 5.7 the Chow theories of the moduli stacks of punctured maps do not change by enlarging the log structures. We can thus freely use the enlarged log structures in discussing gluing.

We are now in position to state the central technical gluing result. It explains how a τ-marked punctured map is equivalent to giving a collection of τ_i-marked punctured maps obeying a logarithmic matching condition.

Theorem 5.8. *Let $X \to B$ be a morphism of fs logarithmic schemes over $\Bbbk$ fulfilling the assumptions stated at the beginning of Chapter 3, and assume X is simple. Let $\tau_1, \ldots, \tau_r$ be the global types of punctured maps (Definition 2.44) obtained by splitting a global type $\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \bar{\mathbf{u}})$ along a subset of edges $\mathbf{E}$. Then the commutative diagram*

$$\begin{array}{ccc} \widetilde{\mathfrak{M}}'(\mathcal{X}/B,\tau) & \xrightarrow{\delta_{\mathfrak{M}}} & \prod_{i=1}^r \widetilde{\mathfrak{M}}'(\mathcal{X}/B,\tau_i) \\ {\scriptstyle \mathrm{ev}_{\mathbf{E}}}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_{\mathbf{L}}} \\ \prod_{E\in\mathbf{E}} \mathcal{X} & \xrightarrow{\Delta} & \prod_{E\in\mathbf{E}} \mathcal{X}\times\mathcal{X} \end{array}$$

with Δ the product of diagonal embeddings and the other arrows defined in (5.4), (5.5), and (5.6), is cartesian in the category of fs log stacks. We remind the reader that all products in this square are taken over B.

An analogous statement holds for τ replaced by a decorated global type $\boldsymbol{\tau} = (\tau, \mathbf{A})$.

Remark 5.9. We note that it is important that we use the weakly marked moduli spaces here. Indeed, there exist simple examples of (strongly) marked punctured maps which may be glued to obtain a punctured map which is only weakly marked. This arises as saturation issues in the above fiber product description may introduce nilpotents. For an explicit example, see [26, Example 4.5]. We also note that this is essentially the same saturation issue as in Remark 3.5, and the examples are closely related.

The proof of the theorem, given further below, is based on the following gluing result for punctures with a section.

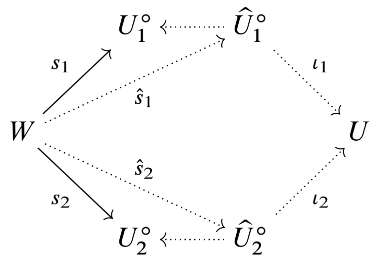
Lemma 5.10. *Let W be an fs log scheme and U_i° a puncturing along $\{0\} \times W$ of strict open neighborhoods $U_1, U_2 \subseteq \mathbb{A}^1 \times W$ of $\{0\} \times W$, $i = 1, 2$. Here $\mathbb{A}^1$ is endowed with its toric log structure. Furthermore, let $s_i : W \to U_i^\circ$ be sections with schematic image $\{0\} \times \underline{W}$ of the composition $U_i^\circ \to U_i \to W$ of the puncturing map and the projection.*

Then there exists an enlarged puncturing $\widehat{U}_i^\circ \to U_i^\circ \to U_i$ through which the sections s_i factor, and a unique log smooth curve $\pi : U \to W$ with maps

$$\iota_1 : \widehat{U}_1^\circ \to U, \quad \iota_2 : \widehat{U}_2^\circ \to U$$

over W inducing an isomorphism of underlying schemes $\underline{U}_1 \amalg_{\{0\}\times\underline{W}} \underline{U}_2 \simeq \underline{U}$, strict away from $\{0\} \times \underline{W}$, and such that $\iota_1 \circ \hat{s}_1 = \iota_2 \circ \hat{s}_2$, with $\hat{s}_i$ the lifts of s_i.

Remark 5.11. The lifting of s_i to $\widehat{U}_i^\circ$ is unique. The enlarged puncturing $\widehat{U}_i^\circ$ is not unique, but may be chosen uniquely if we require that $\widehat{U}_i^\circ \to U_i^\circ \times U$ is prestable. We obtain a pushout diagram up to unique punctured enlargement:



Proof of Lemma 5.10. The statement is about the unique definition of the log structure on $\underline{U}$ near the nodal locus $\{0\} \times \underline{W} \subset \underline{U}$. Since this is a local question we can restrict attention to a neighborhood of a geometric point $\bar{q} = (0, \bar{w})$ of $\{0\} \times \underline{W}$. By the definition of puncturing, the linear coordinate of $\mathbb{A}^1$ defines elements $\sigma_x \in \mathcal{M}_{U_1^\circ,\bar{q}}$, $\sigma_y \in \mathcal{M}_{U_2^\circ,\bar{q}}$.

Now assume that $U = (\underline{U}, \mathcal{M}_U) \to W$ is a log smooth curve with the required properties for some $\widehat{U}_i^\circ$. Since $\widehat{U}_i^\circ$, U_i° are both puncturings of U_i we may identify $\overline{\mathcal{M}}^{\mathrm{gp}}_{\widehat{U}_i^\circ} = \overline{\mathcal{M}}^{\mathrm{gp}}_{U_i^\circ} = \overline{\mathcal{M}}^{\mathrm{gp}}_{U_i}$. Then

$$\bar{\iota}_i^\flat : \overline{\mathcal{M}}^{\mathrm{gp}}_{U,\bar{q}} \to \overline{\mathcal{M}}^{\mathrm{gp}}_{\widehat{U}_i^\circ,\bar{q}} = \overline{\mathcal{M}}^{\mathrm{gp}}_{U_i^\circ,\bar{q}} = \overline{\mathcal{M}}^{\mathrm{gp}}_{W,\bar{w}} \oplus \mathbb{Z}$$

is an isomorphism with $\overline{\mathcal{M}}_{W,\bar{w}} \oplus \mathbb{N} \subseteq \bar{\iota}_i^\flat(\overline{\mathcal{M}}_{U,\bar{q}})$. Thus there exist $\widetilde{\sigma}_x, \widetilde{\sigma}_y \in \mathcal{M}_{U,\bar{q}}$ with

$$\sigma_x = \iota_1^\flat(\widetilde{\sigma}_x), \quad \sigma_y = \iota_2^\flat(\widetilde{\sigma}_y).$$

An important property of log smooth structures at nodes is that logarithmic lifts of given local coordinates at the two branches of the node become unique if one requires their product to lie in $\pi^\flat(\mathcal{M}_{W,\bar{w}})$ [51, Section 3.8]. With this condition imposed on $\widetilde{\sigma}_x, \widetilde{\sigma}_y$, we now obtain a unique element $\sigma_q \in \mathcal{M}_{W,\bar{w}}$ with

$$\widetilde{\sigma}_x \cdot \widetilde{\sigma}_y = \pi^\flat(\sigma_q). \tag{5.7}$$

Under the assumption of the existence of factorizations $\hat{s}_1$, $\hat{s}_2$ of the sections s_1, s_2, we can compute σ_q from σ_x and σ_y as follows: With $\iota_1 \circ \hat{s}_1 = \iota_2 \circ \hat{s}_2$ we obtain

$$\sigma_q = (\iota_1 \circ \hat{s}_1)^\flat(\pi^\flat(\sigma_q)) = (\iota_1 \circ \hat{s}_1)^\flat(\widetilde{\sigma}_x) \cdot (\iota_2 \circ \hat{s}_2)^\flat(\widetilde{\sigma}_y) = s_1^\flat(\sigma_x) \cdot s_2^\flat(\sigma_y).$$

Note also that $\mathcal{M}_{U,\bar{q}}$ is generated by $(\pi^*\mathcal{M}_W)_{\bar{q}}$ and $\widetilde{\sigma}_x, \widetilde{\sigma}_y$, with single relation (5.7).

Conversely, we can define the structure of a log smooth curve at $\bar{q} \to \underline{U}$ with the requested properties simply by defining

$$\sigma_q = s_1^\flat(\sigma_x) \cdot s_2^\flat(\sigma_y), \tag{5.8}$$

and

$$\mathcal{M}_{U,\bar{q}} := (\pi^* \mathcal{M}_W)_{\bar{q}} \oplus_{\mathbb{N}} \mathbb{N}^2,$$

with the generator $1 \in \mathbb{N}$ in the fibered sum mapping to $\pi^\flat(\sigma_q) \in (\pi^* \mathcal{M}_W)_q$ and to $(1,1) \in \mathbb{N}^2$, respectively. The structure morphism

$$\mathcal{M}_{U,\bar{q}} \to \mathcal{O}_{U,\bar{q}}$$

is defined by the structure morphism of W on the first summand, and by mapping $(a,b) \in \mathbb{N}^2$ to $x^a y^b$ when writing $\underline{U} \subseteq W \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{Z}[x,y]/(xy)$. Since the projection $U_i^\circ \setminus (\{0\} \times \underline{W}) \to W$ is strict, this log structure near $\bar{q}$ patches uniquely to the given log structure on $U_i^\circ \setminus (\{0\} \times \underline{W})$ to define the desired log smooth curve $U \to W$.

The morphisms $\iota_i : \widehat{U}_i^\circ \to U$ are then given by

$$\begin{aligned} (\iota_1^\flat)_{\bar{q}} &: \mathcal{M}_{U,\bar{q}} \to \mathcal{M}^{\mathrm{gp}}_{U_1^\circ,\bar{q}}, \quad (1,0) \mapsto \sigma_x, & (0,1) \mapsto \sigma_x^{-1}\pi^\flat(\sigma_q), \\ (\iota_2^\flat)_{\bar{q}} &: \mathcal{M}_{U,\bar{q}} \to \mathcal{M}^{\mathrm{gp}}_{U_2^\circ,\bar{q}}, \quad (1,0) \mapsto \sigma_y^{-1}\pi^\flat(\sigma_q), & (0,1) \mapsto \sigma_y. \end{aligned} \tag{5.9}$$

These definitions are forced upon us by the structure homomorphisms on U_i° and by the defining relation (5.8) for $\mathcal{M}_{U,\bar{q}}$. If $\sigma_x^{-1}\pi^\flat(\sigma_q) \notin \mathcal{M}_{U_1^\circ,q}$, we may have to enlarge the puncturing of U_1° for this map to define $\widehat{U}_1^\circ \to U$, and similarly for $\widehat{U}_2^\circ$; if we choose the enlargement to be generated by $\sigma_x^{-1}\pi^\flat(\sigma_q)$ it is uniquely defined. Note that by (5.8), the image of σ_q under the structure morphism is $xy = 0$, and hence this enlargement of puncturing is possible. Note also that s_1 factors uniquely over this extension of puncturing since by (5.8),

$$(s_1^\flat)^{\mathrm{gp}}\big(\sigma_x^{-1}\pi^\flat(\sigma_q)\big) = s_2^\flat(\sigma_y),$$

and similarly for s_2. Finally, to check the equality $\iota_1 \circ s_1 = \iota_2 \circ s_2$ we compute

$$(s_1^\flat \circ \iota_1^\flat)(1,0) = s_1^\flat(\sigma_x) = s_2^\flat(\sigma_y)^{-1}\sigma_q = s_2^\flat(\sigma_y^{-1}\pi^\flat(\sigma_q)) = (s_2^\flat \circ \iota_2^\flat)(1,0),$$

and similarly for $(1,0)$ replaced by $(0,1)$. This shows the claimed properties for $U \to W$ and ι_1, ι_2. Uniqueness follows from the discussion at the beginning of the proof. ∎

Remark 5.12. It is worthwhile to understand the gluing construction of a pair of punctured points to a node on the level of ghost sheaves and in terms of the dual tropical picture. The relevant monoids are

$$Q = \overline{\mathcal{M}}_{W,\bar{w}}, \quad Q_i = \overline{\mathcal{M}}_{U_i^\circ,\bar{q}} \subset Q \oplus \mathbb{Z},$$

and their duals

$$\omega = \mathrm{Hom}(Q, \mathbb{R}_{\geq 0}), \quad \tau_i = \mathrm{Hom}(Q_i, \mathbb{R}_{\geq 0}) \subset \omega \times \mathbb{R}_{\geq 0}.$$

We choose the embedding $Q_i \subset Q \oplus \mathbb{Z}$ such that $\bar{\pi}^\flat$ identifies Q with $Q \oplus \{0\}$, while the puncturing log structure is generated by $(0, 1) \in Q \oplus \mathbb{Z}$. The sections s_i define left-inverses

$$\bar{s}_i^\flat : Q_i \to Q$$

to $\bar{\pi}^\flat$. Now the point of the gluing construction is that there are exactly two automorphisms ϕ of $Q^{\mathrm{gp}} \oplus \mathbb{Z}$ making the following diagram of monoids commutative:

$$\begin{array}{ccccc}
Q & & \xrightarrow{\mathrm{id}} & & Q \\
\bar{\pi}^\flat \downarrow & & & & \downarrow \bar{\pi}^\flat \\
Q_1 & & & & Q_2 \\
\downarrow & & & & \downarrow \\
Q^{\mathrm{gp}} \oplus \mathbb{Z} & & \xrightarrow{\phi} & & Q^{\mathrm{gp}} \oplus \mathbb{Z} \\
& \bar{s}_1^\flat \searrow & & \swarrow \bar{s}_2^\flat & \\
& & Q^{\mathrm{gp}} & &
\end{array}$$

Indeed, by commutativity of the square, $\phi(m, 0) = (m, 0)$ for all $m \in Q^{\mathrm{gp}}$. Define $\rho_i = \bar{s}_i^\flat(0, 1)$, $i = 1, 2$, and ρ_q by $\phi(0, 1) = (\rho_q, d)$. Then $d = \pm 1$ since $\phi(0, 1)$ together with $Q^{\mathrm{gp}} \oplus \{0\}$ generates $Q^{\mathrm{gp}} \oplus \mathbb{Z}$. This sign determines the two possibilities. Commutativity of the triangle now shows

$$\rho_1 = \bar{s}_1^\flat(0, 1) = \bar{s}_2^\flat(\rho_q, \pm 1) = \rho_q \pm \rho_2.$$

The situation obtained by splitting a node into two punctures produces the negative sign. With this choice we obtain an isomorphism of the submonoid $Q_{12} \subset Q^{\mathrm{gp}} \oplus \mathbb{Z}$ generated by $Q \oplus \mathbb{N}$ and $\phi^{-1}(Q \oplus \mathbb{N})$ with $Q \oplus_{\mathbb{N}} \mathbb{N}^2$, with $1 \in \mathbb{N}$ mapping to ρ_q and $(1, 1)$, respectively. The defining equation $\rho_q = \rho_1 + \rho_2$ retrieves (5.8) in the proof of Lemma 5.10 on the level of ghost sheaves. The change of puncturing of U_1° becomes necessary if $Q_{12} \not\subset Q_1$, and similarly if $\phi(Q_{12}) \not\subset Q_2$ for U_2°. Figure 5.2 provides an illustration.

For the tropical interpretation, illustrated in Figure 5.3, we have two factorizations

$$\omega \xrightarrow{\Sigma(s_i)} \tau_i \xrightarrow{\Sigma(\pi)} \omega,$$

of id_ω. Here the second map is the projection to the first component when writing $\tau_i \subseteq \omega \times \mathbb{R}_{\geq 0}$. Thus $\Sigma(s_i)(h) = (h, \ell_i(h))$ for some piecewise linear map

$$\ell_i : \omega \to \mathbb{R}_{\geq 0}.$$

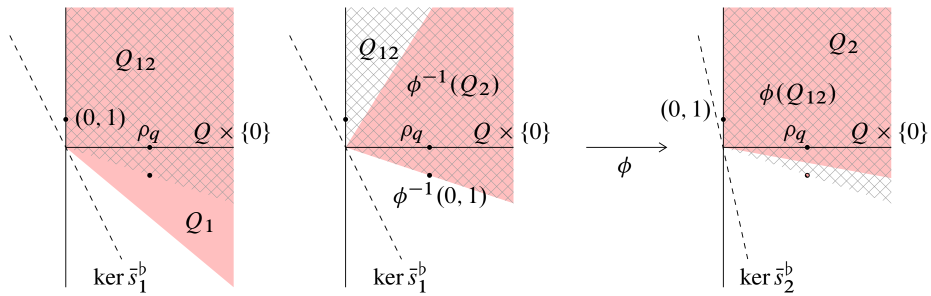


Figure 5.2. The monoids $Q_1, Q_2, Q_{12} \subset Q \oplus \mathbb{Z}$ and their comparison under $\phi : Q \oplus \mathbb{Z} \to Q \oplus \mathbb{Z}$. The hatched area depicts Q_{12}, the solid shading Q_1, Q_2 or $\phi^{-1}(Q_2)$. Note that $\phi(Q_{12}) = Q_{12}$ because both cones are spanned by $Q \times \{0\}$, $(0, 1)$ and $\rho_q - (0, 1)$. In the sketched situation, the puncturing for U_2° has to be enlarged, the one for U_1° does not.

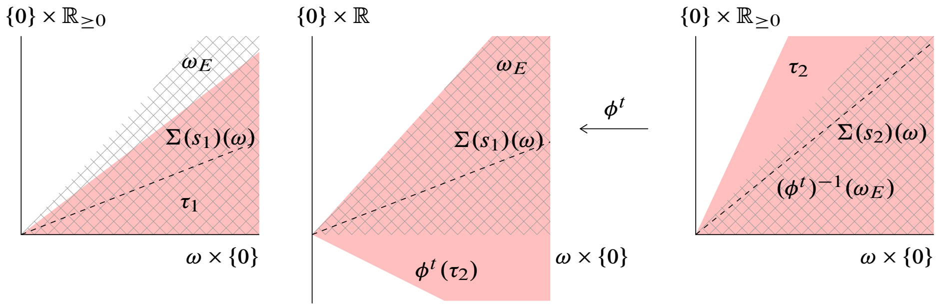


Figure 5.3. The dual tropical picture of Figure 5.2. The hatched area covers ω_E, the solid shading τ_1, τ_2 and $\phi^t(\tau_2)$. The dashed line indicates the image of ω under $\Sigma(s_1)$ or $\Sigma(s_2)$.

Thinking of h as parametrizing a punctured tropical curve, $\ell_i(h)$ specifies a point on the puncturing interval or ray emanating from the unique vertex v_i. The tropical glued curve then produces the metric graph with two vertices v_1, v_2 by joining the two intervals at the specified points, hence producing an edge E of length $\ell_1(h) + \ell_2(h)$. The tropical glued curve over ω thus has edge function $\ell : \omega \to \mathbb{R}_{\geq 0}$ simply defined by

$$\ell = \ell_1 + \ell_2. \tag{5.10}$$

The process of producing the glued cone $\omega_E \subset \omega \times \mathbb{R}_{\geq 0}$ over ω is dual to the statement $Q_{12} = (Q \oplus \mathbb{N}) + \phi^{-1}(Q \oplus \mathbb{N})$:

$$\omega_E = \mathrm{Hom}(Q_{12}, \mathbb{R}_{\geq 0}) = (\omega \times \mathbb{R}_{\geq 0}) \cap \phi^t(\omega \times \mathbb{R}_{\geq 0}).$$

The change of puncturing is necessary if $\ell(h)$ is smaller than either of the length functions obtained by tropicalizing the puncturing, or if either one of $Q_{12} \cap Q_1$, $\phi(Q_{12}) \cap Q_2$ is not saturated.

We now turn to the proof of the gluing theorem for punctured maps to $\mathcal{X}/B$.

Proof of Theorem 5.8. Write $\tau_i = (G_i, \mathbf{g}_i, \sigma_i, \bar{\mathbf{u}}_i)$. We check the universal property of cartesian diagrams.

Step 1: An object of the fibered product. Consider an fs log scheme W with two morphisms

$$W \to \prod_{E \in \mathbf{E}} \mathcal{X}, \quad W \to \prod_{i=1}^{r} \widetilde{\mathfrak{M}}'(\mathcal{X}/B, \tau_i) \tag{5.11}$$

together with an isomorphism of the compositions to $\prod_E \mathcal{X} \times \mathcal{X}$. Spelled out this means that (1) for each $i = 1, \ldots, r$ we have given a weakly τ_i-marked, pre-stable punctured map

$$(\pi_i : C_i^\circ \to W, \mathbf{p}_i, f_i : C_i^\circ \to \mathcal{X})$$

over W and for each leg $(E, v) \in L(G_i)$ a section $s_{E,v} : W \to C_i^\circ$ with image the puncture labeled by the leg in G_i generated by E; and (2) the sections fullfill the logarithmic matching property

$$f_{i(v)} \circ s_{E,v} = f_{i(v')} \circ s_{E,v'}, \tag{5.12}$$

for each edge $E \in \mathbf{E}$ with adjacent vertices v, v'. Write $p_{E,v}$ for the strict closed subspace of C_i° defined by $(E, v) \in L(G_i)$.

Step 2. The glued curve. Denote by $\underline{C}$ the family of nodal curves over $\underline{W}$ obtained by gluing $\coprod_i \underline{C}_i$ schematically along pairs of punctures. Let $E \in \mathbf{E}$ be an edge with vertices v, v', and q_E the nodal section of $\underline{C} \to \underline{W}$ given by the image of the pair of punctures $p_{E,v}$, $p_{E,v'}$. Applying Lemma 5.10 étale locally near the image of q_E provides a local extension of the log structure defined by the C_i° away from q_E to a log smooth curve over W. Thus there is a punctured curve

$$(\pi : C^\circ \to W, \mathbf{p})$$

with underlying scheme $\underline{C}$ that replaces each pair of punctures $p_{E,v}$, $p_{E,v'}$ in $\coprod_i C_i^\circ$, for an edge $E \in \mathbf{E}$ with vertices v, v', by a node q_E. The lemma also provides a morphism of punctured curves $\widehat{C}_i^\circ \to C_i^\circ$ with unique liftings $\hat{s}_{E,v}$ of each section $s_{E,v}$ to $\widehat{C}_{i(v)}^\circ$, and morphisms

$$\iota_i : \widehat{C}_i^\circ \to C^\circ$$

with $\iota_{i(v)} \circ \hat{s}_{E,v} = \iota_{i(v')} \circ \hat{s}_{E,v'}$, and $\widehat{C}_i^\circ$ equal to C_i° possibly up to enlargement of the puncturing. For each edge $E \in \mathbf{E}$ we can thus define the nodal section

$$s_E := \iota_{i(v)} \circ \hat{s}_{E,v} = \iota_{i(v')} \circ \hat{s}_{E,v'} : W \to C^\circ.$$

Step 3. Gluing the tropical map. Denote by $(\widehat{C}_i^\circ, \hat{\mathbf{p}}_i, \hat{f}_i)$ with $\hat{f}_i = f_i \circ (\widehat{C}_i^\circ \to C_i^\circ)$ the punctured stable map with the enlarged punctured structure. It follows from the tropical description of the gluing construction in Remark 5.12 that the tropicalizations

$$\Sigma(\hat{f}_i) : \Sigma(\widehat{C}_i^\circ) \to \Sigma(\mathcal{X})$$

of $\hat{f}_i$ glue to a map of generalized cone complexes

$$\Sigma(C^\circ) \to \Sigma(\mathcal{X})$$

which commutes with the map to $\Sigma(B)$. In fact, restricting to a geometric point $\bar{w}$ of $\underline{W}$ and adopting the notation from Remark 5.12, at an edge $E \in \mathbf{E}$ with vertices v_1, v_2, the cone $\omega_E \subseteq \omega \times \mathbb{R}_{\geq 0}$ of $\Sigma(C_{\bar{w}}^\circ)$ is defined by the length function $\ell_E = \ell_1 + \ell_2$. Denote further $\sigma = (\overline{\mathcal{M}}_{\mathcal{X},\bar{y}}^\vee)_\mathbb{R}$ for

$$\bar{y} = f_{i(v_1)}(s_{E,v_1}(\bar{w})) = f_{i(v_2)}(s_{E,v_2}(\bar{w})).$$

Assuming E oriented from v_1 to v_2, the contact orders obtained from splitting τ at E are related by

$$u_{E,v_1} = u_E = -u_{E,v_2} \in \sigma_\mathbb{Z}^{\mathrm{gp}}.$$

Now the map $\omega_E \to \sigma \in \Sigma(\mathcal{X})$ can be defined by

$$\omega_E \ni (h,\lambda) \mapsto V_1(h) + \lambda \cdot u_{E,v_1} = V_2(h) + (\ell_E(h) - \lambda) \cdot u_{E,v_2}, \qquad (5.13)$$

where $V_\mu : \omega \to \sigma$ is the map for the vertex v_μ given by $\Sigma(f_{i(v_\mu)})$, $\mu = 1, 2$. The image of this map lies in σ since the line segment $\{h\} \times [0, \ell_1(h)]$ is contained in $\tau_1 \subseteq \omega \times \mathbb{R}_{\geq 0}$ and $V_1(h) + \lambda \cdot u_{E,v_1} = \Sigma(f_{i(v_1)})(h,\lambda)$, and similarly for the line segment $\{h\} \times [\ell_1(h), \ell(h)]$ and $\Sigma(f_{i(v_2)})$. The equality in (5.13) holds because

$$\begin{aligned} V_1(h) + \ell_1(h) \cdot u_{E,v_1} &= \Sigma(f_{i(v_1)} \circ s_{E,v_1})(h) \\ &= \Sigma(f_{i(v_2)} \circ s_{E,v_2})(h) = V_2(h) + \ell_2(h) \cdot u_{E,v_2}. \end{aligned}$$

Note this last argument uses the assumption that $\bar{u}_E$ is monodromy-free to assure that $u_{E,v_1} = -u_{E,v_2}$. This finishes the construction of the map $\Sigma(C^\circ) \to \Sigma(\mathcal{X})$.

Step 4. Gluing the punctured map. In view of [3, Proposition 2.10][3], we thus obtain a morphism $C^\circ \to \mathcal{A}_X$ over $\mathcal{A}_B$. By the same token, the composition $C^\circ \to \mathcal{A}_X \to \mathcal{A}_B$ agrees with $C^\circ \to B \to \mathcal{A}_B$. We thus obtain an induced morphism

$$f : C^\circ \to \mathcal{X} = B \times_{\mathcal{A}_B} \mathcal{A}_X,$$

[3]While [3, Proposition 2.10] assumes a more restricted context, the proof only uses that the Artin fan of the codomain is Zariski (Definition A.7). This is true here by simplicity of X and our standing assumptions on B.

commuting with the maps to B. By functoriality of this construction and the tropical description of the gluing process, it holds $f \circ \iota_i = \hat{f}_i$ for all i.

The data $(C^\circ \to W, f, \mathbf{p})$ and the collection of nodal sections s_E now define the desired morphism

$$W \to \widetilde{\mathfrak{M}}'(\mathcal{X}/B, \tau).$$

Indeed, splitting the domain C° at the nodes for edges $E \in \mathbf{E}$ and pre-stabilizing obviously retrieves the collection of pre-stable maps $(C_i^\circ \to W, \mathbf{p}_i, f_i)$ with compatible evaluation maps to $\mathcal{X}$ and sections $s_{E,v}$ that we started with. This finishes the existence part in checking cartesianity.

Uniqueness follows from the uniqueness statement in Lemma 5.10. ■

5.2.4 Relative and absolute maps

We end this section by remarking that in many situations, working with all fiber products over B may be burdensome, as each product in the diagram of Theorem 5.8 is over B. In the standard degeneration situation considered in Section 5.4 below, we might be working over a standard log point b_0, and saturation issues even over b_0 can complicate the fiber product. Thus the following is generally useful.

Proposition 5.13. *Let B be an affine log scheme equipped with a global chart $P \to \mathcal{M}_B$ inducing an isomorphism $P \cong \Gamma(B, \overline{\mathcal{M}}_B)$. Let τ be a global type of punctured tropical map for X/B (Definition 2.44 (1)), with underlying graph connected.*[4] *Then there are isomorphisms $\mathfrak{M}(\mathcal{X}/B, \tau) \cong \mathfrak{M}(\mathcal{X}/\operatorname{Spec} \Bbbk, \tau)$ and $\mathcal{M}(X/B, \tau) \cong \mathcal{M}(X/\operatorname{Spec} \Bbbk, \tau)$.*

Proof. We show the first isomorphism, the second being similar. There is a canonical forgetful morphism $\mathfrak{M}(\mathcal{X}/B, \tau) \to \mathfrak{M}(\mathcal{X}/\operatorname{Spec} \Bbbk, \tau)$, and we need to show it is an isomorphism. For this purpose, it is enough to demonstrate that given a punctured map $f : C^\circ/W \to \mathcal{X}$, there is a unique morphism $h : W \to B$ which fits into a commutative diagram

$$\begin{array}{ccc} C^\circ & \xrightarrow{f} & \mathcal{X} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle g} \\ W & \xrightarrow[h]{} & B \end{array}$$

First, to define the underlying $\underline{h} : \underline{W} \to \underline{B}$ it is sufficient to define $\underline{h}^\# : \Gamma(B, \mathcal{O}_B) \to \Gamma(W, \mathcal{O}_W) \cong \Gamma(W, \pi_* \mathcal{O}_C)$, the latter isomorphism from the fact that π is flat, proper with connected and reduced fibers and [67, Lemma 0E0S]. We take this map to coincide with $(g \circ f)^\# : \Gamma(B, \mathcal{O}_B) \to \Gamma(C, \mathcal{O}_C)$.

[4] Connectedness is generally assumed in this paper, although usually not necessary, but here the result is not true without it.

We next enhance $\underline{h}$ to a log morphism, first by describing the map at the level of ghost sheaves, or equivalently, at the tropical level. Fix $\bar{w}$ a geometric point of $\underline{W}$, and let $\tau' = (G', \mathbf{g}', \boldsymbol{\sigma}', \mathbf{u}')$ be the type of $C_{\bar{w}} \to \mathcal{X}$, so that in particular there is a contraction morphism $\tau' \to \tau$. Since τ' and τ have the same set of legs with the same contact orders, the fact that τ is defined over B implies that the composed map $\Sigma(g \circ f) : \Sigma(C_{\bar{w}}) \to \Sigma(B) = P^{\vee}_{\mathbb{R}}$ contracts all legs. However, $g \circ f$ is a punctured map with underlying schematic map constant, and thus by Proposition 2.27, the restriction of $\Sigma(g \circ f)$ to any fiber of $\Sigma(\pi)$ is a balanced tropical map. Since all legs are contracted, the image of this tropical map is compact. Hence, there must be a hyperplane H in the vector space $P^*_{\mathbb{R}}$ containing the image of a vertex of this map and with the entire image contained in a half-space bounded by H. By balancing, this is impossible unless the tropical map is constant. Hence the desired diagram exists at the tropical level. This shows that the map $P = \Gamma(B, \overline{\mathcal{M}}_B) \to \Gamma(C^{\circ}_{\bar{w}}, \overline{\mathcal{M}}_{C^{\circ}_{\bar{w}}})$ factors uniquely over $\overline{\mathcal{M}}_{W,\bar{w}}$. In particular, we obtain a map $\bar{h}^{\flat} : \Gamma(B, \overline{\mathcal{M}}_B) \to \Gamma(W, \overline{\mathcal{M}}_W)$.

Finally, there is a unique lifting of $\bar{h}^{\flat}$ to $h^{\flat} : \Gamma(B, \mathcal{M}_B) \to \Gamma(W, \mathcal{M}_W)$. Indeed, let $s \in \Gamma(B, \mathcal{M}_B)$ be a section which maps to $\bar{s} \in \Gamma(B, \overline{\mathcal{M}}_B)$. Then because the desired diagram exists at the level of ghost sheaves, $(\bar{f}^{\flat} \circ \bar{g}^{\flat})(\bar{s}) = \bar{\pi}^{\flat}(\bar{t})$ for some $\bar{t} \in \Gamma(W, \overline{\mathcal{M}}_W)$. Thus étale locally on W, we may choose a lift $t \in \mathcal{M}_W$ of $\bar{t}$, and write $(f^{\flat} \circ g^{\flat})(s) = \psi \cdot \pi^{\flat}(t)$ for some $\psi \in \Gamma(\mathcal{O}^{\times}_C)$. However, again by properness of π and connectivity and reducedness of the fibers of π, $\psi = \pi^{\#}(\psi')$ for some invertible function ψ' on W, and we may define $h^{\flat}(s) = \psi' \cdot t$. Because this choice of $h^{\flat}(s)$ is determined uniquely by $(g \circ f)^{\flat}$, this local description patches to give a section $h^{\flat}(s) \in \Gamma(W, \mathcal{M}_W)$, making the diagram commute.

We have thus defined a functor $\mathfrak{M}(\mathcal{X}/\operatorname{Spec}\Bbbk, \tau) \to \mathfrak{M}(\mathcal{X}/B, \tau)$ at the level of objects. By the uniqueness of the construction of the morphism $W \to B$ given $f : C^{\circ}/W \to \mathcal{X}$ above, a morphism in the category $\mathfrak{M}(\mathcal{X}/\operatorname{Spec}\Bbbk, \tau)$ defines a morphism in the category $\mathfrak{M}(\mathcal{X}/B, \tau)$, hence completely defining the functor. This defines the desired morphism $\mathfrak{M}(\mathcal{X}/\operatorname{Spec}\Bbbk, \tau) \to \mathfrak{M}(\mathcal{X}/B, \tau)$ which is inverse to the forgetful morphism $\mathfrak{M}(\mathcal{X}/B, \tau) \to \mathfrak{M}(\mathcal{X}/\operatorname{Spec}\Bbbk, \tau)$. ∎

We note that the two moduli problems, with isomorphic moduli spaces $\mathcal{M}(X/B, \tau)$ and $\mathcal{M}(X/\operatorname{Spec}\Bbbk, \tau)$, have different obstruction theories.

5.3 Evaluation stacks and gluing at the virtual level

While Theorem 5.8 transparently describes the process of gluing a collection of punctured maps at pairs of punctures with matching contact orders, it lacks two crucial properties needed for applications in punctured Gromov–Witten theory. First, since the diagonal map $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is not proper except in trivial cases and neither is the splitting map $\delta_{\mathfrak{M}}$, it is impossible to push forward cycles via $\delta_{\mathfrak{M}}$ for the purpose of splitting computations according to the splitting of τ along the chosen set of edges $\mathbf{E} \subseteq E(G)$. And second, the obvious commutative square lifting the splitting map $\delta_{\mathfrak{M}}$

to a map $\mathcal{M}'(X/B,\tau) \to \prod_i \mathcal{M}'(X/B,\tau_i)$ is far from being cartesian even on the underlying stacks of (pre-) stable maps since it imposes matching at the nodes only on $\mathcal{X}$ rather than on X. (We remind the reader that the products such as $\prod_i$ are all over the base log scheme B in this discussion.) Hence this approach has no hope to be compatible with the virtual formalism.

Both problems are solved by enriching the stacks $\mathfrak{M}(\mathcal{X}/B,\tau)$ and $\mathfrak{M}(\mathcal{X}/B,\tau_i)$ of punctured maps to the relative Artin stack $\mathcal{X}/B$ and their various cousins $\mathfrak{M}'(\mathcal{X}/B,\tau)$, $\widetilde{\mathfrak{M}}'(\mathcal{X}/B,\tau)$ etc., by providing a lift of the underlying evaluations to X. Note that such enriched stacks of maps to $\mathcal{X}$ have already been considered at the beginning of Section 4.2 in the context of obstruction theories with imposed point conditions.

For this discussion we mostly work with the stacks $\mathfrak{M}(\mathcal{X}/B,\tau)$ of marked maps (Definition 3.8), except in the analogue Corollary 5.15 of Theorem 5.8, which requires stacks $\widetilde{\mathfrak{M}}'(\mathcal{X}/B,\tau)$ with weak markings and sections (Section 5.2.2). All other results also hold in the weakly marked and decorated contexts.

We continue to assume that $X \to B$ is a morphism of fs logarithmic schemes over $\Bbbk$ fulfilling the assumptions stated at the beginning of Chapter 3.

Definition 5.14. Let $\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \bar{\mathbf{u}})$ be a global type of punctured maps to X and $\mathbf{S} \subseteq E(G) \cup L(G)$ a subset of edges and legs. The *evaluation stack* of $\mathfrak{M}(\mathcal{X}/B,\tau)$ with respect to $\mathbf{S}$ is the fiber product

$$\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\tau) = \mathfrak{M}(\mathcal{X}/B,\tau) \times_{\prod_{S\in\mathbf{S}} \underline{\mathcal{X}}} \prod_{S\in\mathbf{S}} \underline{X}$$

of $\prod_{S\in\mathbf{S}} \underline{X} \to \prod_{S\in\mathbf{S}} \underline{\mathcal{X}}$ with the evaluation map

$$\underline{\mathrm{ev}}_{\mathbf{S}} : \mathfrak{M}(\mathcal{X}/B,\tau) \to \prod_{S\in\mathbf{S}} \underline{\mathcal{X}}, \quad (C^\circ/W, \mathbf{p}, f) \mapsto (\underline{f} \circ s_S)_{S\in\mathbf{S}},$$

evaluating at the punctured and nodal sections $s_S : \underline{W} \to \underline{C}^\circ$ for $S \in \mathbf{S}$.

Analogous definitions apply in the weakly marked and decorated contexts as in Definition 3.8, or for the stacks $\widetilde{\mathfrak{M}}'(\mathcal{X}/B,\tau)$ of Section 5.2.2.

Note that $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\tau)$ of course depends on the logarithmic scheme X, but we suppress this in the notation as $\mathcal{X}$ always denotes its relative Artin fan. We also suppress $\mathbf{S}$ in the notation of the evaluation stacks and rather specify this subset whenever not clear from the context.

As indicated in the definition, we endow $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\tau)$ with the log structure making the projection to $\mathfrak{M}(\mathcal{X}/B,\tau)$ strict, to obtain the sequence of morphisms of log stacks

$$\mathcal{M}(X/B,\tau) \xrightarrow{\varepsilon} \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\tau) \to \mathfrak{M}(\mathcal{X}/B,\tau)$$

as in (4.14). Recall that the obstruction theory for this sequence of morphisms has been worked out in Section 4.2. It was noted that, as the morphisms are strict, this coincides with the obstruction theory for the underlying stacks. We further saw that the obstruction theory of $\underline{\mathcal{M}}(X/B, \tau)$ over $\underline{\mathfrak{M}}(\mathcal{X}/B, \tau)$ is the composition of an obstruction theory for ε with the trivial obstruction theory in pure degree 0 of the smooth morphism $\underline{\mathfrak{M}^{\mathrm{ev}}}(\mathcal{X}/B, \tau) \to \underline{\mathfrak{M}}(\mathcal{X}/B, \tau)$ of relative dimension $(\dim X - \dim B) \cdot |\mathbf{S}|$.

We now adopt the setup of Section 5.2 and split τ at a subset $\mathbf{E} \subseteq E(G)$ of edges with $\mathbf{E} \subseteq \mathbf{S}$ to obtain global types $\tau_i = (G_i, \mathbf{g}_i, \boldsymbol{\sigma}_i, \bar{\mathbf{u}}_i)$. For the following corollary of Theorem 5.8 for evaluation stacks, we write $\widetilde{\mathfrak{M}}^{\mathrm{ev}}(\mathcal{X}/B, \tau)$ for the evaluation stack of $\widetilde{\mathfrak{M}}(\mathcal{X}/B, \tau)$ with evaluations at all nodes specified by $\mathbf{E}$, thus by Proposition 5.7 having the same underlying stack as $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B, \tau)$, but with the enlarged log structure admitting a logarithmic evaluation map analogous to (5.4). Similarly, we obtain evaluation stack analogues of the evaluation morphism for the τ_i (5.5), still denoted $\mathrm{ev}_{\mathbf{L}}$, and the splitting morphism (5.6), now denoted δ^{ev}.

Corollary 5.15. *In the situation of Theorem 5.8, the commutative diagram*

$$\begin{array}{ccc} \widetilde{\mathfrak{M}}'^{\mathrm{ev}}(\mathcal{X}/B, \tau) & \xrightarrow{\ \delta^{\mathrm{ev}}\ } & \prod_{i=1}^{r} \widetilde{\mathfrak{M}}'^{\mathrm{ev}}(\mathcal{X}/B, \tau_i) \\ \Big\downarrow{\scriptstyle \mathrm{ev}_{\mathbf{E}}} & & \Big\downarrow{\scriptstyle \mathrm{ev}_{\mathbf{L}}} \\ \prod_{E \in \mathbf{E}} X & \xrightarrow{\ \Delta\ } & \prod_{E \in \mathbf{E}} X \times X \end{array}$$

with arrows defined by the above adaptations to the evaluation stacks for $\mathbf{S} \subseteq E(G) \cup L(G)$ *with* $\mathbf{E} \subseteq \mathbf{S}$*, is cartesian in the category of fs log stacks.*

In particular, the splitting morphism δ^{ev} *is finite and representable.*

Proof. The stated commutative square is the front face of the commutative box

$$\begin{array}{ccccc} \widetilde{\mathfrak{M}}'^{\mathrm{ev}}(\mathcal{X}/B, \tau) & & \xrightarrow{\qquad\qquad \delta^{\mathrm{ev}} \qquad\qquad} & & \prod_{i=1}^{r} \widetilde{\mathfrak{M}}'^{\mathrm{ev}}(\mathcal{X}/B, \tau_i) \\ & \searrow & & \swarrow & \\ \Big\downarrow{\scriptstyle \mathrm{ev}_{\mathbf{E}}} & \widetilde{\mathfrak{M}}'(\mathcal{X}/B, \tau) & \dashrightarrow & \prod_{i=1}^{r} \widetilde{\mathfrak{M}}'(\mathcal{X}/B, \tau_i) & \Big\downarrow{\scriptstyle \mathrm{ev}_{\mathbf{L}}} \\ & \vdots & & \vdots & \\ & \prod_{E \in \mathbf{E}} \mathcal{X} & \dashrightarrow & \prod_{E \in \mathbf{E}} \mathcal{X} \times \mathcal{X} & \\ & \nearrow & & \nwarrow & \\ \prod_{E \in \mathbf{E}} X & & \xrightarrow{\qquad\qquad \Delta \qquad\qquad} & & \prod_{E \in \mathbf{E}} X \times X \end{array}$$

with back face the cartesian square from Theorem 5.8 and the sides cartesian squares defining the evaluation stacks. Hence the stated diagram is cartesian.

The claimed properties of the splitting morphism δ^{ev} follow since an fs fiber product is the saturation and integralization of the ordinary fiber product. ∎

Remark 5.16. For systematic reasons we work in the category of log schemes over B in this section, and thus all products in the statement of Corollary 5.15 are fiber products over B. For explicit computations this leads to fibered sums of lattices, which sometimes require an extra treatment of multiplicities due to saturation issues. This additional step can be avoided by observing that the statement of Corollary 5.15 holds unchanged when interpreting the products as absolute products rather than as products over B, but still with $\mathcal{X}$ the relative Artin fan of X/B.

This statement is not a formal consequence of general properties of fiber products, but is due to the connectedness of the graph G given by τ, as in the argument in the proof of Proposition 5.13. To explain this let $\prod_B$ denote the relative fiber product and $\prod$ the absolute one. To check the universal property of the commutative square in Corollary 5.15 with absolute products, let be given a morphism $W \to \prod_i \widetilde{\mathfrak{M}}'^{\mathrm{ev}}(\mathcal{X}/B, \tau_i)$, $i = 1, \ldots, r$, such that the composition with $\mathrm{ev}_{\mathbf{L}}$ factors over Δ. For each leg $L = (E, v) \in L(G_i)$ we obtain an evaluation map $f_i \circ p_L : W \to \mathcal{X}$, and by composing with $\mathcal{X} \to B$ a map $b_L : W \to B$. This map is independent of the choice of $L \in L(G_i)$ since the i-th component of $W \to \prod_i \widetilde{\mathfrak{M}}'^{\mathrm{ev}}(\mathcal{X}/B, \tau_i)$ defines a punctured map over B, but a priori may vary with i. Now the factorization of ev_L over Δ implies that if the i-th and j-th vertex of G are connnected by an edge then the maps $W \to B$ obtained for i and j coincide. Since G is connected we conclude that all these maps agree. Hence the map $W \to \prod_i \widetilde{\mathfrak{M}}'^{\mathrm{ev}}(\mathcal{X}/B, \tau_i)$ factors over $(\prod_B)_i \widetilde{\mathfrak{M}}'^{\mathrm{ev}}(\mathcal{X}/B, \tau_i)$, and in turn the composition with $\mathrm{ev}_{\mathbf{L}}$ factors over $(\prod_B)_E X$. We are then in position to apply Corollary 5.15 in the stated form to obtain the unique lift to $\widetilde{\mathfrak{M}}'^{\mathrm{ev}}(\mathcal{X}/B, \tau)$.

By the corollary, we obtain a proper push-forward homomorphism in Chow theory for algebraic stacks, as defined by Kresch [45], for the evaluation stacks:

$$\delta_*^{\mathrm{ev}} : A_*(\underline{\mathfrak{M}}^{\mathrm{ev}}(\mathcal{X}/B, \tau)) \to A_*\Big(\prod_i \underline{\mathfrak{M}}^{\mathrm{ev}}(\mathcal{X}/B, \tau_i)\Big), \quad \alpha \mapsto \delta_*(\alpha). \tag{5.14}$$

Note that we can work with markings or weak markings here because the corresponding stacks have the same reductions (Proposition 3.33).

It remains to relate δ^{ev} with the splitting morphism for moduli spaces of punctured maps to X rather than $\mathcal{X}$ and to show compatibility with the obstruction theory. Note that these results use the unenhanced, basic log structures on the moduli stacks.

Proposition 5.17. *Let $X \to B$ be a morphism of fs logarithmic schemes over $\Bbbk$ fulfilling the assumptions stated at the beginning of Chapter 3. Let $\tau_1, \ldots, \tau_r$ be the global types of punctured maps (Definition 2.44) obtained by splitting a global type $\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \bar{\mathbf{u}})$ along a subset of edges $\mathbf{E}$. Then there is a cartesian diagram*

$$\begin{array}{ccc} \mathcal{M}(X/B, \tau) & \xrightarrow{\delta} & \prod_{i=1}^r \mathcal{M}(X/B, \tau_i) \\ \big\downarrow{\scriptstyle \varepsilon} & & \big\downarrow{\scriptstyle \widehat{\varepsilon} = \prod_i \varepsilon_i} \\ \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B, \tau) & \xrightarrow{\delta^{\mathrm{ev}}} & \prod_{i=1}^r \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B, \tau_i) \end{array} \tag{5.15}$$

with horizontal arrows the splitting maps from Proposition 5.4, finite and representable by Corollary 5.15, and the vertical arrows the canonical strict morphisms. Here we assume the set of edges and legs $\mathbf{S} \subseteq E(G) \cup L(G)$ used in the definition of the evaluation stacks (Definition 5.14) contains the set $\mathbf{E} \subseteq E(G)$ of splitting edges.

Analogous statements hold for decorated and for weakly marked versions of the moduli stacks (Definition 3.8).

Proof. We argue by spelling out the definitions of the various stacks. Indeed, a pair of morphisms from an fs log scheme W to $\prod_{i=1}^r \mathcal{M}(X/B, \tau_i)$ and to $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B, \tau)$ together with an isomorphism of their images in $\prod_{i=1}^r \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B, \tau_i)$ is equivalent to (1) an ordinary stable map $(\underline{C}/\underline{W}, \underline{p}, \underline{f})$ to $\underline{X}$ marked by the genus-decorated graph $(G, \mathbf{g})$ given by τ, and (2) a punctured map $(C^\circ/W, \mathbf{p}, f_{\mathcal{X}})$ to $\mathcal{X}$ producing the morphism $W \to \prod_{i=1}^r \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B, \tau_i)$ by splitting at the nodes labeled by $\mathbf{E} \subseteq E(G)$. Note that (1) is obtained by the schematic matching condition at the paired marked points provided by the evaluation stacks. Since $X \to \mathcal{X}$ is strict, $\underline{f}$ and $f_{\mathcal{X}}$ together are the same as a log morphism $f : C^\circ \to X$. Moreover, a marking by τ is equivalent to markings by τ_i of the punctured maps $(C_i^\circ/W, \mathbf{p}_i, f_i)$ obtained by splitting. The correspondence is also easily seen to be functorial. Thus the fiber categories over W of the cartesian product and of $\mathcal{M}(X/B, \tau)$ are equivalent. ■

5.3.1 Notation for obstruction theories

To bring in the perfect obstruction theories discussed in Chapter 4, we now in addition to $\tau, \tau_i, \mathbf{E}, \mathbf{S}$ as in Proposition 5.17 assume $X \to B$ to be log smooth. To analyze the obstruction theories in (5.15), we introduce the following short-hand notation:[5]

$$\begin{aligned}
&\mathcal{M}_{\mathrm{gl}} := \mathcal{M}(X/B, \tau), &&\mathcal{M}_i := \mathcal{M}(X/B, \tau_i), &&\mathcal{M}_{\mathrm{spl}} := \prod_{i=1}^r \mathcal{M}_i \\
&\mathfrak{M}_{\mathrm{gl}} := \mathfrak{M}(\mathcal{X}/B, \tau), &&\mathfrak{M}_i := \mathfrak{M}(\mathcal{X}/B, \tau_i), &&\mathfrak{M}_{\mathrm{spl}} := \prod_{i=1}^r \mathfrak{M}_i \qquad (5.16)\\
&\mathfrak{M}^{\mathrm{ev}}_{\mathrm{gl}} := \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B, \tau), &&\mathfrak{M}^{\mathrm{ev}}_i := \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B, \tau_i), &&\mathfrak{M}^{\mathrm{ev}}_{\mathrm{spl}} := \prod_{i=1}^r \mathfrak{M}^{\mathrm{ev}}_i
\end{aligned}$$

Denote further by $\overline{C}_i^\circ \to \mathcal{M}_i$ and by $C^\circ \to \mathcal{M}_{\mathrm{gl}}$ the universal curves over $\mathcal{M}_i$ and $\mathcal{M}_{\mathrm{gl}}$, respectively, by $C_i^\circ \to \mathcal{M}_{\mathrm{spl}}$ the pullback of $\overline{C}_i^\circ$ under the projection from the product $\mathcal{M}_{\mathrm{spl}} \to \mathcal{M}_i$, and write $\pi_{\mathrm{spl}} : C^\circ_{\mathrm{spl}} = \coprod_i C_i^\circ \to \mathcal{M}_{\mathrm{spl}}$. We also have universal morphisms $f : C^\circ \to X$, $f_{\mathrm{spl}} : C^\circ_{\mathrm{spl}} \to X$, and the subspaces of special points to be considered

[5] For the sake of being specific we work with the marked versions here. Analogous results also hold for the weakly marked cases.

$\iota : Z \to C^\circ$, $\iota_{\mathrm{spl}} : Z_{\mathrm{spl}} \to C^\circ_{\mathrm{spl}}$ with projections $p = \pi \circ \iota$ and $p_{\mathrm{spl}} = \pi_{\mathrm{spl}} \circ \iota_{\mathrm{spl}}$ to $\mathcal{M}_{\mathrm{gl}}$ and $\mathcal{M}_{\mathrm{spl}}$, respectively. Here Z is the union of the images of the punctured and nodal sections labeled by $\mathbf{S} \subseteq E(G) \cup L(G)$, while Z_{spl} is the union of punctured and nodal sections given by $\mathbf{S}_i \subseteq E(G_i) \cup L(G_i)$, $i = 1, \dots, r$, obtained from $\mathbf{S}$ by splitting, both endowed with the induced log structures making ι, ι_{spl} strict.

5.3.2 The fundamental diagram

We consider the following commutative diagram:

$$
\begin{array}{ccccc}
\widetilde{C}^\circ & \longrightarrow & C^\circ_{\mathrm{spl}} = \coprod_i C^\circ_i & \xrightarrow{f_{\mathrm{spl}}} & X \\
\downarrow{\scriptstyle\kappa} & & \Big| & & \\
C^\circ & & \Big| & \xrightarrow{f} & X \\
\downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi_{\mathrm{spl}}} & & \\
\mathcal{M}_{\mathrm{gl}} & \xrightarrow{\delta} & \mathcal{M}_{\mathrm{spl}} = \prod_i \mathcal{M}_i & & \\
\downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\widehat{\varepsilon}=\prod_i \varepsilon_i} & & \\
\mathfrak{M}^{\mathrm{ev}}_{\mathrm{gl}} & \xrightarrow{\delta^{\mathrm{ev}}} & \mathfrak{M}^{\mathrm{ev}}_{\mathrm{spl}} = \prod_i \mathfrak{M}^{\mathrm{ev}}_i . & &
\end{array}
\qquad \tilde{f} : \widetilde{C}^\circ \to X, \quad \tilde{\pi} : \widetilde{C}^\circ \to \mathcal{M}_{\mathrm{gl}}
\tag{5.17}
$$

The lower square is the cartesian square from Proposition 5.17 with strict vertical arrows.

The strict map $\kappa : \widetilde{C}^\circ \to C^\circ$ is the map induced by splitting the nodal sections of $C^\circ \to \mathcal{M}_{\mathrm{gl}}$ given by $\mathbf{E} \subseteq \mathbf{S}$ according to Proposition 5.4. The underlying morphism $\underline{\kappa}$ of ordinary stacks is therefore the corresponding partial normalization from Definition 5.1.

The upper square thus identifies the pullback of C°_{spl} with the pre-stabilization of $\widetilde{C}^\circ$ (Definition 2.6). This part of the diagram is a pullback of nodal curves, cartesian only in the category of stacks, because of the pre-stabilization.

The morphism $\tilde{f}$ is as defined by the diagram. There is also the closed substack $\widetilde{Z} = \kappa^{-1}(Z) \to \widetilde{C}^\circ$ of special points on $\widetilde{C}^\circ$ with projection $\tilde{p} : \widetilde{Z} \to \mathcal{M}_{\mathrm{gl}}$, endowed with the log structure making $\widetilde{Z} \to \widetilde{C}^\circ$ strict.

5.3.3 An obstruction theory for ε and $\widehat{\varepsilon}$

The discussion in Section 4.2 provides obstruction theories $\mathbb{G} \to \mathbb{L}_{\mathcal{M}_{\mathrm{gl}}/\mathfrak{M}^{\mathrm{ev}}_{\mathrm{gl}}}$ for $\varepsilon : \mathcal{M}_{\mathrm{gl}} \to \mathfrak{M}^{\mathrm{ev}}_{\mathrm{gl}}$ and $\mathbb{G}_{\mathrm{spl}} \to \mathbb{L}_{\mathcal{M}_{\mathrm{spl}}/\mathfrak{M}^{\mathrm{ev}}_{\mathrm{spl}}}$ for $\widehat{\varepsilon} : \mathcal{M}_{\mathrm{spl}} \to \mathfrak{M}^{\mathrm{ev}}_{\mathrm{spl}}$ with

$$
\mathbb{G} = R\tilde{\pi}_*\big(\tilde{f}^*\Omega_{X/B} \otimes \omega_{\tilde{\pi}}(\widetilde{Z})\big), \quad \mathbb{G}_{\mathrm{spl}} = R\pi_{\mathrm{spl}*}\big(f^*_{\mathrm{spl}}\Omega_{X/B} \otimes \omega_{\pi_{\mathrm{spl}}}(Z_{\mathrm{spl}})\big). \tag{5.18}
$$

Recall that this obstruction theory is obtained by taking the cone of a morphism of perfect obstruction theories provided by Proposition 4.3:

$$\begin{array}{ccc} L\widehat{\varepsilon}^*\mathbb{F}_{\mathrm{spl}} & \longrightarrow & \mathbb{E}_{\mathrm{spl}} \\ {\scriptstyle L\widehat{\varepsilon}^*\Psi}\downarrow & & \downarrow{\scriptstyle \Phi} \\ L\widehat{\varepsilon}^*\mathbb{L}_{\mathfrak{M}^{\mathrm{ev}}_{\mathrm{spl}}/\mathfrak{M}_{\mathrm{spl}}} & \longrightarrow & \mathbb{L}_{\mathcal{M}_{\mathrm{spl}}/\mathfrak{M}_{\mathrm{spl}}} \end{array}$$

5.3.4 A justice of obstructions[6]

We now have four deformation/obstruction situations with corresponding perfect obstruction theories. Given $T \to \mathcal{M}_{\mathrm{gl}}$ a morphism from an affine scheme and $f_T : C^\circ_T \to X$, $h_T : Z_T \to X$, $\tilde{f}_T : \widetilde{C}^\circ_T \to X$, $\tilde{h}_T : \widetilde{Z}_T \to X$ the respective base-changes to T of the universal morphisms from the universal curve and universal sections, pulled back to $\mathcal{M}_{\mathrm{gl}}$ in the last two instances, these are as follows. All deformation situations are relative $\mathfrak{M}_{\mathrm{gl}}$, with the last two pulled back from a deformation situation relative $\mathfrak{M}_{\mathrm{spl}}$.

$(\mathcal{M}_{\mathrm{gl}}/\mathfrak{M}_{\mathrm{gl}})$ Deforming $f_T : C^\circ_T \to X$:

$$\mathbb{E} = R\pi_*(f^*\Omega_{X/B} \otimes \omega_\pi) \to \mathbb{L}_{\mathcal{M}_{\mathrm{gl}}/\mathfrak{M}_{\mathrm{gl}}}.$$

$(\mathfrak{M}^{\mathrm{ev}}_{\mathrm{gl}}/\mathfrak{M}_{\mathrm{gl}})$ Deforming $h_T : Z_T \to X$:

$$L\varepsilon^*\mathbb{F} = p_*(h^*\Omega_{X/B}) \to L\varepsilon^*\mathbb{L}_{\mathfrak{M}^{\mathrm{ev}}_{\mathrm{gl}}/\mathfrak{M}_{\mathrm{gl}}}.$$

$(\mathcal{M}_{\mathrm{spl}}/\mathfrak{M}_{\mathrm{spl}})$ Deforming $\tilde{f}_T : \widetilde{C}^\circ_T \to X$:

$$L\delta^*\mathbb{E}_{\mathrm{spl}} = R\tilde{\pi}_*(\tilde{f}^*\Omega_{X/B} \otimes \omega_{\tilde{\pi}}) \to L\delta^*\mathbb{L}_{\mathcal{M}_{\mathrm{spl}}/\mathfrak{M}_{\mathrm{spl}}}.$$

$(\mathfrak{M}^{\mathrm{ev}}_{\mathrm{spl}}/\mathfrak{M}_{\mathrm{spl}})$ Deforming $\tilde{h}_T : \widetilde{Z}_T \to X$:

$$L\delta^*L\widehat{\varepsilon}^*\mathbb{F}_{\mathrm{spl}} = \tilde{p}_*(\tilde{h}^*\Omega_{X/B}) \to L\delta^*L\widehat{\varepsilon}^*\mathbb{L}_{\mathfrak{M}^{\mathrm{ev}}_{\mathrm{spl}}/\mathfrak{M}_{\mathrm{spl}}}.$$

Lemma 5.18. *There is a morphism of distinguished triangles*

$$\begin{array}{ccccccc} L\delta^*L\widehat{\varepsilon}^*\mathbb{F}_{\mathrm{spl}} & \longrightarrow & L\delta^*\mathbb{E}_{\mathrm{spl}} & \longrightarrow & \mathbb{G} & \longrightarrow & L\delta^*L\widehat{\varepsilon}^*\mathbb{F}_{\mathrm{spl}}[1] \\ \downarrow & & \downarrow & & \| & & \downarrow \\ L\varepsilon^*\mathbb{F} & \longrightarrow & \mathbb{E} & \longrightarrow & \mathbb{G} & \longrightarrow & L\varepsilon^*\mathbb{F}[1] \end{array}$$

with $\mathbb{G} = L\delta^*\mathbb{G}_{\mathrm{spl}} = R\tilde{\pi}_*(\tilde{f}^*\Omega_{X/B} \otimes \omega_{\tilde{\pi}}(\widetilde{Z}))$.

[6] Our Babel of coauthors proposes this collective noun for a system of compatible obstructions.

Proof. The lower row in the claimed diagram was produced in (4.17) in the proof of Proposition 4.5 by applying $R\pi_*$ to (4.16) tensored with $f^*\Omega_{X/B}$. We claim that (4.16) appears as the lower row in the following commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \kappa_*\omega_{\tilde\pi} & \longrightarrow & \kappa_*(\omega_{\tilde\pi}(\tilde Z)) & \longrightarrow & \kappa_*\tilde\iota_*\mathcal{O}_{\tilde Z}[1] & \longrightarrow & 0\\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & \omega_\pi & \longrightarrow & \kappa_*(\omega_{\tilde\pi}(\tilde Z)) & \longrightarrow & \iota_*\mathcal{O}_Z[1] & \longrightarrow & 0.
\end{array} \tag{5.19}
$$

Away from the nodal locus $Z'' \subset Z$ the upper and lower rows are identical, and this identification defines the diagram there. Étale locally near a node, the arrow $\kappa_*(\omega_{\tilde\pi}(\tilde Z)) \to \iota_*\mathcal{O}_Z[1]$ takes the difference of the residues of a differential with at most simple poles along the two components of $\tilde Z'' \simeq Z'' \amalg Z''$ defined by the two branches at the node. This map factors as κ_* of the residue map

$$\rho\colon \omega_{\tilde\pi}(\tilde Z'') \to \tilde\iota_*\mathcal{O}_{\tilde Z''}[1]$$

and the [1]-twist of the difference map

$$\kappa_*\tilde\iota_*\mathcal{O}_{\tilde Z''} = \iota_*\mathcal{O}_{Z''} \oplus \iota_*\mathcal{O}_{Z''} \to \iota_*\mathcal{O}_{Z''}, \quad (a,b) \mapsto a-b.$$

The kernel of $\kappa_*\rho$ selects differentials without poles, that is, $\kappa_*\omega_{\tilde\pi}$. This extends the construction of Diagram (5.19) over the nodal locus.

To produce the morphism of triangles in the statement it remains to show that tensoring the upper row of (5.19) with $f^*\Omega_{X/B}$ and applying $R\pi_*$ leads to the upper row of the claimed diagram. From Proposition 4.5 we already know that the middle term leads to $\mathbb{G}$:

$$\mathbb{G} \overset{(5.18)}{=} R\tilde\pi_*\big(\tilde f^*\Omega_{X/B} \otimes \omega_{\tilde\pi}(\tilde Z)\big) = R\pi_*\big(f^*\Omega_{X/B} \otimes \kappa_*(\omega_{\tilde\pi}(\tilde Z))\big).$$

The other two terms are readily obtained by the projection formula for κ using $\tilde\pi = \pi \circ \kappa$, $\tilde f = f \circ \kappa$, $\tilde h = \tilde f \circ \tilde\iota$, $\tilde p = \tilde\pi \circ \tilde\iota$:

$$
\begin{aligned}
L\delta^*\mathbb{E}_{\mathrm{spl}} &= R\tilde\pi_*\big(\tilde f^*\Omega_{X/B} \otimes \omega_{\tilde\pi}\big) = R\pi_*(f^*\Omega_{X/B} \otimes \kappa_*\omega_{\tilde\pi})\\
L\delta^* L\hat\varepsilon^*\mathbb{F}_{\mathrm{spl}} &= \tilde p_*(\tilde h^*\Omega_{X/B}) = R\tilde\pi_*\big(\tilde f^*\Omega_{X/B} \otimes \tilde\iota_*\mathcal{O}_{\tilde Z}\big)\\
&= R\pi_*(f^*\Omega_{X/B} \otimes \kappa_*\tilde\iota_*\mathcal{O}_{\tilde Z}).
\end{aligned}
$$

■

Theorem 5.19. *Let $X \to B$ be a log smooth morphism of fs logarithmic schemes over $\Bbbk$ fulfilling the assumptions stated at the beginning of Chapter 3, and τ, τ_i,* **E**, **S**

as in Proposition 5.17. Then with the notation of (5.16), we have

(1) *The obstruction theory $\mathbb{G} \to \mathbb{L}_{\mathcal{M}_{\mathrm{gl}}/\mathfrak{M}^{\mathrm{ev}}_{\mathrm{gl}}}$ for*

$$\mathcal{M}_{\mathrm{gl}} = \mathcal{M}_{\mathrm{gl}}(X/B,\tau) \to \mathfrak{M}^{\mathrm{ev}}_{\mathrm{gl}} = \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\tau)$$

coincides with the pullback of one of the obstruction theories $\mathbb{G}_{\mathrm{spl}} \to \mathbb{L}_{\mathcal{M}_{\mathrm{spl}}/\mathfrak{M}^{\mathrm{ev}}_{\mathrm{spl}}}$ (Remark 4.4) for

$$\mathcal{M}_{\mathrm{spl}} = \prod_i \mathcal{M}(X/B,\tau_i) \to \mathfrak{M}^{\mathrm{ev}}_{\mathrm{spl}} = \prod_i \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\tau_i)$$

described in Section 5.3.3.

(2) *If $\widehat{\varepsilon}^!$ and $\varepsilon^!$ denote Manolache's virtual pullback defined using the two given obstruction theories for the vertical arrows in diagram (5.15), then for $\alpha \in A_*(\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\tau))$, we have the identity*

$$\widehat{\varepsilon}^! \delta^{\mathrm{ev}}_*(\alpha) = \delta_* \varepsilon^!(\alpha).$$

Proof. (1) The morphism between the obstruction theories in question appear as the joint middle square in the following diagram of two adjacent cubes:

$$\begin{array}{ccccccc}
L\delta^*\mathbb{E}_{\mathrm{spl}} & \longrightarrow & \mathbb{G} & \longrightarrow & L\delta^* L\widehat{\varepsilon}^*\mathbb{F}_{\mathrm{spl}}[1] & & \\
& \searrow & \| & \searrow & & \searrow & \\
& & L\delta^*\mathbb{L}_{\mathcal{M}_{\mathrm{spl}}/\mathfrak{M}_{\mathrm{spl}}} & \longrightarrow & L\delta^*\mathbb{L}_{\mathcal{M}_{\mathrm{spl}}/\mathfrak{M}^{\mathrm{ev}}_{\mathrm{spl}}} & \longrightarrow & L\delta^* L\widehat{\varepsilon}^*\mathbb{L}_{\mathfrak{M}^{\mathrm{ev}}_{\mathrm{spl}}/\mathfrak{M}_{\mathrm{spl}}}[1] \\
\downarrow & & \| & & \downarrow & & \\
\mathbb{E} & \longrightarrow & \mathbb{G} & \longrightarrow & L\varepsilon^*\mathbb{F}[1] & & \downarrow \\
& \searrow & \downarrow & \searrow & \downarrow & \searrow & \\
& & \mathbb{L}_{\mathcal{M}_{\mathrm{gl}}/\mathfrak{M}_{\mathrm{gl}}} & \longrightarrow & \mathbb{L}_{\mathcal{M}_{\mathrm{gl}}/\mathfrak{M}^{\mathrm{ev}}_{\mathrm{gl}}} & \longrightarrow & L\varepsilon^*\mathbb{L}_{\mathfrak{M}^{\mathrm{ev}}_{\mathrm{gl}}/\mathfrak{M}_{\mathrm{gl}}}[1]
\end{array}$$

The back face is the morphism of triangles from Lemma 5.18. The bottom face is commutative by the construction of the obstruction theory with point conditions $\mathbb{G} \to \mathbb{L}_{\mathcal{M}_{\mathrm{gl}}/\mathfrak{M}^{\mathrm{ev}}_{\mathrm{gl}}}$ in (4.15) based on Proposition 4.3. Similarly, the top face is commutative as the pullback by δ of the corresponding diagram for $\mathbb{G}_{\mathrm{spl}} \to \mathbb{L}_{\mathcal{M}_{\mathrm{spl}}/\mathfrak{M}^{\mathrm{ev}}_{\mathrm{spl}}}$. The front face of the diagram is the morphism of distinguished triangles of cotangent complexes for the compositions $\mathcal{M}_{\mathrm{gl}} \to \mathfrak{M}^{\mathrm{ev}}_{\mathrm{gl}} \to \mathfrak{M}_{\mathrm{gl}}$ and $\mathcal{M}_{\mathrm{spl}} \to \mathfrak{M}^{\mathrm{ev}}_{\mathrm{spl}} \to \mathfrak{M}_{\mathrm{spl}}$, and hence is commutative as well.

For commutativity of the left face we argue in two steps. First apply functoriality of obstruction theories, Lemma 4.1, to compare the pulled-back obstruction theory $(\mathcal{M}_{\mathrm{spl}}/\mathfrak{M}_{\mathrm{spl}})$ for f_{spl} with the obstruction theory for $\tilde{f}$, both relative $\mathfrak{M}_{\mathrm{spl}}$, to obtain the commutative square

$$\begin{array}{ccc}
L\delta^*\mathbb{E}_{\mathrm{spl}} & \longrightarrow & L\delta^*\mathbb{L}_{\mathcal{M}_{\mathrm{spl}}/\mathfrak{M}_{\mathrm{spl}}} \\
\downarrow & & \downarrow \\
\widetilde{\mathbb{E}} & \longrightarrow & \mathbb{L}_{\mathcal{M}_{\mathrm{gl}}/\mathfrak{M}_{\mathrm{gl}}}.
\end{array} \tag{5.20}$$

Here $\widetilde{\mathbb{E}} = R\tilde{\pi}_*(\tilde{f}^*\Omega_{X/B} \otimes \omega_{\tilde{\pi}})$ and we replaced the lower right-hand corner $\mathbb{L}_{\mathcal{M}_{\mathrm{gl}}/\mathfrak{M}_{\mathrm{spl}}}$ by $\mathbb{L}_{\mathcal{M}_{\mathrm{gl}}/\mathfrak{M}_{\mathrm{gl}}}$ using functoriality of the cotangent complex. Note also that the proof of Lemma 4.1 did not use the general assumption in Section 4.1 that M is an open substack of the stack of diagrams described in (4.1), so does apply to the non-universal family over $\mathcal{M}_{\mathrm{gl}}$ given by $\tilde{f}$.

We are then in the situation of Section 4.1.7 with $Y \to S$ the universal curve over $\mathfrak{M}_{\mathrm{gl}}$, Z the partial normalization of this curve, and $M = N = \mathcal{M}_{\mathrm{gl}}$. Thus Proposition 4.3 provides the commutative square

$$\begin{array}{ccc} \widetilde{\mathbb{E}} & \longrightarrow & \mathbb{L}_{\mathcal{M}_{\mathrm{gl}}/\mathfrak{M}_{\mathrm{gl}}} \\ \downarrow & & \downarrow \\ \mathbb{E} & \longrightarrow & \mathbb{L}_{\mathcal{M}_{\mathrm{gl}}/\mathfrak{M}_{\mathrm{gl}}}. \end{array} \tag{5.21}$$

Again, this result did not use universality of the family of maps over $\mathcal{M}_{\mathrm{gl}}$ given by $\tilde{f}$. Composing the two squares (5.20) and (5.21) proves commutativity of the left face of our big diagram of adjacent cubes.

An analogous argument for the nodal locus Z and its pullback $\tilde{Z} \subset \tilde{C}^\circ$ instead of C° and $\tilde{C}^\circ$ also shows commutativity of the right face.

Thus the whole diagram is commutative except possibly the middle, separating square that describes the morphism of interest from the pullback of the obstruction theory for $(\mathcal{M}_{\mathrm{spl}}/\mathfrak{M}^{\mathrm{ev}}_{\mathrm{spl}})$ to $(\mathcal{M}_{\mathrm{gl}}/\mathfrak{M}^{\mathrm{ev}}_{\mathrm{gl}})$.

However, chasing the diagram, we see that the two morphisms from $\mathbb{G}$ to the front right corner $L\varepsilon^*\mathbb{L}_{\mathfrak{M}^{\mathrm{ev}}_{\mathrm{gl}}/\mathfrak{M}_{\mathrm{gl}}}[1]$, one via the top dashed arrow, the other via the bottom dashed arrow, agree. Their difference factors over a homomorphism

$$\mathbb{G} \to \mathbb{L}_{\mathcal{M}_{\mathrm{gl}}/\mathfrak{M}_{\mathrm{gl}}}.$$

The set of such homomorphisms acts transitively on the set of dashed arrows on the bottom face defining the obstruction theory for $(\mathcal{M}_{\mathrm{gl}}/\mathfrak{M}^{\mathrm{ev}}_{\mathrm{gl}})$ as discussed in Remark 4.4 Thus there is a choice of dashed bottom arrow making the separating middle square of the diagram commutative, as claimed.

(2) This follows from the morphism δ^{ev} being finite and representable, hence projective, and the push-pull formula of [50, Theorem 4.1 (iii)]. ■

5.3.5 Gluing by the numbers

We now achieve a numerical gluing formula for Gromov–Witten invariants for classes in $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B, \tau)$ whose push-forward to $\prod \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B, \tau_i)$ decomposes as a sum of products of classes. This is for example the case for point classes in $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B, \tau)$, or if all gluing strata are toric [71].

Corollary 5.20. *In the situation of Theorem 5.19 let* $\alpha \in A_*(\mathfrak{M}(\mathcal{X}/B, \tau))$ *and assume that there exists* $\alpha_{i,\mu} \in A_*(\mathfrak{M}(\mathcal{X}/B, \tau_i))$, $i = 1, \ldots, r$, $\mu = 1, \ldots, m$, *with*

$$\delta_*^{\mathrm{ev}}(\alpha) = \sum_{\mu=1}^{m} \alpha_{1,\mu} \times \cdots \times \alpha_{r,\mu}.$$

Then writing $\varepsilon_i : \mathcal{M}(X/B, \tau_i) \to \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B, \tau_i)$ *for the canonical map, the following equality of associated virtual classes holds in* $A_*(\prod_i \mathcal{M}(X/B, \tau_i))$*:*

$$\delta_* \varepsilon^!(\alpha) = \sum_{\mu=1}^{m} \varepsilon_1^!(\alpha_{1,\mu}) \times \cdots \times \varepsilon_r^!(\alpha_{r,\mu}).$$

Proof. The claimed formula follows readily from Theorem 5.19 (2) by observing that

$$\hat{\varepsilon}^!(\alpha_{1,\mu} \times \cdots \times \alpha_{r,\mu}) = \varepsilon_1^!(\alpha_{1,\mu}) \times \cdots \times \varepsilon_r^!(\alpha_{r,\mu}). \qquad \blacksquare$$

5.3.6 Compatibility with contractions of types

We end this section by noting that the relative obstruction theories are also compatible with contraction morphisms relating different global types (Definition 2.44 (1)).

Proposition 5.21. *Let* $X \to B$ *be as in Theorem 5.19 and assume* $\tau' \to \tau$ *is a contraction morphism of global types. Then the commutative diagram*

$$\begin{array}{ccc} \mathcal{M}(X/B, \tau') & \longrightarrow & \mathcal{M}(X/B, \tau) \\ \Big\downarrow{\scriptstyle \varepsilon'} & & \Big\downarrow{\scriptstyle \varepsilon} \\ \mathfrak{M}(\mathcal{X}/B, \tau') & \longrightarrow & \mathfrak{M}(\mathcal{X}/B, \tau) \end{array}$$

is cartesian, and the relative obstruction theory for ε *pulls back to the relative obstruction theory for* ε'*. Taking curve classes into consideration, if* $\boldsymbol{\tau} = (\tau, \mathbf{A})$*, the commutative diagram*

$$\begin{array}{ccc} \coprod_{\boldsymbol{\tau}'=(\tau',\mathbf{A}')} \mathcal{M}(X/B, \boldsymbol{\tau}') & \longrightarrow & \mathcal{M}(X/B, \boldsymbol{\tau}) \\ \Big\downarrow{\scriptstyle \varepsilon'} & & \Big\downarrow{\scriptstyle \varepsilon} \\ \mathfrak{M}(\mathcal{X}/B, \tau') & \longrightarrow & \mathfrak{M}(\mathcal{X}/B, \tau) \end{array} \tag{5.22}$$

is cartesian, and the same statement on relative obstruction theories holds. Here, the disjoint union is over all decorations $\boldsymbol{\tau}'$ *of* τ' *such that the contraction morphism* $\tau' \to \tau$ *induces a contraction morphism* $\boldsymbol{\tau}' \to \boldsymbol{\tau}$*.*

Analogous results hold for weakly marked versions of the stacks (Definition 3.8), and for evaluation stacks on the bottom (Definition 5.14).

Proof. That the diagrams are Cartesian follows from the definition of markings and decorated markings of punctured maps (Definition 3.8).

The statement about obstruction theories then follows from the functoriality statement Lemma 4.1 and the construction in Section 4.2 of the relative obstruction theory for $\mathcal{M}(X/B,\tau) \to \mathfrak{M}(\mathcal{X}/B,\tau)$. ∎

Remark 5.22. The formalism for gluing presented here was found after many futile attempts leading to practically useless gluing procedures. With hindsight compatibility with the virtual formalism provides the strongest guiding principle that rules out many alternative approaches. From this point of view one discovers the imperative that one work with obstruction theories relative to a class of unobstructed base stacks that induce the gluing.

A first attempt would work with moduli stacks $\mathfrak{M}(\mathcal{X}/B)$ of punctured maps to the relative Artin fan $\mathcal{X} \to B$. This approach does indeed work, but it is often problematic for practical applications because the gluing map $\mathfrak{M}(\mathcal{X}/B,\boldsymbol{\tau}) \to \prod_i \mathfrak{M}(\mathcal{X}/B,\boldsymbol{\tau}_i)$ is neither representable nor proper, hence does not allow pushing forward of cycles.

The key insight is to use evaluation stacks to add just enough information to get rid of the stacky nature of the gluing in $\mathfrak{M}(\mathcal{X}/B)$, thus leading to a finite and representable splitting map δ^{ev}. In addition, δ^{ev} fits into the expected gluing diagram stated in Corollary 5.15 thus providing a practical path to explicit computations.

5.4 Gluing in the degeneration setup

We now apply our gluing theorems to the degeneration situation previously studied in [3]. In this case B is a smooth affine curve over $\operatorname{Spec}\Bbbk$ with log structure trivial except at a marked point $b_0 \in B$, and $\mathcal{A}_X$ is assumed Zariski. Base change to b_0 produces a log smooth space X_0 over the standard log point $\operatorname{Spec}(\mathbb{N} \to \Bbbk)$. Let $\beta = (g,\bar{\mathbf{u}},A)$ be a class of punctured maps to X. Note that $\Sigma(X_0) = \Sigma(X)$, so we can view β also as a class of punctured maps to X_0. The fiber of the tropicalization $\Sigma(X_0) \to \Sigma(b_0) = \mathbb{R}_{\geq 0}$ of the projection $X_0 \to b_0$ over $1 \in \mathbb{R}_{\geq 0}$ defines a polyhedral complex $\Delta(X_0) = \Delta(X)$. Restricting to this fiber turns our cone complexes into the polyhedral complexes of traditional tropical geometry.

The main result of [3] gives the following decomposition of the virtual fundamental class of $\mathcal{M}(X_0/b_0,\beta)$ in terms of rigid tropical maps to $\Delta(X_0)$. We emphasize that this result uses the marked rather than weakly marked versions of the moduli stacks.

Theorem 5.23. *Let β be a class of stable logarithmic maps to X_0/b_0. Then we have the following equality of Chow classes on $\mathcal{M}(X_0/b_0,\beta)$:*

$$[\mathcal{M}(X_0/b_0,\beta)]^{\mathrm{virt}} = \sum_{\boldsymbol{\tau}=(\tau,\mathbf{A})} \frac{m_\tau}{|\operatorname{Aut}(\boldsymbol{\tau})|} j_{\boldsymbol{\tau}*}[\mathcal{M}(X_0/b_0,\boldsymbol{\tau})]^{\mathrm{virt}}.$$

The sum runs over representatives of isomorphism classes of realizable global types $\boldsymbol{\tau}$ *of punctured maps to* X_0 *over* b_0 *of total class* $(g, \bar{\mathbf{u}}, A)$ *and with basic monoid* $Q_\tau \simeq \mathbb{N}$. *The multiplicity* m_τ *is the index of the image of the homomorphism* $\mathbb{N} \to Q_\tau$ *given by the map* $\mathcal{M}(X_0/b_0, \beta) \to b_0$. *The morphism* $j_{\boldsymbol{\tau}} : \mathcal{M}(X_0/b_0, \boldsymbol{\tau}) \to \mathcal{M}(X_0/b_0, \beta)$ *is induced by the contraction morphism* $\tau \to \beta$. *Finally,* $\mathrm{Aut}(\boldsymbol{\tau})$ *denotes the group of automorphisms of the decorated type* $\boldsymbol{\tau}$, *i.e., automorphisms of the underlying graph* G *preserving* $\mathbf{g}$, $\boldsymbol{\sigma}$, $\mathbf{u}$ *and* $\mathbf{A}$.

5.4.1 Degenerate types

Theorem 5.25 below is an analogous result in the punctured case, which also provides a stratified version in the case without punctures. Before stating this result we need some preparations concerning types in degeneration situations. Since one works with log spaces over b_0 and $\Sigma(b_0) = \mathbb{R}_{\geq 0}$, all tropical objects come with a map to $\mathbb{R}_{\geq 0}$. We denote all these maps by p in the following. Assuming $X_0 \to b_0$ is the fiber over the unique marked point $b_0 \to B$ in a log smooth curve B over the trivial log point, the tropicalization of a punctured map over the generic point $\eta \in \underline{B}$ maps to $0 \in \Sigma(B) = \Sigma(b_0)$ under p. Degenerations of families of punctured maps over η to b_0 then provide a contraction morphism of the associated types (Definition 2.44 (1)). This motivates the following definition.

Definition 5.24. Let $\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u})$ be a realizable global type (Definition 2.44 (2)) of punctured maps to X_0/b_0 (Definition 3.28) and Q_τ the associated basic monoid.

1. We call τ *generic* if $(Q_\tau^\vee)_\mathbb{R}$ and $\boldsymbol{\sigma}(x)$ for each $x \in V(G) \cup E(G) \cup L(G)$ map to $\{0\} \subset \mathbb{R}_{\geq 0}$ under p.
2. A *degeneration* of a realizable global type τ is a contraction morphism $\tau' \to \tau$ between realizable global types with $p : Q_{\tau'}^\vee \to \mathbb{N}$ non-constant. The *codimension* of $\tau' \to \tau$ is defined as $\mathrm{rk}\, Q_{\tau'}^{\mathrm{gp}} - \mathrm{rk}\, Q_\tau^{\mathrm{gp}}$. In the case of codimension one we define the *multiplicity* $m_{\tau'}$ as the index of $p^{\mathrm{gp}}(Q_{\tau'}^*)$ in $\mathbb{Z}$. Finally, $\mathrm{Aut}(\tau'/\tau)$ denotes the group of automorphisms of τ' commuting with $\tau' \to \tau$.

Analogous notions are used in the decorated case (Definition 3.8).

5.4.2 Degenerate types decompose

Let now $\boldsymbol{\tau} = (G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u}, \mathbf{A})$ be a generic realizable decorated global type for X/B. By the assumption $p(\boldsymbol{\sigma}(x)) = 0$ we can view $\boldsymbol{\tau}$ also as a decorated global type for X_b/b for $b \neq b_0$. The analogue to the main results of [3] is:

Theorem 5.25. *In the above situation, additionally assuming* X *is simple, the following holds.*

1. *For any point* $j_b : \{b\} \hookrightarrow B$, *one has* $j_b^![\mathcal{M}(X/B, \boldsymbol{\tau})]^{\mathrm{virt}} = [\mathcal{M}(X_b/b, \boldsymbol{\tau})]^{\mathrm{virt}}$.

(2) *The following equation holds:*

$$[\mathcal{M}(X_0/b_0,\boldsymbol{\tau})]^{\mathrm{virt}} = \sum_{\boldsymbol{\tau}'=(\tau',\mathbf{A}')} \frac{m_{\tau'}}{|\operatorname{Aut}(\boldsymbol{\tau}'/\boldsymbol{\tau})|} j_{\boldsymbol{\tau}'*}[\mathcal{M}(X_0/b_0,\boldsymbol{\tau}')]^{\mathrm{virt}} \quad (5.23)$$

The sum runs over representatives of isomorphism classes of degenerations $\boldsymbol{\tau}' = (\tau', \mathbf{A}') \to \boldsymbol{\tau}$ *of realizable global types of punctured maps to* X/B *of codimension one, with* $m_{\tau'}$ *its multiplicity.*

Proof. By Proposition 3.29, $\boldsymbol{\tau}$ can be viewed both as a type realizable over B and as a type realizable over $b \in B$ for $b \neq b_0$. Thus $\mathfrak{M}(\mathcal{X}/B,\boldsymbol{\tau})$ is non-empty and $\mathfrak{M}(\mathcal{X}_b/b,\boldsymbol{\tau}) = \mathfrak{M}(\mathcal{X}/B,\boldsymbol{\tau}) \times_B b$ is non-empty for $b \in B \setminus \{b_0\}$. By Proposition 3.30, $\mathfrak{M}(\mathcal{X}/B,\boldsymbol{\tau})$ is pure-dimensional. Further, by the same proposition, every irreducible component of $\mathfrak{M}(\mathcal{X}/B,\boldsymbol{\tau})$ contains a point whose corresponding punctured map has tropical type $\boldsymbol{\tau}$, as all other strata are of lower dimension. By genericity of the type $\boldsymbol{\tau}$, the stratum of $\mathfrak{M}(\mathcal{X}/B,\boldsymbol{\tau})$ of points with type $\boldsymbol{\tau}$ maps to the open stratum of B. Thus the restriction of $\mathfrak{M}(\mathcal{X}/B,\boldsymbol{\tau}) \to B$ to each irreducible component is dominant. There are no embedded components by the local description in Remark 3.27. We conclude that the structure map $\mathfrak{M}(\mathcal{X}/B,\boldsymbol{\tau}) \to B$ is flat.

(1) then follows immediately from general properties of virtual pull-backs.

For (2), as in the proof of [3, Theorem 3.11], we begin by showing the corresponding decomposition as Chow classes

$$[\mathfrak{M}(\mathcal{X}_0/b_0,\tau)] = \sum_{\tau'\to\tau} \frac{m_{\tau'}}{|\operatorname{Aut}(\tau'/\tau)|} \iota_{\tau'*}[\mathfrak{M}(\mathcal{X}_0/b_0,\tau')]. \quad (5.24)$$

Here τ is the underlying global type of $\boldsymbol{\tau}$, and $\tau' \to \tau$ runs over all contraction morphisms as in the statement of the theorem (without the decoration). Finally, $\iota_{\tau'}$: $\mathfrak{M}(\mathcal{X}_0/b_0,\tau') \to \mathfrak{M}(\mathcal{X}_0/b_0,\tau)$ is the natural morphism. However, using the smooth local description of $\mathfrak{M}(\mathcal{X}/B,\tau)$ given in Remark 3.27 and the fact that $|\operatorname{Aut}(\tau'/\tau)|$ is the degree of the finite map $\iota_{\tau'}$ onto its image, we easily obtain the result using standard toric geometry. We leave the details to the reader.

We now make use of the diagram (5.22) for a given choice of contraction $\tau' \to \tau$, and we see by the push-pull result of [50, Theorem 4.1] that

$$\begin{aligned} \varepsilon^! \iota_{\tau'*}[\mathfrak{M}(\mathcal{X}_0/b_0,\tau')] &= \sum_{\boldsymbol{\tau}'=(\tau',\mathbf{A}')} j_{\tau'*}(\varepsilon')^![\mathfrak{M}(\mathcal{X}_0/b_0,\boldsymbol{\tau}')] \\ &= \sum_{\boldsymbol{\tau}'=(\tau',\mathbf{A}')} j_{\tau'*}[\mathcal{M}(X_0/b_0,\boldsymbol{\tau}')]^{\mathrm{virt}}, \end{aligned} \quad (5.25)$$

where the sum is over all choices of decorations $\boldsymbol{\tau}'$ of τ' giving a contraction morphism $\boldsymbol{\tau}' \to \boldsymbol{\tau}$ compatible with $\tau' \to \tau$. On the other hand, $\operatorname{Aut}(\tau'/\tau)$ acts on the set of all such decorations, with the orbit of a decoration $\boldsymbol{\tau}'$ having stabilizer $\mathrm{Aut}(\boldsymbol{\tau}'/\boldsymbol{\tau})$. Thus we may rewrite the last summation of (5.25) as

$$\sum_{\boldsymbol{\tau}'=(\tau',\mathbf{A}')} j_{\tau'*}[\mathcal{M}(X_0/b_0,\boldsymbol{\tau}')]^{\mathrm{virt}} \frac{|\mathrm{Aut}(\tau'/\tau)|}{|\mathrm{Aut}(\boldsymbol{\tau}'/\boldsymbol{\tau})|},$$

where now the sum is over a set of representatives of isomorphism classes of type $\boldsymbol{\tau}'$ with a contraction morphism $\boldsymbol{\tau}' \to \boldsymbol{\tau}$. Combining this with the relation (5.24) then gives the desired result. ■

5.4.3 Splitting and factoring decomposed degenerate types

As a corollary of Theorem 5.19 we now obtain a formula for the computation of each summand $[\mathcal{M}(X_0/b_0,\boldsymbol{\tau}')]^{\mathrm{virt}}$ in (5.23) in terms of punctured Gromov–Witten theory of the strata. For the statement note that if τ_v is a global type with only one vertex, with associated stratum $\sigma \in \Sigma(X)$, then a τ_v-marked punctured map $(C^\circ/W,\mathbf{p},f)$ to X has a factorization

$$f : C^\circ \xrightarrow{f_\sigma} X_\sigma \to X,$$

where the stratum X_σ is now endowed with the log structure making the embedding $X_\sigma \to X$ strict. The composition with this strict closed embedding in fact induces an isomorphism

$$\mathcal{M}(X_0/b_0,\tau_v) \xrightarrow{\simeq} \mathcal{M}(X_\sigma/b_0,\tau_v).$$

Similarly, we obtain

$$\mathfrak{M}(\mathcal{X}_0/b_0,\tau_v) \simeq \mathfrak{M}(\mathcal{X}_\sigma/b_0,\tau_v) \quad \text{and} \quad \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}_0/b_0,\tau_v) \simeq \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}_\sigma/b_0,\tau_v).$$

Note also that $X_\sigma \to \mathcal{X}_\sigma$ is strict and smooth despite $\mathcal{X}_\sigma$ being only idealized log smooth over b_0 (see Proposition 2.48). Thus the obstruction theory developed in Section 4.2 still applies with target $X_\sigma \to \mathcal{X}_\sigma \to b_0$ and yields the same result as with $X_0 \to \mathcal{X}_0 \to b_0$. Theorem 5.19 applied to our degeneration situation can therefore be stated as follows.

Corollary 5.26. *Let $(G,\mathbf{g},\boldsymbol{\sigma},\mathbf{u},\mathbf{A})$ be a decorated type of punctured maps with basic monoid $Q_\tau \simeq \mathbb{N}$ and $\boldsymbol{\tau} = (G,\mathbf{g},\boldsymbol{\sigma},\bar{\mathbf{u}},\mathbf{A})$ the associated decorated global type. Denote by $\boldsymbol{\tau}_v$, $v \in V(G)$, the decorated global types obtained by splitting $\boldsymbol{\tau}$ at all edges, that is, for $\mathbf{E} = E(G)$. Then the diagram*

$$\begin{array}{ccc} \mathcal{M}(X_0/b_0,\boldsymbol{\tau}) & \xrightarrow{\delta} & \prod_{v\in V(G)} \mathcal{M}(X_{\boldsymbol{\sigma}(v)}/b_0,\boldsymbol{\tau}_v) \\ \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \hat{\varepsilon}=\prod_{v\in V(G)}\varepsilon_v} \\ \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}_0/b_0,\boldsymbol{\tau}) & \xrightarrow{\delta^{\mathrm{ev}}} & \prod_{v\in V(G)} \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}_{\boldsymbol{\sigma}}(v)/b_0,\boldsymbol{\tau}_v) \end{array}$$

with horizontal arrows the splitting maps from Proposition 5.4 finite and representable, is cartesian, and it holds

$$\delta_*[\mathcal{M}(X_{\mathbf{0}}/b_{\mathbf{0}}, \boldsymbol{\tau})]^{\mathrm{virt}} = \widehat{\varepsilon}^! \delta_*^{\mathrm{ev}}[\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}_{\mathbf{0}}/b_{\mathbf{0}}, \boldsymbol{\tau})].$$

As in Corollary 5.20, a numerical formula in terms of punctured Gromov–Witten invariants of the strata X_σ of X can be derived assuming $\delta_*^{\mathrm{ev}}[\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}_{\mathbf{0}}/b_{\mathbf{0}}, \boldsymbol{\tau})]$ decomposes into a sum of products. This is the case for example if all gluing strata $X_{\boldsymbol{\sigma}(E)}$, $E \in E(G)$, are toric, as proved in [71] based on Corollary 5.15.