Appendix A

Contact orders

Here we give a somewhat more sophisticated universal view on contact orders. This was the point of view we originally planned to give, but for most current applications, the simpler approach exposited in Section 2.4 suffices. Nevertheless, that approach obscures some of the subtleties of contact orders, and at times it may be worth having this more precise point of view.

For a target X with fs log structure, consider the following étale sheaves over $\underline{X}$:

$$\overline{\mathcal{M}}_X^\vee = \mathcal{H}om(\overline{\mathcal{M}}_X, \mathbb{N}) \quad \text{and} \quad \overline{\mathcal{M}}_X^* = \mathcal{H}om(\overline{\mathcal{M}}_X, \mathbb{Z}) \cong \mathcal{H}om\big(\overline{\mathcal{M}}_X^{\mathrm{gp}}, \mathbb{Z}\big).$$

Definition A.1. A *family of contact orders of* X consists of a strict morphism $Z \to X$ and a section $\mathbf{u} \in \Gamma(Z, \overline{\mathcal{M}}_Z^*)$ satisfying the following condition. Let $u : \mathcal{M}_Z \to \overline{\mathcal{M}}_Z \xrightarrow{\mathbf{u}} \mathbb{Z}$ be the composite homomorphism associated to $\mathbf{u}$. Then the map $\alpha : \mathcal{M}_Z \to \mathcal{O}_Z$ sends $u^{-1}(\mathbb{Z} \setminus \{0\})$ to 0.

We call the ideal $\mathcal{I}_\mathbf{u} \subset \mathcal{M}_Z$ generated by $u^{-1}(\mathbb{Z} \setminus \{0\})$ the *contact log-ideal* associated to $\mathbf{u}$, and denote by $\overline{\mathcal{I}}_\mathbf{u}$ the corresponding *contact ideal* in $\overline{\mathcal{M}}_Z$. These are coherent sheaves of ideals.

The family of contact orders is said to be *connected* if Z is connected.

For simplicity, we will refer to $\mathbf{u}$ as the contact order when there is no confusion about the strict morphism $Z \to X$. Given a family of contact orders $\mathbf{u} \in \Gamma(Z, \overline{\mathcal{M}}_Z^*)$ of X, the *pullback* of $\mathbf{u}$ along a strict morphism $Z' \to Z$ defines a family of contact orders $\mathbf{u}' \in \Gamma(Z', \overline{\mathcal{M}}_{Z'}^*)$.

Example A.2. To motivate this definition, consider a punctured map $f : C^\circ \to X$ over W, and a punctured section $p \in \mathbf{p}$. Take $\underline{Z} := \underline{W}$, and give $\underline{Z}$ the log structure given by pullback of $\mathcal{M}_X$ via $\underline{f} \circ p$, so that $Z \to X$ is strict. Let $\mathbf{u}$ be the following composition

$$\overline{\mathcal{M}}_Z \xrightarrow{\bar{f}^\flat} p^* \overline{\mathcal{M}}_{C^\circ} \to \overline{\mathcal{M}}_W \oplus \mathbb{Z} \to \mathbb{Z}, \tag{A.1}$$

where the middle arrow is the inclusion and the last arrow is the projection to the second factor.

We claim that $\mathbf{u}$ defines a family of contact orders of X. Indeed, let $\delta \in \mathcal{M}_Z$ and represent $f^\flat(\delta) = (e_\delta, \sigma^{u_p(\delta)})$, where σ is the element of $\mathcal{M}_C$ corresponding to a local defining equation of the section p.

If $u_p(\delta) > 0$ then

$$\alpha_Z(\delta) = p^*\alpha_C(f^\flat(\delta)) = p^*\alpha_C(e_\delta) \cdot p^*\alpha_C\big(\sigma^{u_p(\delta)}\big) = 0$$

since $p^*\alpha_C(\sigma) = 0$.

If $u_p(\delta) < 0$ then $f^\flat(\delta) \notin \mathcal{M}_C$ and hence, by Definition 2.1 (2) we have $\alpha_Z(\delta) = 0$.

The goal now is to define a universal family of contact orders for the Artin fan $\mathcal{A}_X$ of X.

A.1 Family of contact orders of Artin cones

Let $(Z \to X, \mathbf{u} \in \Gamma(Z, \overline{\mathcal{M}}_Z^*))$ be a family of contact orders of X. For any strict morphism $X \to Y$, $\mathbf{u}$ is naturally a family of contact orders of Y via the composition $Z \to X \to Y$. Conversely, we can pull back a contact order on Y to X by base-change: if we denote by $\mathcal{Z}_X$ and $\mathcal{Z}_Y$ the sheaves over **Sch** of families of contact orders on X and Y, respectively, and if we denote by $\mathcal{Z}_X \to X$ the map forgetting the section $\mathbf{u}$, then $\mathcal{Z}_X = \mathcal{Z}_Y \times_Y X$. Thus we may parameterize contact orders of the Artin fan $\mathcal{A}_X$ instead of X: pulling back such parametrization gives a parametrization of contact orders on X. This is the approach taken here, which is achieved in Proposition A.8 and Definition A.9. We first study the local case.

Consider a toric monoid P with $\sigma = \operatorname{Hom}(P, \mathbb{R}_{\geq 0})$, $N_\sigma = \operatorname{Hom}(P, \mathbb{Z}) = P^*$. This gives the toric variety $\mathbb{A}_\sigma = \operatorname{Spec}(P \xrightarrow{\alpha} \Bbbk[P])$, torus $T_\sigma := \operatorname{Spec}(\Bbbk[P^{\mathrm{gp}}])$ and Artin cone

$$\mathcal{A}_\sigma = [\mathbb{A}_\sigma / T_\sigma]. \tag{A.2}$$

Choose an integral vector $u \in N_\sigma$, which we view as $u \in \operatorname{Hom}(P, \mathbb{Z})$. Let I_u be the ideal of P generated by $u^{-1}(\mathbb{Z} \setminus \{0\})$. This generates a T_σ-invariant ideal in $\Bbbk[P]$, defining an invariant closed subscheme $Z_{u,\sigma} \subseteq \mathbb{A}_\sigma$ with quotient a closed substack $\mathcal{Z}_{u,\sigma} \subseteq \mathcal{A}_\sigma$. We proceed to construct a family of contact orders parametrized by $\mathcal{Z}_{u,\sigma}$.

For each face $\tau \prec \sigma$ (where $\prec$ denotes an inclusion of faces) consider the prime ideal $\mathcal{K}_{\tau \prec \sigma} = P \setminus \tau^\perp$. It defines a toric stratum $Z_{\tau \prec \sigma} := V(\alpha(\mathcal{K}_{\tau \prec \sigma})) \subseteq \mathbb{A}_\sigma$ where the duals of the stalks of $\overline{\mathcal{M}}_{Z_{\tau \prec \sigma}}$ are identified with the faces of σ containing τ. Note that the torus T_σ acts on $Z_{\tau \prec \sigma}$. Denote by $\mathcal{Z}_{\tau \prec \sigma} := [Z_{\tau \prec \sigma} / \operatorname{Spec}(\Bbbk[P^{\mathrm{gp}}])] \subseteq \mathcal{A}_\sigma$.

Lemma A.3. *We have* $(\mathcal{Z}_{u,\sigma})_{\mathrm{red}} = \bigcup_{\tau^{\mathrm{gp}} \ni u} \mathcal{Z}_{\tau \prec \sigma} \subseteq \mathcal{A}_\sigma$.

Proof. The ideal $\sqrt{I_u}$ defines some union of strata and we identify those strata $\mathcal{Z}_{\tau \prec \sigma}$ on which it vanishes. If $u \notin \tau^{\mathrm{gp}}$ there is an element $p \in \tau^\perp \cap P$ such that $u(p) \neq 0$. Therefore $p \in I_u$ but the monomial z^p does not vanish at the generic point of $\mathcal{Z}_{\tau \prec \sigma}$. Hence $(\mathcal{Z}_{u,\sigma})_{\mathrm{red}}$ is contained in the given union of strata. Conversely, if $u \in \tau^{\mathrm{gp}}$, and if $p \in u^{-1}(\mathbb{Z} \setminus \{0\})$, then $p \notin \tau^\perp \cap P$, hence z^p vanishes along $\mathcal{Z}_{\tau \prec \sigma}$. Thus $\mathcal{Z}_{\tau \prec \sigma}$ is contained in $(\mathcal{Z}_{u,\sigma})_{\mathrm{red}}$, proving the result. ∎

Since $\overline{\mathcal{M}}^*_{(\mathcal{Z}_{u,\sigma})_{\mathrm{red}}}$ is the pullback of $\overline{\mathcal{M}}^*_{\mathcal{Z}_{u,\sigma}}$ under the reduction $(\mathcal{Z}_{u,\sigma})_{\mathrm{red}} \to \mathcal{Z}_{u,\sigma}$, and reduction induces an isomorphism of étale sites, we have

$$\Gamma\big(\mathcal{Z}_{u,\sigma}, \overline{\mathcal{M}}^*_{\mathcal{Z}_{u,\sigma}}\big) = \Gamma\big((\mathcal{Z}_{u,\sigma})_{\mathrm{red}}, \overline{\mathcal{M}}^*_{(\mathcal{Z}_{u,\sigma})_{\mathrm{red}}}\big).$$

We define an element $\mathbf{u}_{u,\sigma}$ of this group by defining it on stalks in a manner compatible with generization. For a point z in the dense stratum of $\mathcal{Z}_{\tau\prec\sigma}$, with $F_\tau = P\cap\tau^\perp$, we have $\overline{\mathcal{M}}_{\mathcal{Z}_{u,\sigma},z} = (P+F_\tau^{\mathrm{gp}})/F_\tau^{\mathrm{gp}}$. Thus the condition $u\in\tau^{\mathrm{gp}}$ guarantees that $u: P\to\mathbb{Z}$ descends to $u:\overline{\mathcal{M}}_{\mathcal{Z}_{u,\sigma},z}\to\mathbb{Z}$. Being induced by the same element u, this is compatible with generization. Note that the scheme $\mathcal{Z}_{u,\sigma}$ was defined in such a way so that $\alpha_{\mathcal{Z}_{u,\sigma}}(\mathcal{I}_{\mathbf{u}_{u,\sigma}}) = 0$, so that $\mathcal{Z}_{u,\sigma}$ acquires the structure of an idealized log stack.

Thus u defines a family of contact orders of $\mathcal{A}_\sigma$

$$\mathbf{u}_{u,\sigma}\in\Gamma\big(\mathcal{Z}_{u,\sigma},\overline{\mathcal{M}}^*_{\mathcal{Z}_{u,\sigma}}\big). \tag{A.3}$$

It is connected since the most degenerate stratum $\mathcal{Z}_{\sigma\prec\sigma}$ is contained in the closure of $\mathcal{Z}_{\tau\prec\sigma}$ for each face τ.

Lemma A.4. *For any connected family of contact orders $\mathbf{u}\in\Gamma(Z,\overline{\mathcal{M}}^*_Z)$ of $\mathcal{A}_\sigma$, there exists a unique $u\in N_\sigma$ such that $\psi: Z\to\mathcal{A}_\sigma$ factors uniquely through $\mathcal{Z}_{u,\sigma}$, and $\mathbf{u}_{u,\sigma}$ pulls back to $\mathbf{u}$.*

Proof. The global chart $P\to\overline{\mathcal{M}}_{\mathcal{A}_\sigma}$ over $\mathcal{A}_\sigma$ pulls back to a global chart $P\to\overline{\mathcal{M}}_Z$ over Z. The composition $P\to\overline{\mathcal{M}}_Z\xrightarrow{\mathbf{u}}\mathbb{Z}$ defines an integral vector $u\in N_\sigma$. Consider the sheaf of monoid ideals $\mathcal{J}_u\subset\mathcal{M}_{\mathcal{A}_\sigma}$ generated by I_u. By definition, the contact log-ideal $\mathcal{I}_{\mathbf{u}}$ is generated by $\psi^{-1}\mathcal{J}_u$. Since $\alpha_Z(\mathcal{I}_{\mathbf{u}})=0$ and since $\mathcal{J}_u$ defines $\mathcal{Z}_{u,\sigma}\subseteq\mathcal{A}_\sigma$, we have the factorization $Z\to\mathcal{Z}_{u,\sigma}$ of ψ, with $\mathbf{u}$ the pullback of $\mathbf{u}_{u,\sigma}$. ∎

We can now assemble all the $\mathcal{Z}_{u,\sigma}$ by defining

$$\mathcal{Z}_\sigma = \coprod_{u\in N_\sigma}\mathcal{Z}_{u,\sigma},$$

and write $\psi_\sigma:\mathcal{Z}_\sigma\to\mathcal{A}_\sigma$ for the morphism which restricts to the closed embedding $\mathcal{Z}_{u,\sigma}\hookrightarrow\mathcal{A}_\sigma$ on each connected component $\mathcal{Z}_{u,\sigma}$ of $\mathcal{Z}_\sigma$. Then the $\mathbf{u}_{u,\sigma}$ yield a section $\mathbf{u}_\sigma\in\Gamma(\mathcal{Z}_\sigma,\overline{\mathcal{M}}^*_{\mathcal{Z}_\sigma})$, giving the universal family, over $\mathcal{Z}_\sigma$, of contact orders of $\mathcal{A}_\sigma$. This follows immediately from Lemma A.4 by restricting to connected components.

Proposition A.5. *Assume Z is locally connected. For any family of contact orders $\mathbf{u}\in\Gamma(Z,\overline{\mathcal{M}}^*_Z)$ of $\mathcal{A}_\sigma$, $\psi: Z\to\mathcal{A}_\sigma$ factors uniquely through $\mathcal{Z}_\sigma$, and $\mathbf{u}_\sigma$ pulls back to $\mathbf{u}$.*

Corollary A.6. *If τ is a face of σ, viewing $\mathcal{A}_\tau$ naturally as an open substack of $\mathcal{A}_\sigma$ we then have $\mathcal{Z}_\tau\cong\psi_\sigma^{-1}(\mathcal{A}_\tau)$, and the section $\mathbf{u}_\sigma\in\Gamma(\mathcal{Z}_\sigma,\overline{\mathcal{M}}^*_{\mathcal{Z}_\sigma})$ pulls back to the section $\mathbf{u}_\tau\in\Gamma(\mathcal{Z}_\tau,\overline{\mathcal{M}}^*_{\mathcal{Z}_\tau})$.*

Proof. The statement is immediate from the universal property stated in Proposition A.5. ∎

A.2 Family of contact orders of Zariski Artin fans

We now consider the case of an Artin fan $\mathcal{A}_X$. Recall that $\mathcal{A}_X$ has an étale cover by Artin cones. It was constructed in [5, Proposition 3.1.1] as a colimit of Artin cones $\mathcal{A}_\sigma$, viewed as sheaves over Log.

Definition A.7. We say that the Artin fan $\mathcal{A}_X$ is *Zariski* if it admits a Zariski cover by Artin cones.

A sufficient condition for $\mathcal{A}_X$ to be Zariski is that X is simple, because then $\mathcal{A}_X$ is the Artin fan associated to the ordinary cone complex $\Sigma(X)$ [14, Theorem 6.11]. Proposition C.11 shows that X is simple provided X has Zariski log structure and is log smooth over a simple B. The case B a trivial log point has previously been treated in [3, Lemma 2.6].

Fix a Zariski Artin fan $\mathcal{A}_X$. Let $\mathcal{Z}$ be the colimit of the $\mathcal{Z}_\sigma$ viewed as sheaves over $\mathcal{A}_X$. Note that $\mathcal{Z}$ is obtained by gluing together the local model $\mathcal{Z}_\sigma$ for each Zariski open $\mathcal{A}_\sigma \subseteq \mathcal{A}_X$ via the canonical identification given by Corollary A.6.[1]

The following proposition classifies contact orders on $\mathcal{A}_X$ by globalizing Proposition A.5.

Proposition A.8. *There is a section $\mathbf{u}_{\mathcal{X}} \in \Gamma(\mathcal{Z}, \bar{\mathcal{M}}^*_{\mathcal{Z}})$ making $\mathcal{Z}$ into a family of contact orders for $\mathcal{A}_X$. This family of contact orders is universal in the sense that for any family of contact orders $\mathbf{u} \in \Gamma(Z, \bar{\mathcal{M}}^*_Z)$ of $\mathcal{A}_X$, $\psi : Z \to \mathcal{A}_X$, there is a unique factorization of ψ through $\mathcal{Z} \to \mathcal{A}_X$ such that $\mathbf{u}$ is the pullback of $\mathbf{u}_{\mathcal{X}}$.*

Proof. If $\mathcal{A}_\sigma \to \mathcal{A}_X$ is a Zariski open set, then by the construction of $\mathcal{Z}$,

$$\mathcal{Z} \times_{\mathcal{A}_X} \mathcal{A}_\sigma = \mathcal{Z}_\sigma.$$

By Corollary A.6, the sections $\mathbf{u}_\sigma$ glue to give a section $\mathbf{u}_{\mathcal{X}} \in \Gamma(\mathcal{Z}, \bar{\mathcal{M}}^*_{\mathcal{Z}})$, yielding a family of contact orders in $\mathcal{A}_X$.

Consider a family of contact orders $Z \to \mathcal{A}_X$, $\mathbf{u}$. To show the desired factorization, it suffices to prove the existence and uniqueness locally on each Zariski open subset $\mathcal{A}_\sigma \to \mathcal{A}_X$, which follows from Proposition A.5. ∎

Definition A.9. A *connected contact order* for X is a choice of connected component of $\mathcal{Z}$.

We end this discussion with a couple of properties of the space $\mathcal{Z}$ of contact orders.

[1] It should be possible to carry this process out for more general Artin fans.

Proposition A.10. *Suppose that the Artin fan $\mathcal{A}_X$ of X is Zariski. There is a one-to-one correspondence between irreducible components of $\mathcal{Z}$ and pairs (u, σ) where $\sigma \in \Sigma(X)$ is a minimal cone such that $u \in \sigma^{\mathrm{gp}}$.*

Proof. Since $\mathcal{Z}_\sigma \subseteq \mathcal{Z}$ is Zariski open, an irreducible component of $\mathcal{Z}$ is the closure of an irreducible component of some $\mathcal{Z}_\sigma$, so we may assume $\mathcal{A}_X = \mathcal{A}_\sigma$. Then the statement follows from the description of $\mathcal{Z}_{u,\sigma}$ in Lemma A.3. ■

Remark A.11. Note that if $u \in N_\sigma$ with $u \in \sigma$ or $-u \in \sigma$, then $\mathcal{Z}_{u,\sigma}$ is irreducible and reduced. In fact, topologically $\mathcal{Z}_{u,\sigma}$ is the closure of the stratum $\mathcal{Z}_{\tau \prec \sigma}$ where $\tau \subseteq \sigma$ is the minimal face containing u. Further, the ideal generated by $u^{-1}(\mathbb{Z} \setminus \{0\})$ is precisely $P \setminus F_\tau$, so that $\mathcal{Z}_{u,\sigma}$ is reduced. In the case that $u \in \sigma$, it is the contact orders associated to ordinary marked points, as developed in [2, 15, 30].

For a simple non-reduced example let $P = \sigma_{\mathbb{Z}}^\vee$ be the submonoid of $\mathbb{N}^2$ generated by $(e, 0)$, $(0, e)$, $(1, 1)$ and $u : P \to \mathbb{Z}$ given by $u(a, b) = a - b$. Then I_u is generated by $(e, 0)$, $(0, e)$, and $\Bbbk[P]/I_u \simeq \Bbbk[t]/(t^e)$ is non-reduced for $e > 1$.

Thus the situation for more general contact orders associated to punctures may be more complex than that for marked points.

Example A.12. Even in the Zariski case, there may be monodromy which creates a difference between the point of view taken on contact orders in this appendix and that taken in Section 2.4. See Example 2.38 for a simple example with monodromy. There, taking $u = (0, 1, 0)$ as a tangent vector to any of the top-dimensional cones of $\Sigma(X)$, the corresponding connected contact order is a double cover of a one-dimensional closed subscheme of X. Explicitly, X contains ℓ strata isomorphic to $\mathbb{P}^1$, forming a cycle, i.e., a nodal elliptic curve. Then u induces a family of contact orders $Z \to X$ which is a double cover of this elliptic curve. This curve has 2ℓ irreducible components, in one-to-one correspondence with the set of pairs of the form (u, σ) and $(-u, \sigma)$ for σ running over two-dimensional cones of $\Sigma(X)$ tangent to u.

Example 2.39 provides an X with $\mathcal{A}_X$ Zariski where a similar monodromy produces connected contact orders with an infinite number of irreducible components. In this case one sees connected components of moduli spaces of punctured maps with an infinity of irreducible components.

By the discussion in Remark A.11 above, additional hypotheses are usually needed to obtain good control of moduli spaces of punctured maps. Here is a simple criterion that often suffices in practice.

Proposition A.13. *Suppose $\overline{\mathcal{M}}_X^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is generated by its global sections. Assume further X quasi-compact, with locally connected logarithmic strata. Then every connected component of contact orders of $\mathcal{A}_X$ has finitely many irreducible components.*

Proof. Let $V = \Gamma(\underline{X}, \overline{\mathcal{M}}_X^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{R})$, so that the induced map $|\Sigma(X)| \to V^*$ is injective on each $\sigma \in \Sigma(X)$ as in Proposition 3.13. Suppose $\mathbf{u} \in \Gamma(\mathcal{Z}, \overline{\mathcal{M}}_{\mathcal{Z}}^*)$ is a connected component of contact orders of $\mathcal{A}_X$. Denote the composition $V \to \overline{\mathcal{M}}_{\mathcal{Z}}^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\mathbf{u}} \mathbb{R}$ by $v \in V^*$. For each irreducible component of $\mathcal{Z}$, its corresponding vector u as in Proposition A.10 is then uniquely determined by v. By Proposition A.10 again $\mathcal{Z}$ has finitely many irreducible components, as $\Sigma(X)$ has finitely many cones by quasi-compactness. ■

A.3 Connection with the global contact orders of Section 2.4

We continue to work with a Zariski X. For simplicity in this discussion, let us also assume that X is log smooth over $\operatorname{Spec} \Bbbk$, so in particular associated to any $\sigma \in \Sigma(X)$ is a closed stratum $X_\sigma \subseteq X$ such that the dual cone of the stalk of $\overline{\mathcal{M}}_X$ at the generic point of X_σ is σ.

In this case, $\mathcal{A}_X$ is Zariski, and $\mathcal{A}_{X_\sigma}$, the Artin fan of X_σ, is also Zariski. Let $\mathcal{Z}^\sigma \to \mathcal{A}_{X_\sigma}$ be the universal family of contact orders for $\mathcal{A}_{X_\sigma}$. Further, write $\mathcal{X}_\sigma$ for the reduced closed stratum of $\mathcal{A}_{X_\sigma}$ corresponding to σ. In this situation, we have:

Proposition A.14. *There is a one-to-one correspondence between $\mathfrak{C}_\sigma(X)$ and the set of connected components of $\mathcal{Z}^\sigma \times_{\mathcal{A}_{X_\sigma}} \mathcal{X}_\sigma$.*

Proof. By the construction of the colimit of sets, $\mathfrak{C}_\sigma(X)$ is the quotient of the set $\coprod_{\sigma \prec \sigma' \in \Sigma(X)} N_{\sigma'}$ by the equivalence relation $\sim$ generated by the following set of relations. Whenever given inclusions of faces $\sigma \prec \sigma' \prec \sigma''$ in $\Sigma(X)$, one obtains an induced map $\iota_{\sigma'\sigma''} : N_{\sigma'} \to N_{\sigma''}$. Then for $x \in N_{\sigma'}$, we have $x \sim \iota_{\sigma'\sigma''}(x)$.

On the other hand, we may cover $\mathcal{A}_{X_\sigma}$ with Zariski open sets $\mathcal{A}_{\sigma'}$ with σ' running over $\sigma' \in \Sigma(X)$ with $\sigma \prec \sigma'$. Note that by the construction of the universal contact order of $\mathcal{A}_{\sigma'}$, there is a one-to-one correspondence between $N_{\sigma'}$ and the set of connected components of $\mathcal{Z}_{\sigma'} = \mathcal{Z}^\sigma \times_{\mathcal{A}_{X_\sigma}} \mathcal{A}_{\sigma'}$, with $u \in N_{\sigma'}$ corresponding to $\mathcal{Z}_{u,\sigma'}$. Note the same is then true of the set of connected components of $\mathcal{Z}_{\sigma'} \times_{\mathcal{A}_{X_\sigma}} \mathcal{X}_\sigma$, with $u \in N_{\sigma'}$ corresponding to $\mathcal{Z}_{u,\sigma'} \times_{\mathcal{A}_{X_\sigma}} \mathcal{X}_\sigma$.

Define another equivalence relation $\approx$ on $\coprod_{\sigma \prec \sigma'} N_{\sigma'}$ as follows. Suppose $u' \in N_{\sigma'}$, $u'' \in N_{\sigma''}$. Then $u' \approx u''$ if $\mathcal{Z}_{u',\sigma'} \times_{\mathcal{A}_{X_\sigma}} \mathcal{X}_\sigma$ and $\mathcal{Z}_{u'',\sigma''} \times_{\mathcal{A}_{X_\sigma}} \mathcal{X}_\sigma$ are open substacks of the same connected component of $\mathcal{Z}^\sigma \times_{\mathcal{A}_{X_\sigma}} \mathcal{X}_\sigma$. The statement follows once we show that the two equivalence relations $\sim$ and $\approx$ are equal.

Note that $\mathcal{A}_{\sigma'} \cap \mathcal{A}_{\sigma''} \cap \mathcal{X}_\sigma$ may be covered with sets $\mathcal{A}_\tau \cap \mathcal{X}_\sigma$ with $\tau \in \Sigma(X)$ running over those τ with $\sigma \prec \tau \prec \sigma' \cap \sigma''$. This makes it clear that $\approx$ is also generated by the following relations. Suppose given $\sigma \prec \sigma' \prec \sigma''$ with $u' \in N_{\sigma'}$, $u'' \in N_{\sigma''}$. Then because of the inclusion $\mathcal{Z}_{\sigma'} \subseteq \mathcal{Z}_{\sigma''}$ of Corollary A.6, $\mathcal{Z}_{\sigma',u'} \times_{\mathcal{A}_{X_\sigma}} \mathcal{X}_\sigma$ and $\mathcal{Z}_{\sigma'',u''} \times_{\mathcal{A}_{X_\sigma}} \mathcal{X}_\sigma$ may be both viewed as open substacks of $\mathcal{Z}^\sigma \times_{\mathcal{A}_{X_\sigma}} (\mathcal{X}_\sigma \times_{\mathcal{A}_{X_\sigma}}$

$\mathscr{A}_{\sigma''}$). If these two open substacks are not disjoint, then $u' \approx u''$. However, it follows from Proposition A.5 that this is the case precisely when $\iota_{\sigma'\sigma''}(u') = u''$. Thus $\sim$ and $\approx$ are the same equivalence relation, since they are generated by the same set of relations. ■