Appendix B

Charts for morphisms of log stacks

We discuss here properties of charts of morphisms of algebraic log stacks, due to a lack of a good reference. There are many standard results involving existence and properties of charts for morphisms between fs log schemes étale locally, e.g., [52, Section II.2], as well as local descriptions of log smooth or étale morphisms, e.g., [52, Section IV.3.3]. However, to apply these results to morphisms of log stacks, one would need to pass to smooth neighborhoods, which destroys any discussion of the more delicate condition of being log étale. Thus it is far more convenient to think of charts as being given by maps to toric stacks rather than toric varieties. The results of this appendix are used in Section 3.5 to describe local models for our moduli spaces of punctured maps to Artin fans, but are also used extensively elsewhere, e.g., in [33].

Here we fix a ground field $\Bbbk$ of characteristic 0, as usual, and all schemes and stacks are defined over $\operatorname{Spec}\Bbbk$. We define, given P a fine monoid and $K \subseteq P$ a monoid ideal,

$$\mathcal{A}_P := [\operatorname{Spec}\Bbbk[P]/\operatorname{Spec}\Bbbk[P^{\mathrm{gp}}]], \quad \mathcal{A}_{P,K} = [(\operatorname{Spec}\Bbbk[P]/K)/\operatorname{Spec}\Bbbk[P^{\mathrm{gp}}]]. \quad \text{(B.1)}$$

Here both stacks carry a canonical log structure coming from P, and the second stack carries a canonical idealized log structure induced by the monoid ideal K, see [52, Section III.1.3].

Remark B.1. If P is a fine monoid, define $\overline{P} = P/P^\times$. Then $\mathcal{A}_P \cong \mathcal{A}_{\overline{P}}$.

Proposition B.2. *Let $f : X \to Y$ be a morphism of (idealized) fs log stacks over* $\operatorname{Spec}\Bbbk$, *with coherent sheaves of ideals $\mathcal{K}_X$ and $\mathcal{K}_Y$ in the idealized case. Let $\bar{x}$ be a geometric point of $\underline{X}$, $\bar{y} = f(\bar{x})$, $P = \overline{\mathcal{M}}_{X,\bar{x}}$, $Q = \overline{\mathcal{M}}_{Y,\bar{y}}$, ($K = \overline{\mathcal{K}}_{X,\bar{x}}$, $J = \overline{\mathcal{K}}_{Y,\bar{y}}$ in the idealized case). Then in the two cases, there are strict étale neighborhoods X' and Y' of $\bar{x}$ and $\bar{y}$ respectively and commutative diagrams*

$$\begin{array}{ccc} X' & \longrightarrow & \mathcal{A}_P \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & \mathcal{A}_Q \end{array} \qquad\qquad \begin{array}{ccc} X' & \longrightarrow & \mathcal{A}_{P,K} \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & \mathcal{A}_{Q,J} \end{array}$$

with horizontal arrows (idealized) strict. If further Y is already equipped with a strict morphism $Y \to \mathcal{A}_Q$, we may take $Y = Y'$.

Proof. If Y is equipped with a strict morphism $Y \to \mathcal{A}_Q$, we take $Y = Y'$. Otherwise, there is a tautological morphism $Y \to \mathrm{Log}_\Bbbk$. By [53, Corollary 5.25], $\mathrm{Log}_\Bbbk$ has a strict étale cover by stacks of the form $\mathcal{A}_Q$ for various monoids Q. Thus we may choose a strict étale morphism $\mathcal{A}_Q \to \mathrm{Log}_{\Bbbk}$ whose image contains the image of $\bar{y}$, and take Y' to be the étale neighborhood $Y \times_{\mathrm{Log}_{\Bbbk}} \mathcal{A}_Q$ of $\bar{y}$. If the image of $\bar{y}$ in $\mathcal{A}_Q$ is not the deepest stratum of $\mathcal{A}_Q$, we may replace $\mathcal{A}_Q$ with a Zariski open subset of the form $\mathcal{A}_{Q'}$ where Q' is a localization of Q along some face and such that $\bar{y}$ maps to the deepest stratum of $\mathcal{A}_{Q'}$. By Remark B.1, $\mathcal{A}_{Q'} \cong \mathcal{A}_{Q'/(Q')^\times}$, so we may assume that $Q = \overline{\mathcal{M}}_{Y,\bar{y}}$.

Let $X'' = X \times_Y Y'$ be the corresponding étale neighborhood of $\bar{x}$. Similarly, we have a tautological strict morphism $X'' \to \mathrm{Log}_{Y'}$. Now $\mathrm{Log}_{Y'}$ can be covered by strict étale morphisms of the form $\mathcal{A}_P \times_{\mathcal{A}_Q} Y' \to \mathrm{Log}_{Y'}$ for various P such that the projection to $\mathcal{A}_P$ is strict, again by [53, Corollary 5.25]. Here we range over fs monoids P and morphisms $\theta : Q \to P$. Take $X' = X'' \times_{\mathrm{Log}_{Y'}} (\mathcal{A}_P \times_{\mathcal{A}_Q} Y')$ for a suitable choice of P and $\theta : Q \to P$ so that X' is an étale neighborhood of $\bar{x}$. The projection of this stack to $\mathcal{A}_P$ then yields the desired strict morphism $X' \to \mathcal{A}_P$ making the diagram commutative. As before, we can pass to a Zariski open substack of $\mathcal{A}_P$ to be able to assume that $P = \overline{\mathcal{M}}_{X,\bar{x}}$.

In the idealized case, the morphisms $X' \to \mathcal{A}_P$, $Y' \to \mathcal{A}_Q$ factor through $\mathcal{A}_{P,K}$ and $\mathcal{A}_{Q,J}$ respectively, and the factored morphisms are idealized strict. ∎

Lemma B.3. *Let $\theta : Q \to P$ be a morphism of fs monoids, $J \subseteq Q$, $K \subseteq P$ with $\theta(J) \subseteq K$. Then the induced morphism $\theta : \mathcal{A}_{P,K} \to \mathcal{A}_{Q,J}$ is idealized log étale.*

Proof. The morphism θ is clearly locally of finite presentation, so we need only verify the formal lifting criterion. Suppose given a diagram

$$\begin{array}{ccccc} T_0 & \xrightarrow{g_0} & \mathcal{A}_{P,K} & \longrightarrow & \mathcal{A}_P \\ {\scriptstyle i}\downarrow & \overset{g'}{\nearrow} & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta} \\ T & \xrightarrow[g]{} & \mathcal{A}_{Q,J} & \longrightarrow & \mathcal{A}_Q \end{array}$$

Here i is a strict and idealized strict closed immersion with ideal sheaf having square zero, and the right-hand horizontal arrows are strict, but not idealized strict, closed immersions, with the right-hand square commutative. We wish to show there is a unique g' making the diagram commute.

By [53, Corollary 5.23], $\mathcal{A}_P \to \mathcal{A}_Q$ is log étale. Thus forgetting the idealized structure on T, we obtain a unique morphism $h : T \to \mathcal{A}_P$ in the above diagram making everything commute. Let $\mathcal{K}_{T_0}$, $\mathcal{K}_T$ be the coherent sheaves of monoid ideals for T_0 and T respectively giving the idealized structure. Note $\mathcal{K}_{T_0}$ is the pullback of a sheaf of ideals $\overline{\mathcal{K}}_{T_0} \subseteq \overline{\mathcal{M}}_{T_0}$ under the projection $\mathcal{M}_{T_0} \to \overline{\mathcal{M}}_{T_0}$, and similarly for T. By strictness, $\overline{\mathcal{K}}_{T_0} = \overline{\mathcal{K}}_T$. Since $\bar{g}_0^\flat(K) \subseteq \overline{\mathcal{K}}_{T_0}$, as g_0 is an idealized morphism, we have by commutativity that $\bar{h}^\flat(K) \subseteq \overline{\mathcal{K}}_T$. Hence α_T vanishes on any lift to $\mathcal{M}_T$ of an element $\bar{h}^\flat(k)$ for $k \in K$. It then follows that h factors through the closed immersion $\mathcal{A}_{P,K} \to \mathcal{A}_P$, and this factorization yields the unique lifting g'. ∎

Proposition B.4. *With the hypotheses of Proposition B.2, suppose in addition that f is log smooth (resp. log étale, idealized log smooth, idealized log étale). Then in the non-idealized case, the induced morphism $X' \to Y' \times_{\mathcal{A}_Q} \mathcal{A}_P$ is smooth (resp. étale) and in the idealized case, the induced morphism $X' \to Y' \times_{\mathcal{A}_{Q,J}} \mathcal{A}_{P,K}$ is smooth (resp. étale).*

Proof. In the non-idealized case, [53, Theorem 4.6], shows that $X \to Y$ is log smooth (étale) if and only if the tautological morphism $\underline{X} \to \mathrm{Log}_Y$ is smooth (étale). It follows immediately by base-change from the construction of the proof of Proposition B.2 that, if f is log smooth (étale), the morphism $X'' \to Y'$ is log smooth (étale) and hence $\underline{X}'' \to \mathrm{Log}_{Y'}$ is smooth (étale). Thus by another base-change, we see that the projection $X' \to Y' \times_{\mathcal{A}_Q} \mathcal{A}_P$ is smooth (étale).

The idealized case requires a little bit more work because the analogous statement for idealized log smooth (étale) morphisms does not seem to appear in the literature. First, by [53, Lemma 4.8], $X'' \to \mathrm{Log}_{Y'}$ is locally of finite presentation, as $X \to Y$ is locally of finite presentation, being idealized log étale, and thus $X' \to Y' \times_{\mathcal{A}_Q} \mathcal{A}_P$ is locally of finite presentation. However, as in the proof of Proposition B.2, $X' \to Y' \times_{\mathcal{A}_Q} \mathcal{A}_P$ factors through the closed immersion $Y' \times_{\mathcal{A}_{Q,J}} \mathcal{A}_{P,K} \hookrightarrow Y' \times_{\mathcal{A}_Q} \mathcal{A}_P$, and thus $X' \to Y' \times_{\mathcal{A}_{Q,J}} \mathcal{A}_{P,K}$ is also locally of finite presentation. So we just need to show the formal lifting criterion, i.e., given a diagram

$$
\begin{array}{ccc}
\underline{T}_0 & \xrightarrow{\underline{g}_0} & \underline{X}' \\
{\scriptstyle \underline{i}}\downarrow & \nearrow & \downarrow{\scriptstyle \underline{f}'} \\
\underline{T} & \xrightarrow[\underline{g}]{} & \underline{Y' \times_{\mathcal{A}_{Q,J}} \mathcal{A}_{P,K}}
\end{array}
\tag{B.2}
$$

where i is a closed immersion with ideal of square zero, there is, étale locally on $\underline{T}_0$, a dotted line as indicated, unique in the étale case. Give $\underline{T}_0$ and $\underline{T}$ the idealized log structure making all arrows in the above square strict and idealized strict. Via composition of g with the projection to Y', we obtain a diagram

$$
\begin{array}{ccc}
T_0 & \xrightarrow{g_0} & X' \\
{\scriptstyle i}\downarrow & \overset{g'}{\nearrow} & \downarrow \\
T & \xrightarrow[g'']{} & Y'
\end{array}
$$

Formal idealized log smoothness (idealized log étaleness) then implies, étale locally, a (unique) lift g'. It is then sufficient to show that $f' \circ g'$ coincides with g in (B.2).

However, by Lemma B.3, the projection $Y' \times_{\mathcal{A}_{Q,J}} \mathcal{A}_{P,K} \to Y'$ is idealized log étale, and hence by uniqueness in the formal lifting criterion for idealized log étale morphisms, $f' \circ g' = g$. ■