Appendix C

Functorial tropicalization and the category of points

Various definitions of tropicalization in logarithmic geometry are available in the literature [1, 3, 14, 30, 42, 69]. The purpose of this appendix is to spell out the construction of tropicalization as a functor from the category of fine log algebraic stacks to the category of generalized cone complexes generalizing [69, Proposition 6.3] to cases with monodromy, and closer in spirit to [30, Appendix B]. This refines the discussion in [3, Section 2.1].

We adopt the definition from [43, Section II.1], [64, Section 2], [1, Section 2.2] of a *generalized cone complex* Σ as a topological space $|\Sigma|$ together with a *presentation* given by a homeomorphism with the colimit in the category of topological spaces of a diagram in **Cones** with all arrows face morphisms. Here we use the topology induced by embedding a cone σ into its vector space $N_\sigma \otimes_{\mathbb{Z}} \mathbb{R}$. For any cone σ in a presentation we always include all face embeddings $\tau \to \sigma$ in the diagram. The *strata* of $|\Sigma|$ are the images of the interiors of cones from the presentation. We consider generalized cone complexes up to equivalence generated by adding more cones to a presentation. A morphism of cone complexes $\Sigma \to \Sigma'$ is given by a continuous map $|\Sigma| \to |\Sigma'|$ that locally lifts to a morphism of diagrams of presentations. Unlike the cited references, we do not impose any finiteness conditions since we want to admit situations with infinitely many strata.

C.1 Tropicalization of fine log schemes

We begin by recalling the definition of the category of geometric points $\mathbf{Pt}(X)$ of a scheme X with arrows defined by specialization, following [9, Section VIII.7], see also [67, Section 0GJ2]. An object in $\mathbf{Pt}(X)$ is a morphism $\bar{x} : \operatorname{Spec} \kappa \to X$ with $\kappa = \kappa(\bar{x})$ an algebraically closed field. Given $\bar{x}$ we have the associated local scheme $X(\bar{x}) = \operatorname{Spec} \mathcal{O}_{X,\bar{x}}$. A *specialization* arrow $\bar{x} \to \bar{y}$ is an X-morphism $\operatorname{Spec} \kappa(\bar{x}) \to X(\bar{y})$ or, equivalently by [9, Section VIII.7, Proposition 7.4], an X-morphism $X(\bar{x}) \to X(\bar{y})$.

Composition with a morphism $f : X \to Y$ defines a functor

$$f_* : \mathbf{Pt}(X) \to \mathbf{Pt}(Y)$$

compatible with composition, so **Pt** is a functor from the category of schemes to the category of categories **Cat**.

For each étale sheaf of sets $\mathcal{F}$ on X, a specialization arrow $\bar{x} \to \bar{y}$ in $\mathbf{Pt}(X)$ induces a generization map[1]

$$\mathcal{F}_{\bar{y}} \to \mathcal{F}_{\bar{x}}. \tag{C.1}$$

This assignment is compatible with morphisms of sheaves. Thus if $\mathbf{Sh}(X_{\text{ét}})$ denotes the étale topos of X, we obtain a functor

$$\text{Stalks} : \mathbf{Sh}(X_{\text{ét}}) \to \text{Func}(\mathbf{Pt}(X)^{\text{op}}, \mathbf{Sets}) \tag{C.2}$$

associating to an étale sheaf its functor of stalks, a diagram in **Sets** indexed by $\mathbf{Pt}(X)^{\text{op}}$. We emphasize that the generization homomorphism (C.1) does not only depend on $\bar{x}$, $\bar{y}$, but on the choice of specialization arrow $\bar{x} \to \bar{y}$.

Example C.1. Let C be the nodal cubic. If $\overline{\eta}$ denotes a geometric generic point and $\bar{x}$ a geometric point over the node, there are two different C-morphisms

$$\operatorname{Spec} \kappa(\overline{\eta}) \to X(\bar{x})$$

that reflect the specialization along the two branches of C at $\bar{x}$. This statement can most easily be seen by going over to the usual two-fold étale cover $\pi : \widetilde{C} \to C$, and observing that each of the two lifts $\tilde{\bar{x}} \in \mathbf{Pt}(\widetilde{C})$ of $\bar{x}$ has generization homomorphisms to both lifts of $\overline{\eta}$.

Charts for the log structure define a locally finite stratification of $\underline{X}$ with a *stratum* a maximal connected locally closed subset $Z \subseteq |\underline{X}|$ with $\overline{\mathcal{M}}_X|_Z$ locally constant. Denote by $\text{Strata}(X)$ the set of strata of X. For each $Z \in \text{Strata}(X)$ choose a geometric point $\bar{x} = \bar{x}_Z$ of Z and define

$$\sigma_Z = \text{Hom}\big(\overline{\mathcal{M}}_{X,\bar{x}}, \mathbb{R}_{\geq 0}\big) \in \mathbf{Cones}. \tag{C.3}$$

Different choices of $\bar{x}$ lead to isomorphic σ_Z, but the isomorphism is only unique up to the monodromy action of the étale fundamental group $\pi_1(Z, \bar{x})$ of the stratum on $\overline{\mathcal{M}}_{X,\bar{x}}$. More precisely, since the automorphism group of a fine monoid is finite, arguing with [67, Lemma 0DV5] shows the following. There exists a finite connected étale Galois cover

$$f : \widetilde{Z} \to Z$$

with $f^{-1}\overline{\mathcal{M}}_X$ a constant sheaf. Lifting $\bar{x}$ to $\widetilde{Z}$ yields an isomorphism

$$\Gamma\big(\widetilde{Z}, f^{-1}\overline{\mathcal{M}}_X\big) \xrightarrow{\simeq} \overline{\mathcal{M}}_{X,\bar{x}} \tag{C.4}$$

[1] We prefer "generization map" over the common "specialization map" in this context since the map goes from the stalk at the more special point to the stalk at the more generic point.

by restriction. Now by definition, $\pi_1(Z, \bar{x})$ acts on f, and the induced action on $\Gamma(\widetilde{Z}, f^{-1}\bar{\mathcal{M}}_X)$ by pullback corresponds to the action of $\pi_1(Z, \bar{x})$ on $\bar{\mathcal{M}}_{X,\bar{x}}$ via (C.4). The minimal choice of f with $f^{-1}\bar{\mathcal{M}}_X$ a constant sheaf has connected $\widetilde{Z}$ and is a Galois cover. Moreover, in the minimal case, the action of $\pi_1(Z, \bar{x})$ on $\bar{\mathcal{M}}_{X,\bar{x}}$ factors over a faithful action of the Galois group $\mathrm{Aut}(\widetilde{Z}/Z)$.

For each stratum Z with chosen geometric point $\bar{x} = \bar{x}_Z$ denote by

$$G_Z \subseteq \mathrm{Aut}(\bar{\mathcal{M}}_{X,\bar{x}}) \tag{C.5}$$

the image of the monodromy action of $\pi_1(Z, \bar{x})$ on $\bar{\mathcal{M}}_{X,\bar{x}}$. By the previous discussion, $G_Z \simeq \mathrm{Aut}(\widetilde{Z}/Z)$ for any minimal connected Galois cover $f : \widetilde{Z} \to Z$ with $f^{-1}\bar{\mathcal{M}}_X$ a constant sheaf.

Now if $W \in \mathrm{Strata}(X)$ is another stratum, and $\bar{w}$ is a geometric point of $W \cap \mathrm{cl}(Z)$, there exists a geometric point $\bar{\eta}$ of Z and a specialization arrow $\chi : \bar{\eta} \to \bar{w}$ [67, Section 0BUP], hence a generization homomorphism $\bar{\mathcal{M}}_{X,\bar{w}} \to \bar{\mathcal{M}}_{X,\bar{\eta}}$. Since $\bar{\mathcal{M}}_X$ is locally constant on the strata there are also isomorphisms

$$\bar{\mathcal{M}}_{X,\bar{w}} \xrightarrow{\simeq} \bar{\mathcal{M}}_{\bar{x}_W}, \quad \bar{\mathcal{M}}_{X,\bar{\eta}} \xrightarrow{\simeq} \bar{\mathcal{M}}_{\bar{x}_Z}, \tag{C.6}$$

for $\bar{x}_Z$, $\bar{x}_W$ the chosen reference points for the two strata. These isomorphisms are unique up to composing with elements of G_W and G_Z, respectively. We call any morphism

$$\iota : \sigma_Z \to \sigma_W \tag{C.7}$$

obtained by applying $\mathrm{Hom}(\cdot, \mathbb{R}_{\geq 0})$ to any of the compositions

$$\bar{\mathcal{M}}_{\bar{x}_W} \xrightarrow{\simeq} \bar{\mathcal{M}}_{X,\bar{w}} \xrightarrow{\chi} \bar{\mathcal{M}}_{X,\bar{\eta}} \xrightarrow{\simeq} \bar{\mathcal{M}}_{\bar{x}_Z}$$

a *specialization morphism* or *specialization arrow*. Note that ι also depends on the choice of $\bar{w}$, and hence the actions of G_Z and G_W on the set of specialization arrows may not be transitive. For $Z = W$ the set of specialization arrows equals $G_Z = G_W$.

If $f : X \to Y$ is a morphism of fine log schemes, $Z \in \mathrm{Strata}(X)$ and $f(\bar{x}_Z)$ a geometric point of $Z' \in \mathrm{Strata}(Y)$, then $\bar{f}^\flat : f^{-1}\bar{\mathcal{M}}_Y \to \bar{\mathcal{M}}_X$ together with a choice of isomorphism $\bar{\mathcal{M}}_{Y,f(\bar{x}_Z)} \simeq \bar{\mathcal{M}}_{Y,\bar{x}_{Z'}}$ in (C.6) defines a morphism

$$\varphi : \sigma_Z \to \sigma_{Z'} \tag{C.8}$$

in **Cones** by the composition

$$\mathrm{Hom}\big(\bar{\mathcal{M}}_{X,\bar{x}_Z}, \mathbb{R}_{\geq 0}\big) \to \mathrm{Hom}\big(\bar{\mathcal{M}}_{Y,f(\bar{x}_Z)}, \mathbb{R}_{\geq 0}\big) \xrightarrow{\simeq} \mathrm{Hom}\big(\bar{\mathcal{M}}_{Y,\bar{x}_{Z'}}, \mathbb{R}_{\geq 0}\big).$$

Note such φ are not in general face morphisms. The set of all such arrows is compatible with specialization in the sense that if $\iota : \sigma_Z \to \sigma_W$ is a specialization morphism (C.7) in X then there exists a specialization morphism $\iota' : \sigma_{Z'} \to \sigma_{W'}$ in Y and morphisms $\varphi : \sigma_Z \to \sigma_{Z'}$, $\psi : \sigma_W \to \sigma_{W'}$ as in (C.8) making the following diagram commute:

$$\begin{array}{ccc} \sigma_Z & \xrightarrow{\iota} & \sigma_W \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} \\ \sigma_{Z'} & \xrightarrow{\iota'} & \sigma_{W'}. \end{array} \tag{C.9}$$

We are then in position to define the tropicalization of X as a generalized cone complex.

Definition C.2. Let $X = (\underline{X}, \mathcal{M}_X)$ be a fine log scheme. The *tropicalization* $\Sigma(X)$ of X is the generalized cone complex defined by the diagram in **Cones** with one object σ_Z from (C.3) for each stratum $Z \subset X$ and face morphisms $\sigma_Z \to \sigma_W$ the set of specialization morphisms from (C.7).

A morphism $f : X \to Y$ of fine log schemes induces the morphism

$$\Sigma(f) : \Sigma(X) \to \Sigma(Y)$$

defined by all arrows $\varphi : \sigma_Z \to \sigma_{Z'}$ as in (C.8).

Note that diagrams of specialization arrows as in (C.9) show that the map of topological spaces $|\Sigma(X)| \to |\Sigma(Y)|$ is well defined and continuous, and that it lifts locally to a morphism of presentations. Thus $\Sigma(f)$ indeed is a morphism of generalized cone complexes.

We need to check that our definition of tropicalization does not depend on the choices of a geometric point $\bar{x}_Z$ for each stratum Z of X.

Lemma C.3. *The definition of tropicalization in Definition C.2 is independent of choices.*

Proof. Let Z be a logarithmic stratum of X and $\bar{x}'_Z$ another choice of geometric point. Since $\bar{\mathcal{M}}_X|_Z$ is locally constant there exists an isomorphism

$$\varphi : \sigma_Z = \mathrm{Hom}\big(\bar{\mathcal{M}}_{X,\bar{x}_Z}, \mathbb{R}_{\geq 0}\big) \xrightarrow{\simeq} \sigma'_Z = \mathrm{Hom}\big(\bar{\mathcal{M}}_{X,\bar{x}'_Z}, \mathbb{R}_{\geq 0}\big)$$

that is unique up to the action of $\pi_1(Z)$ on σ_Z. Replacing σ_Z by σ'_Z and all arrows involving σ_Z by composition with φ or φ^{-1} as appropriate, gives an alternative presentation of $|\Sigma(X)|$ as a colimit of a diagram in **Cones**. By construction, both diagrams are locally isomorphic, and hence they lead to the same generalized cone complex. This argument is local to each geometric point, thus also applies to any two different sets of choices of geometric points. ■

We finally check functoriality of this notion of tropicalization.

Proposition C.4. *If $f : X \to Y$ and $g : Y \to Z$ are morphisms of fine log schemes then $\Sigma(g \circ f) = \Sigma(g) \circ \Sigma(f)$.*

Proof. Given a specialization morphism $\iota : Z \to W$ of strata of X there exist two commutative diagrams of the form (C.9) with horizontal arrows specialization morphisms $\iota' : Z' \to W'$ and $\iota'' : Z'' \to W''$ of strata in Y and Z, respectively. The two small commutative squares now define the local liftings of $\Sigma(f)$ and $\Sigma(g)$ to presentations, while their composition defines the lifting of $\Sigma(g \circ f)$. The result is now obvious. ∎

Remark C.5. A canonical and obviously functorial definition of $\Sigma(X)$ runs as follows. The composition of the functor Stalks in (C.2) with $\mathrm{Hom}(\cdot, \mathbb{R}_{\geq 0})$ defines a diagram

$$\mathbf{Pt}(X)^{\mathrm{op}} \to \mathbf{Cones} \tag{C.10}$$

with all morphisms face inclusions. The reasoning in the proof of Lemma C.3 shows that the associated generalized cone complex is canonically isomorphic to $\Sigma(X)$. We preferred to base our definition on the more explicit treatment with one cone for each stratum.

Remark C.6. One might think that a slightly refined definition could also give a functorial notion of tropicalization as a diagram of cones associated to strata. This is, however, not the case. The problem appears already with locally constant sheaves in the étale topology, which can not be described by groupoids of sets obtained from the associated representations of the étale fundamental group. The étale fundamental group of a scheme X depends on the choice of a geometric point and is otherwise only defined up to non-unique isomorphism. Thus a functorial definition would have to involve at least a skeleton of $\mathbf{Pt}(X)$, and hence completely loses the combinatorial flavor of tropicalization.

C.2 Tropicalization of fine log algebraic stacks

Now let X be a fine log algebraic stack, with $\mathcal{M}_X$ and $\overline{\mathcal{M}}_X$ sheaves in the lisse-étale topology. To define the tropicalization $\Sigma(X)$ let

$$h : U \to X$$

be a strict smooth surjection from a log scheme. Then $U \times_X U$ is a scheme that comes with two projections to U. Tropicalizing defines a double arrow of generalized cone complexes

$$\Sigma(U \times_X U) \rightrightarrows \Sigma(U). \tag{C.11}$$

For a geometric point $\bar{x}$ of $U \times_X U$, composition with the two projections defines two geometric points $\bar{x}_1, \bar{x}_2$ of U. Since both projections $U \times_X U \to U$ are strict, we have two isomorphisms

$$\overline{\mathcal{M}}_{U,\bar{x}_i} \to \overline{\mathcal{M}}_{U \times_X U, \bar{x}}, \quad i = 1, 2. \tag{C.12}$$

These isomorphisms induce an equivalence relation on Strata(U), and provide isomorphisms between stalks of $\overline{\mathcal{M}}_U$ at pairs of geometric points in equivalent strata. The quotient Strata(U)$/\sim$ can easily be seen to be independent of the choice of smooth cover $U \to X$, and in fact defines the set Strata(X) of strata of the log algebraic stack X.

To define the tropicalization $\Sigma(X)$, we add $\mathrm{Hom}(\cdot, \mathbb{R}_{\geq 0})$ of the isomorphisms in (C.12) to the set of arrows in the diagram defining $\Sigma(U)$.

Definition C.7. The tropicalization $\Sigma(X)$ of the fine log algebraic stack X is the generalized cone complex defined by the diagram of $\Sigma(U)$ with the added isomorphisms induced by the tropicalization of (C.12).

Restricting the diagram defining $\Sigma(X)$ to one cone for each stratum of X gives an alternative presentation with index category Strata(X).

We need to check independence of our definition of $\Sigma(X)$ from choices.

Lemma C.8. *The definition of $\Sigma(X)$ is independent of the choice of strict smooth cover $U \to X$.*

Proof. It suffices to consider the composition of $U \to X$ with a strict smooth surjection $V \to U$. We obtain the following commutative diagram of strict smooth surjections of log schemes:

$$\begin{array}{ccc} V \times_X V & \longrightarrow & U \times_X U \\ \downarrow\downarrow & & \downarrow\downarrow \\ V & \longrightarrow & U \end{array} \tag{C.13}$$

Now all arrows are surjective on geometric points. Since smooth maps are open, all arrows are also surjective on the set of generizations. Thus each cone and arrow of $\Sigma(V)$ maps isomorphically to a cone or arrow of $\Sigma(U)$, and each cone or arrow of $\Sigma(U)$ arises as an image. Moreover, if two cones σ_1, σ_2 in $\Sigma(U)$ belong to the same stratum in X, that is, are isomorphic images of a cone σ in $\Sigma(U \times_X U)$ appearing from a geometric point in $U \times_X U$, then lifting this geometric point to $V \times_X V$ provides a cone $\widetilde{\sigma}$ in $\Sigma(V \times_X V)$ mapping to cones $\widetilde{\sigma}_1, \widetilde{\sigma}_2$ in $\Sigma(V)$. The tropicalization of (C.13) now shows that the diagram of cones

$$\begin{array}{ccccc} \widetilde{\sigma}_1 & \longleftarrow & \widetilde{\sigma} & \longrightarrow & \widetilde{\sigma}_2 \\ \downarrow & & \downarrow & & \downarrow \\ \sigma_1 & \longleftarrow & \sigma & \longrightarrow & \sigma_2 \end{array}$$

commutes up to composing the lower horizontal arrows with isomorphisms in $\Sigma(U)$.

Taken together we see that the diagram defining $\Sigma(X)$ from $V \to X$ just adds a number of isomorphic cones to the diagram defining $\Sigma(X)$ from $U \to X$. Thus the corresponding generalized cone complexes are equivalent. ■

The proof of functoriality of this notion of tropicalization now follows by local lifting to a presentation as in Proposition C.4. We omit the details.

Proposition C.9. *If $f : X \to Y$ and $g : Y \to Z$ are morphisms of fine log algebraic stacks then $\Sigma(g \circ f) = \Sigma(g) \circ \Sigma(f)$.*

C.3 Tropicalization in the log smooth case

We end this section with some facts on logarithmic strata and tropicalization in the Zariski log smooth case.

Lemma C.10. *Let $f : X \to B$ be a log smooth morphism of fine log schemes. Assume that B is locally noetherian with geometrically unibranch logarithmic strata. Then the logarithmic strata of X are irreducible and geometrically unibranch.*

Proof. First note that X is locally noetherian since B is locally noetherian and f is locally of finite presentation by the definition of log smoothness. Thus a locally irreducible connected subset of $|X|$ is irreducible. It thus suffices to show the stronger statement that each logarithmic stratum Z of X is geometrically unibranch.

Let $z \in |Z|$ and $Z_B \subseteq B$ the logarithmic stratum containing $\underline{f}(z)$. Being geometrically unibranch is a local property that is stable under étale morphisms. By [52, Theorem IV.3.3.1] we may thus replace X and B by étale neighborhoods of z and $\underline{f}(z)$ to obtain a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{g} & B \times_{A_Q} A_P & \xrightarrow{k} & A_P \\ & \searrow^{f} & \downarrow & & \downarrow{\scriptstyle A_\theta} \\ & & B & \xrightarrow{h} & A_Q \end{array}$$

with $A_P = \operatorname{Spec} \mathbb{Z}[P]$, $A_Q = \operatorname{Spec} \mathbb{Z}[Q]$, A_θ the morphism induced by a homomorphism $\theta : Q \to P$ of fine monoids, all horizontal arrrows strict, the square cartesian, g étale, and h a neat chart at z. Thus $Z_B = (h^{-1}(O))_{\text{red}}$, where $O \subseteq A_Q$ is the closed torus orbit defined by the monoid ideal $Q \setminus \{0\}$.

Since $k \circ g$ is strict, the composition $Z \to B \times_{A_Q} A_P \to A_P$ factors over the inclusion of a logarithmic stratum $Z_P \subseteq A_P$. Now toric morphisms respect the decomposition into logarithmic strata. Thus $\underline{A_\theta}(Z_P)$ is contained in a logarithmic stratum of A_Q. But $\underline{h}(\underline{f}(z)) \in \underline{A_\theta}(Z_P)$, so this latter stratum is the closed stratum $O \subseteq A_Q$.

This shows that $g(Z)$ is contained in

$$Z_B \times_{A_Q} A_P = Z_B \times_{A_Q} Z_P = Z_B \times_O Z_P = Z_B \times_{\mathbb{Z}} Z_P.$$

Since $Z_B \times_{\mathbb{Z}} Z_P$ has constant ghost sheaf $\bar{\mathcal{M}}$ it follows that $Z = g^{-1}(Z_B \times_{\mathbb{Z}} Z_P)$, and hence Z is étale over $Z_B \times_{\mathbb{Z}} Z_P$. Here we are using that the preimage of a reduced subscheme under an étale morphism remains reduced [67, Proposition 0250]. Finally, $Z_B \times_{\mathbb{Z}} Z_P$ is geometrically unibranch by the assumption on the strata of B. This shows that Z is geometrically unibranch at z. ∎

Proposition C.11. *Let $f : X \to B$ be a log smooth morphism of fine log schemes with B locally noetherian and with geometrically unibranch logarithmic strata. Assume that B is simple, that is, $\Sigma(B)$ is a cone complex rather than a generalized cone complex, and that the log structure of X is defined in the Zariski topology. Then X is simple as well, and the logarithmic strata of X are irreducible and geometrically unibranch.*

Proof. Lemma C.10 shows the statement on the log strata of X. Thus each logarithmic stratum Z has a unique generic point η_Z. It is then obvious that there is an arrow $\sigma_Z \to \sigma_W$ if and only if $\eta_W \in \mathrm{cl}(\eta_Z)$. Moreover, since $\mathcal{M}_X$ is a sheaf on the Zariski site, $\sigma_Z \to \sigma_W$ must then be the dual of the generization homomorphism $\mathcal{M}_{X,\eta_W} \to \mathcal{M}_{X,\eta_Z}$. Thus there is at most one such arrow, and hence $\Sigma(X)$ is a cone complex. ∎