Chapter 1

Introduction

1.1 Overview

In this paper, we study the Φ^3_3-measure on the three-dimensional torus on $\mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$, formally written as

$$d\rho(u) = Z^{-1} \exp\bigg(\frac{\sigma}{3}\int_{\mathbb{T}^3} u^3 dx\bigg) d\mu(u), \tag{1.1.1}$$

and its associated stochastic quantization. Here, μ is the massive Gaussian free field on $\mathbb{T}^3$ and the coupling constant $\sigma \in \mathbb{R} \setminus \{0\}$ measures the strength of the cubic interaction. The associated energy functional for the Φ^3_3-measure ρ in (1.1.1) is given by

$$E(u) = \frac{1}{2}\int_{\mathbb{T}^3} |\langle\nabla\rangle u|^2 dx - \frac{\sigma}{3}\int_{\mathbb{T}^3} u^3 dx, \tag{1.1.2}$$

where $\langle\nabla\rangle = \sqrt{1-\Delta}$. Since u^3 is not sign definite, the sign of σ does not play any role and, in particular, the problem is not defocusing even if $\sigma < 0$.

Our main goal in this paper is to study the construction of the Φ^3_3-measure and its associated dynamics, following the program on the (non-)construction of focusing[1] Gibbs measures, initiated by Lebowitz, Rose, and Speer [44]. Let us first go over the known results. In the seminal work [44], Lebowitz, Rose, and Speer studied the one-dimensional case and constructed the one-dimensional focusing Gibbs measures[2] in the L^2-(sub)critical setting (i.e. $2 < p \le 6$) with an L^2-cutoff:

$$d\rho(u) = Z^{-1}\mathbf{1}_{\{\int_{\mathbb{T}} |u|^2 dx \le K\}} \exp\bigg(\frac{1}{p}\int_{\mathbb{T}} |u|^p dx\bigg) d\mu(u) \tag{1.1.3}$$

or with a taming by the L^2-norm:

$$d\rho(u) = Z^{-1} \exp\bigg(\frac{1}{p}\int_{\mathbb{T}} |u|^p dx - A\bigg(\int_{\mathbb{T}} u^2 dx\bigg)^q\bigg) d\mu(u) \tag{1.1.4}$$

[1]By "focusing", we also mean the non-defocusing (non-repulsive) case, such as the cubic interaction appearing in (1.1.1), such that the interaction potential (for example, $\frac{\sigma}{3}\int_{\mathbb{T}^3} u^3 dx$ in (1.1.1)) is unbounded from above.

[2]As pointed out by Carlen, Fröhlich, and Lebowitz [17, p. 315], there is in fact an error in the Gibbs measure construction in [44], which was amended by Bourgain [9] (for $2 < p < 6$ with any $K > 0$ and $p = 6$ with $0 < K \ll 1$) and the first and third authors with Sosoe [62] (for $p = 6$ and $K \le \|Q\|^2_{L^2(\mathbb{R})}$). See [62] for a further discussion.

for some appropriate $q = q(p)$, where μ denotes the periodic Wiener measure on $\mathbb{T}$. See [44, Remark 2.1]. Here, the parameter $A > 0$ denotes the so-called (generalized) chemical potential and the expression (1.1.4) is referred to as the generalized grand-canonical Gibbs measure. See also the work by Carlen, Fröhlich, and Lebowitz [17] for a further discussion, where they describe the details of the construction of the generalized grand-canonical Gibbs measure in (1.1.4) in the L^2-subcritical setting ($2 < p < 6$). In [44], Lebowitz, Rose, and Speer also proved non-normalizability of the focusing Gibbs measure ρ in (1.1.3):

$$\mathbb{E}_\mu\bigg[\mathbf{1}_{\{\int_\mathbb{T} |u|^2 dx \le K\}} \exp\bigg(\frac{1}{p}\int_\mathbb{T} |u|^p dx\bigg)\bigg] = \infty$$

in (i) the L^2-supercritical case ($p > 6$) for any $K > 0$ and (ii) the L^2-critical case ($p > 6$), provided that $K > \|Q\|^2_{L^2(\mathbb{R})}$, where Q is the (unique[3]) optimizer for the Gagliardo–Nirenberg–Sobolev inequality on $\mathbb{R}$ such that

$$\|Q\|^6_{L^6(\mathbb{R})} = 3\|Q'\|^2_{L^2(\mathbb{R})}.$$

In a recent work [62], the first and third authors with Sosoe proved that the focusing L^2-critical Gibbs measure ρ in (1.1.3) (with $p = 6$) is indeed constructible at the optimal mass threshold $K = \|Q\|^2_{L^2(\mathbb{R})}$, thus answering an open question posed by Lebowitz, Rose, and Speer [44] and completing the program in the one-dimensional case.

In the two-dimensional setting, Brydges and Slade [16] continued the study on the focusing Gibbs measures and showed that with the quartic interaction ($p = 4$), the focusing Gibbs measure ρ in (1.1.3) (even with proper renormalization on the potential energy $\frac{1}{4}\int_{\mathbb{T}^2} |u|^4 dx$ and on the L^2-cutoff) is not normalizable as a probability measure. See also [61] for an alternative proof. In view of

$$\mathbf{1}_{\{|\cdot| \le K\}}(x) \le \exp\big(-A|x|^\gamma\big)\exp\big(AK^\gamma\big) \tag{1.1.5}$$

for any $K > 0$, $\gamma > 0$, and $A > 0$, this non-normalizability result of the focusing Gibbs measure on $\mathbb{T}^2$ with the quartic interaction ($p = 4$) also applies to the generalized grand-canonical Gibbs measure in (1.1.4). Furthermore, the same non-normalizability applies for higher order interaction (for an integer $p \ge 5$).

In [12], Bourgain reported Jaffe's construction of a Φ^3_2-measure endowed with a Wick-ordered L^2-cutoff:

$$d\rho = Z^{-1}\mathbf{1}_{\{\int_{\mathbb{T}^2} :u^2: \, dx \le K\}} e^{\frac{1}{3}\int_{\mathbb{T}^2} :u^3: \, dx} d\mu(u),$$

[3] Up to the symmetries.

where $:u^2:$ and $:u^3:$ denote the Wick powers of u, and μ denotes the massive Gaussian free field on $\mathbb{T}^2$. See also [61]. We point out that such a Gibbs measure with a (Wick-ordered) L^2-cutoff is not suitable for stochastic quantization in the heat and wave settings due to the lack of the L^2-conservation. In [12], Bourgain instead constructed the following generalized grand-canonical formulation of the Φ_2^3-measure:

$$d\rho(u) = Z^{-1} e^{\frac{1}{3}\int_{\mathbb{T}^2} :u^3: dx - A(\int_{\mathbb{T}^2} :u^2: dx)^2} d\mu(u)$$

for sufficiently large $A > 0$. See [35, 37, 53, 64] for the associated (stochastic) nonlinear wave dynamics.

In this paper, we consider the three-dimensional case and complete the focusing Gibbs measure construction program initiated by Lebowitz, Rose, and Speer [44]. More precisely, we consider the following generalized grand-canonical formulation of the Φ_3^3-measure (namely, with a taming by the Wick-ordered L^2-norm):

$$d\rho(u) = Z^{-1} \exp\bigg(\frac{\sigma}{3}\int_{\mathbb{T}^3} :u^3: \, dx - A\bigg|\int_{\mathbb{T}^3} :u^2: \, dx\bigg|^{\gamma}\bigg) d\mu(u) \tag{1.1.6}$$

for suitable $A, \gamma > 0$. We now state our first main result in a somewhat formal manner. See Theorem 1.2.1 for the precise statement.

Theorem 1.1.1. *The following phase transition holds for the Φ_3^3-measure in* (1.1.6).

(i) (weakly nonlinear regime). *Let $0 < |\sigma| \ll 1$ and $\gamma = 3$. Then, by introducing a further renormalization, the Φ_3^3-measure ρ in* (1.1.6) *exists as a probability measure, provided that $A = A(\sigma) > 0$ is sufficiently large. In this case, the resulting Φ_3^3-measure ρ and the massive Gaussian free field μ on $\mathbb{T}^3$ are mutually singular.*

(ii) (strongly nonlinear regime). *When $|\sigma| \gg 1$, the Φ_3^3-measure in* (1.1.6) *is not normalizable for any $A > 0$ and $\gamma > 0$. Furthermore, the truncated Φ_3^3-measures ρ_N (see* (1.2.11) *below) do not have a weak limit, as measures on $\mathcal{C}^{-\frac{3}{4}}(\mathbb{T}^3)$, even up to a subsequence.*

Theorem 1.1.1 shows that the Φ_3^3-model is critical in terms of the measure construction. In the case of a higher order focusing interaction on $\mathbb{T}^3$ (replacing $:u^3:$ by $:u^p:$ in (1.1.6) for an integer $p \geq 4$ with $\sigma > 0$ when p is even), or the Φ_4^3-model on the four-dimensional torus $\mathbb{T}^4$, the focusing nonlinear interaction gets only worse and thus we expect that the same approach would yield non-normalizability. Hence, in view of the previous results [9, 16, 44, 61, 62], Theorem 1.1.1 completes the focusing Gibbs measure construction program, thus answering an open question posed by Lebowitz, Rose, and Speer (see "Extension to higher dimensions" in [44, Section 5]). See also our companion paper [54], where we completed the program on the (non-)construction of the focusing Hartree Gibbs measures in the three-dimensional setting. See Remark 1.1.3 for a further discussion.

We point out that in the weakly nonlinear regime, the Φ_3^3-measure ρ is constructed only as a weak limit of the truncated Φ_3^3-measures. Moreover, we prove that there exists a shifted measure with respect to which the Φ_3^3-measure is absolutely continuous; see Appendix A. As for the non-normalizability result in Theorem 1.1.1 (ii), our proof is based on a refined version of the machinery introduced by the authors [54] and the first and third authors with Seong [61], which was in turn inspired by the work of the third author and Weber [75] on the non-construction of the Gibbs measure for the focusing cubic nonlinear Schrödinger equation (NLS) on the real line, giving an alternative proof of Rider's result [67]. We, however, point out that there is an additional difficulty in proving Theorem 1.1.1 (ii) due to the singularity of the Φ_3^3-measure with respect to the base massive Gaussian free field μ. (Note that the focusing Gibbs measures considered in [54, 61] are equivalent to the base Gaussian measures.) In order to overcome this difficulty, we first introduce a reference measure[4] ν_δ and construct a σ-finite version of the Φ_3^3-measure (expressed in terms of the reference measure ν_δ). We then show that this σ-finite version of the Φ_3^3-measure is not normalizable. See Chapter 4.

Remark 1.1.2. (i) As the name suggests, the Φ_3^3-measure is of interest from the point of view of constructive quantum field theory. In the defocusing case ($\sigma < 0$) with a quartic interaction (u^4 in place of u^3), the measure ρ in (1.1.1) corresponds to the well-studied Φ_3^4-measure. The construction of the Φ_3^4-measure is one of the early achievements in constructive quantum field theory. For an overview of the constructive program, see the introductions in [1, 33].

(ii) In the one- and two-dimensional cases, the non-normalizability of the focusing Gibbs measures emerges in the L^2-critical case ($p = 6$ when $d = 1$ and $p = 4$ when $d = 2$), suggesting its close relation to the finite time blowup phenomena of the associated focusing NLS. See [62] for a further discussion. In the three-dimensional case, it is interesting to note that the Φ_3^3-model is L^2-*subcritical* and yet we have the non-normalizability (in the strongly nonlinear regime). Thus, the non-normalizability of the Φ_3^3-measure is not related to a blowup phenomenon. Note that, unlike the focusing Φ_1^6- and Φ_2^4-models which make sense in the complex-valued setting, the Φ_3^3-model makes sense only in the real-valued setting. It seems of interest to investigate a possible relation to the following Gagliardo–Nirenberg inequality:

$$\int_{\mathbb{R}^3} |u(x)|^3 dx \lesssim \|u\|_{L^2(\mathbb{R}^3)}^{\frac{3}{2}} \|u\|_{\dot{H}^1(\mathbb{R}^3)}^{\frac{3}{2}}.$$

[4]This reference measure is introduced as a tamed version of the Φ_3^3-measure and is not to be confused with the shifted measure mentioned above. See Proposition 4.1.1.

(iii) Consider a Φ^3_3-measure with a Wick-ordered L^2-cutoff:[5]

$$d\rho(u) = Z^{-1}\mathbf{1}_{\{|\int_{\mathbb{T}^3} :u^2:\, dx| \le K\}} \exp\bigg(\frac{\sigma}{3}\int_{\mathbb{T}^3} :u^3:\, dx\bigg) d\mu(u). \tag{1.1.7}$$

Then, an analogue of Theorem 1.1.1 holds for the Φ^3_3-measure in (1.1.7). In view of (1.1.5), Theorem 1.1.1 implies normalizability of the Φ^3_3-measure in (1.1.7) (with a further renormalization) in the weakly nonlinear regime ($0 < |\sigma| \ll 1$). On the other hand, in the strongly nonlinear regime ($|\sigma| \gg 1$), a modification of the proof of Theorem 1.1.12 (ii) (see also [54, 61]) yields non-normalizability of the Φ^3_3-measure in (1.1.7) for any $K > 0$.

Remark 1.1.3. In [11], Bourgain studied the invariant Gibbs dynamics for the focusing Hartree NLS on $\mathbb{T}^3$ (with $\sigma > 0$):

$$i\partial_t u + (1-\Delta)u - \sigma(V * |u|^2)u = 0, \tag{1.1.8}$$

where $V = \langle\nabla\rangle^{-\beta}$ is the Bessel potential of order $\beta > 0$. In [11], Bourgain first constructed the focusing Gibbs measure with a Hartree-type interaction (for complex-valued u), endowed with a Wick-ordered L^2-cutoff:

$$d\rho(u) = Z^{-1}\mathbf{1}_{\{\int_{\mathbb{T}^3} :|u|^2:\, dx \le K\}} e^{\frac{\sigma}{4}\int_{\mathbb{T}^3}(V * :|u|^2:):|u|^2:\, dx} d\mu(u)$$

for $\beta > 2$ and then constructed the invariant Gibbs dynamics for the associated dynamical problem.[6] In [54], we continued the study of the focusing Hartree Φ^4_3-measure in the generalized grand-canonical formulation (with $\sigma > 0$):

$$d\rho(u) = Z^{-1}\exp\bigg(\frac{\sigma}{4}\int_{\mathbb{T}^3}(V * :u^2:)\, :u^2:\, dx - A\bigg|\int_{\mathbb{T}^3} :u^2:\, dx\bigg|^{\gamma}\bigg) d\mu(u) \tag{1.1.9}$$

and established a phase transition in two respects (i) the focusing Hartree Φ^4_3-measure ρ in (1.1.9) is constructible for $\beta > 2$, while it is not for $\beta < 2$ and (ii) when $\beta = 2$, the focusing Hartree Φ^4_3-measure is constructible for $0 < \sigma \ll 1$, while it is not for $\sigma \gg 1$. See [54] for the precise statements. These results in [54] in particular show the critical nature of the focusing Hartree Φ^4_3-model when $\beta = 2$. In the same work, we also constructed the invariant Gibbs dynamics for the associated (canonical) stochastic

[5] With a slight modification, one may also consider ρ in (1.1.7) with a slightly different cutoff $\mathbf{1}_{\{\int_{\mathbb{T}^3} :u^2:\, dx \le K\}}$, i.e. without an absolute value, and prove the same (non-)normalizability results. See [54, Remark 5.10].

[6] By combining the construction of the focusing Hartree Gibbs measure in the critical case ($\beta = 2$) with $0 < \sigma \ll 1$ in [54] and the well-posedness result in [22], this result on the focusing Hartree NLS (1.1.8) by Bourgain [11] can be extended to the critical case $\beta = 2$ (in the weakly nonlinear regime $0 < \sigma \ll 1$).

quantization equation. See also [13, 14, 54] for the defocusing case ($\sigma < 0$). Note that when $\beta = 0$, the defocusing Hartree Φ^4_3-measure reduces to the usual Φ^4_3-measure.

In terms of scaling, the focusing Hartree Φ^4_3-model with $\beta = 2$ corresponds to the Φ^3_3-model and as such, they share some common features. For example, they are both critical with a phase transition, depending on the size of the coupling constant σ. At the same time, however, there are some differences. While the focusing Hartree Φ^4_3-measure with $\beta = 2$ is absolutely continuous with respect to the base massive Gaussian free field μ, the Φ^3_3-measure studied in this paper is singular with respect to the base massive Gaussian free field μ. As mentioned above, this singularity of the Φ^3_3-measure causes an additional difficulty in proving non-normalizability in the strongly nonlinear regime $|\sigma| \gg 1$.

Next, we discuss the dynamical problem associated with the Φ^3_3-measure constructed in Theorem 1.1.1. In the following, we consider the canonical stochastic quantization equation [66, 68] for the Φ^3_3-measure in (1.1.6) (with $\gamma = 3$). More precisely, we study the following stochastic damped nonlinear wave equation (SdNLW) with a quadratic nonlinearity, posed on $\mathbb{T}^3$:

$$\partial_t^2 u + \partial_t u + (1 - \Delta)u - \sigma u^2 = \sqrt{2}\xi, \quad (x,t) \in \mathbb{T}^3 \times \mathbb{R}_+, \tag{1.1.10}$$

where $\sigma \in \mathbb{R} \setminus \{0\}$, u is an unknown function, and ξ denotes a (Gaussian) space-time white noise on $\mathbb{T}^3 \times \mathbb{R}_+$ with the space-time covariance given by

$$\mathbb{E}\big[\xi(x_1,t_1)\xi(x_2,t_2)\big] = \delta(x_1 - x_2)\delta(t_1 - t_2).$$

In this introduction, we keep our discussion at a formal level and do not worry about various renormalizations required to give a proper meaning to the equation (1.1.10).

With $\vec{u} = (u, \partial_t u)$, define the energy $\mathcal{E}(\vec{u})$ by

$$\begin{aligned} \mathcal{E}(\vec{u}) &= E(u) + \frac{1}{2}\int_{\mathbb{T}^3} (\partial_t u)^2 dx \\ &= \frac{1}{2}\int_{\mathbb{T}^3} |\langle \nabla \rangle u|^2 dx + \frac{1}{2}\int_{\mathbb{T}^3} (\partial_t u)^2 dx - \frac{\sigma}{3}\int_{\mathbb{T}^3} u^3 dx, \end{aligned}$$

where $E(u)$ is as in (1.1.2). This is precisely the energy (= Hamiltonian) of the (deterministic) nonlinear wave equation (NLW) on $\mathbb{T}^3$ with a quadratic nonlinearity:

$$\partial_t^2 u + (1 - \Delta)u - \sigma u^2 = 0. \tag{1.1.11}$$

Then, by letting $v = \partial_t u$, we can write (1.1.10) as the first order system:

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{\partial \mathcal{E}}{\partial v} \\ -\frac{\partial \mathcal{E}}{\partial u} \end{pmatrix} + \begin{pmatrix} 0 \\ -v + \sqrt{2}\xi \end{pmatrix},$$

which shows that the SdNLW dynamics (1.1.10) is given as a superposition of the deterministic NLW dynamics (1.1.11) and the Ornstein–Uhlenbeck dynamics for $v = \partial_t u$:

$$\partial_t v = -v + \sqrt{2}\xi.$$

Now, consider the Gibbs measure $\vec{\rho}$, formally given by

$$\begin{aligned} d\vec{\rho}(\vec{u}) &= Z^{-1} e^{-\mathcal{E}(\vec{u})} d\vec{u} = d\rho \otimes d\mu_0(\vec{u}) \\ &= Z^{-1} \exp\Big(\frac{\sigma}{3}\int_{\mathbb{T}^3} u^3 dx\Big) d(\mu \otimes \mu_0)(u, v), \end{aligned} \tag{1.1.12}$$

where ρ is the Φ^3_3-measure in (1.1.1) and μ_0 denotes the white noise measure; see (1.2.1). See Remark 1.2.6 for the precise definition of the Gibbs measure $\vec{\rho}$. Then, the observation above shows that $\vec{\rho}$ is expected to be invariant under the dynamics of the quadratic SdNLW (1.1.10). Indeed, from the stochastic quantization point of view, the equation (1.1.10) is the so-called canonical stochastic quantization equation (namely, the Hamiltonian stochastic quantization) for the Φ^3_3-measure; see [68]. For this reason, it is natural to refer to (1.1.10) as the *hyperbolic* Φ^3_3*-model*.

Let us now state our main dynamical result in a somewhat formal manner. See Theorem 1.3.2 for the precise statement.

Theorem 1.1.4. *Let $\gamma = 3$ and $0 < |\sigma| \ll 1$. Suppose that $A = A(\sigma) > 0$ is sufficiently large as in Theorem* 1.1.1 (i). *Then, the hyperbolic Φ^3_3-model* (1.1.10) *on the three-dimensional torus $\mathbb{T}^3$* (*with a proper renormalization*) *is almost surely globally well-posed with respect to the random initial data distributed by the* (*renormalized*) *Gibbs measure $\vec{\rho} = \rho \otimes \mu_0$ in* (1.1.12). *Furthermore, the Gibbs measure $\vec{\rho}$ is invariant under the resulting dynamics.*

In view of the critical nature of the Φ^3_3-measure, Theorem 1.1.4 is sharp in the sense that almost sure global well-posedness does not extend to SdNLW with a focusing nonlinearity of a higher order. The construction of the Φ^3_3-measure in Theorem 1.1.1 requires us to introduce several renormalizations together with the taming by the Wick-ordered L^2-norm. This introduces modifications to the equation (1.1.10). See Section 1.3 and Chapters 5 and 6 for the precise formulation of the problem.

Over the last five years, stochastic nonlinear wave equations (SNLW) in the singular setting have been studied extensively in various settings:[7]

$$\partial_t^2 u + \partial_t u + (1 - \Delta) u + \mathcal{N}(u) = \xi \tag{1.1.13}$$

for a power-type nonlinearity [14, 23, 24, 35–37, 52–54, 59, 65, 73] and for trigonometric and exponential nonlinearities [57, 58, 60]. We mention the works [14, 55, 56, 64]

[7] Some of the works mentioned below are on SNLW without damping.

on nonlinear wave equations with rough random initial data. In [36], by combining the paracontrolled calculus, originally introduced in the parabolic setting [18,34,47], with the multilinear harmonic analytic approach, more traditional in studying dispersive equations, Gubinelli, Koch, and the first author studied the quadratic SNLW (1.1.10) (without the damping). The paracontrolled approach in the wave setting was also used in our previous work [54] and was further developed by Bringmann [14]. In order to prove local well-posedness of the hyperbolic Φ^3_3-model (1.1.10), we also follow the paracontrolled approach, in particular combining the analysis in [36, 54]. See Chapter 5. As for the globalization part, a naive approach would be to apply Bourgain's invariant measure argument [9, 10]. However, due to the singularity of the Φ^3_3-measure ρ with respect to the base massive Gaussian free field μ (and the fact that the truncated Φ^3_3-measure ρ_N converges to ρ only weakly), there is an additional difficulty to overcome for the hyperbolic Φ^3_3-model. Hence, Bourgain's invariant measure argument is not directly applicable. In the context of the defocusing Hartree cubic NLW on $\mathbb{T}^3$, Bringmann [14] encountered a similar difficulty and developed a new globalization argument. While it is possible to adapt Bringmann's analysis to our current setting, we instead introduce a new alternative argument, which is conceptually simple and straightforward. In particular, we extensively use the variational approach and also use ideas from theory of optimal transport to directly estimate a probability with respect to the limiting Gibbs measure $\vec{\rho}$ (in particular, without going through shifted measures as in [14]). See Section 1.3 and Chapter 6 for details.

Remark 1.1.5. A slight modification of our proof of Theorem 1.1.4 yields the corresponding results (namely, almost sure global well-posedness and invariance of the associated Gibbs measure) for the (deterministic) quadratic NLW (1.1.11) on $\mathbb{T}^3$ in the weakly nonlinear regime.

Remark 1.1.6. We point out that an analogue of Theorem 1.1.4 also holds for the parabolic Φ^3_3-model, namely, the stochastic nonlinear heat equation with a quadratic nonlinearity:

$$\partial_t u + (1-\Delta)u - \sigma u^2 = \sqrt{2}\xi, \quad (x,t) \in \mathbb{T}^3 \times \mathbb{R}_+. \tag{1.1.14}$$

Thanks to the strong smoothing of the heat propagator, the well-posedness of (1.1.14) follows from elementary analysis based on the first order expansion (also known as the Da Prato–Debussche trick [20]). See for example [25]. While there is an extra term coming from the taming by the Wick-ordered L^2-norm (see, for example, (1.3.1) in the hyperbolic case), this term does not cause any issue in the parabolic setting.

Remark 1.1.7. In [72], the third author introduced a new approach to establish unique ergodicity of Gibbs measures for stochastic dispersive/hyperbolic equations. This was further developed in [74] to prove ergodicity of the hyperbolic Φ^4_2-model, namely (1.1.13) on $\mathbb{T}^2$ with $\mathcal{N}(u) = u^3$. See also [28] by the third author and Forlano on the asymptotic Feller property of the invariant Gibbs dynamics for the cubic SNLW on $\mathbb{T}^2$ with a slightly smoothed noise. The ergodic property of the hyperbolic Φ_3^3-model is a challenging problem, in particular due to its non-defocusing nature.

1.2 Construction of the Φ_3^3-measure

In this section, we describe a renormalization procedure and also a taming by the Wick-ordered L^2-norm required to construct the Φ_3^3-measure in (1.1.6) and make a precise statement (Theorem 1.2.1). For this purpose, we first fix some notations. Given $s \in \mathbb{R}$, let μ_s denote a Gaussian measure with the Cameron–Martin space $H^s(\mathbb{T}^3)$, formally defined by

$$d\mu_s = Z_s^{-1} e^{-\frac{1}{2}\|u\|_{H^s}^2} du = Z_s^{-1} \prod_{n \in \mathbb{Z}^3} e^{-\frac{1}{2}\langle n \rangle^{2s} |\hat{u}(n)|^2} d\hat{u}(n), \tag{1.2.1}$$

where $\langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}}$. When $s = 1$, the Gaussian measure μ_s corresponds to the massive Gaussian free field, while it corresponds to the white noise measure μ_0 when $s = 0$. For simplicity, we set

$$\mu = \mu_1 \quad \text{and} \quad \vec{\mu} = \mu \otimes \mu_0. \tag{1.2.2}$$

Define the index sets Λ and Λ_0 by

$$\Lambda = \bigcup_{j=0}^{2} \mathbb{Z}^j \times \mathbb{N} \times \{0\}^{2-j} \quad \text{and} \quad \Lambda_0 = \Lambda \cup \{(0,0,0)\} \tag{1.2.3}$$

such that $\mathbb{Z}^3 = \Lambda \cup (-\Lambda) \cup \{(0,0,0)\}$. Then, let $\{g_n\}_{n \in \Lambda_0}$ and $\{h_n\}_{n \in \Lambda_0}$ be sequences of mutually independent standard complex-valued[8] Gaussian random variables and set $g_{-n} := \overline{g_n}$ and $h_{-n} := \overline{h_n}$ for $n \in \Lambda_0$. Moreover, we assume that $\{g_n\}_{n \in \Lambda_0}$ and $\{h_n\}_{n \in \Lambda_0}$ are independent from the space-time white noise ξ in (1.1.10). We now define random distributions $u = u^\omega$ and $v = v^\omega$ by the following Gaussian Fourier series:[9]

$$u^\omega = \sum_{n \in \mathbb{Z}^3} \frac{g_n(\omega)}{\langle n \rangle} e_n \quad \text{and} \quad v^\omega = \sum_{n \in \mathbb{Z}^3} h_n(\omega) e_n, \tag{1.2.4}$$

where $e_n = e^{in\cdot x}$. Denoting by $\mathrm{Law}(X)$ the law of a random variable X (with respect to the underlying probability measure $\mathbb{P}$), we then have

$$\mathrm{Law}(u, v) = \vec{\mu} = \mu \otimes \mu_0$$

[8] This means that $g_0, h_0 \sim \mathcal{N}_\mathbb{R}(0,1)$ and $\mathrm{Re}\, g_n, \mathrm{Im}\, g_n, \mathrm{Re}\, h_n, \mathrm{Im}\, h_n \sim \mathcal{N}_\mathbb{R}(0, \frac{1}{2})$ for $n \neq 0$.

[9] By convention, we endow $\mathbb{T}^3$ with the normalized Lebesgue measure $dx_{\mathbb{T}^3} = (2\pi)^{-3} dx$.

for (u, v) in (1.2.4). Note that $\mathrm{Law}(u, v) = \vec{\mu}$ is supported on

$$\mathcal{H}^s(\mathbb{T}^3) := H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)$$

for $s < -\frac{1}{2}$ but not for $s \geq -\frac{1}{2}$ (and more generally in $W^{s,p}(\mathbb{T}^3) \times W^{s-1,p}(\mathbb{T}^3)$ for any $1 \leq p \leq \infty$ and $s < -\frac{1}{2}$).

We now consider the Φ^3_3-measure formally given by (1.1.1). Since u in the support of the massive Gaussian free field μ is merely a distribution, the cubic potential energy in (1.1.1) is not well defined and thus a proper renormalization is required to give a meaning to the potential energy. In order to explain the renormalization process, we first study the regularized model.

Given $N \in \mathbb{N}$, we denote by $\pi_N = \pi_N^{\text{cube}}$ the frequency projector onto the (spatial) frequencies $\{n = (n_1, n_2, n_3) \in \mathbb{Z}^3 : \max_{j=1,2,3} |n_j| \leq N\}$, defined by

$$\pi_N f = \pi_N^{\text{cube}} f = \sum_{n \in \mathbb{Z}^3} \chi_N(n) \hat{f}(n) e_n, \tag{1.2.5}$$

associated with a Fourier multiplier $\chi_N = \chi_N^{\text{cube}}$:

$$\chi_N(n) = \chi_N^{\text{cube}}(n) = \mathbf{1}_Q\big(N^{-1}n\big), \tag{1.2.6}$$

where Q denotes the cube of side length 2 in $\mathbb{R}^3$ centered at the origin:

$$Q = \Big\{\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \max_{j=1,2,3} |\xi_j| \leq 1\Big\}. \tag{1.2.7}$$

It turns out that, due to the critical nature of the Φ^3_3-measure, a choice of frequency projectors makes a difference. See Remark 1.2.2 and Section 1.4 below for discussions on different frequency projectors. In comparing different frequency projectors, we refer to $\pi_N = \pi_N^{\text{cube}}$ in (1.2.5) as the cube frequency projector in the following.

Let u be as in (1.2.4) and set $u_N = \pi_N u$. For each fixed $x \in \mathbb{T}^3$, $u_N(x)$ is a mean-zero real-valued Gaussian random variable with variance

$$\sigma_N = \mathbb{E}\big[u_N^2(x)\big] = \sum_{n \in \mathbb{Z}^3} \frac{\chi_N^2(n)}{\langle n \rangle^2} \sim N \to \infty, \tag{1.2.8}$$

as $N \to \infty$. Note that σ_N is independent of $x \in \mathbb{T}^3$ due to the stationarity of μ. We define the Wick powers $:u_N^2:$ and $:u_N^3:$ by setting

$$:u_N^2: = H_2(u_N; \sigma_N) = u_N^2 - \sigma_N \quad \text{and} \quad :u_N^3: = H_3(u_N; \sigma_N) = u_N^3 - 3\sigma_N u_N,$$

where $H_k(x, \sigma)$ denotes the Hermite polynomial of degree k with variance parameter σ defined by the generating function:

$$e^{tx - \frac{1}{2}\sigma t^2} = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x; \sigma).$$

This suggests us to consider the following renormalized potential energy:

$$R_N(u) = -\frac{\sigma}{3}\int_{\mathbb{T}^3} :u_N^3:\, dx + A\bigg|\int_{\mathbb{T}^3} :u_N^2:\, dx\bigg|^\gamma. \tag{1.2.9}$$

As in the case of the Φ^4_3-measure in [3], the renormalized potential energy $R_N(u)$ in (1.2.9) is divergent (as $N \to \infty$) and thus we need to introduce a further renormalization. This leads to the following renormalized potential energy:

$$R_N^\diamond(u) = R_N(u) + \alpha_N, \tag{1.2.10}$$

where α_N is a diverging constant (as $N \to \infty$) defined in (3.2.6) below. Finally, we define the truncated (renormalized) Φ^3_3-measure ρ_N by

$$d\rho_N(u) = Z_N^{-1} e^{-R_N^\diamond(u)} d\mu(u), \tag{1.2.11}$$

where the partition function Z_N is given by

$$Z_N = \int e^{-R_N^\diamond(u)} d\mu(u). \tag{1.2.12}$$

Then, we have the following construction and non-normalizability of the Φ^3_3-measure. Due to the singularity of the Φ^3_3-measure with respect to the base Gaussian measure $\vec{\mu}$, we need to state our non-normalizability result in a careful manner. Compare this with [54, Theorem 1.15] and [61, Theorem 1.3]. See the beginning of Chapter 4 for a further discussion.

Theorem 1.2.1. *There exist $\sigma_1 \geq \sigma_0 > 0$ such that the following statements hold.*

(i) (weakly nonlinear regime). *Let $0 < |\sigma| < \sigma_0$. Then, by choosing $\gamma = 3$ and $A = A(\sigma) > 0$ sufficiently large, we have the uniform exponential integrability of the density*:

$$\sup_{N\in\mathbb{N}} Z_N = \sup_{N\in\mathbb{N}} \big\| e^{-R_N^\diamond(u)} \big\|_{L^1(\mu)} < \infty \tag{1.2.13}$$

and the truncated Φ^3_3-measure ρ_N in (1.2.11) converges weakly to a unique limit ρ, formally given by[10]

$$d\rho(u) = Z^{-1} \exp\bigg(\frac{\sigma}{3}\int_{\mathbb{T}^3} :u^3:\, dx - A\bigg|\int_{\mathbb{T}^3} :u^2:\, dx\bigg|^3 - \infty\bigg) d\mu(u). \tag{1.2.14}$$

In this case, the resulting Φ^3_3-measure ρ and the base massive Gaussian free field μ are mutually singular.

[10] By hiding α_N in (1.2.11) into the partition function Z_N, we could also say that the limiting Φ^3_3-measure ρ is formally given by (1.1.6) (with $\gamma = 3$).

(ii) (strongly nonlinear regime). *Let* $|\sigma| > \sigma_1$ *and* $\gamma \geq 3$. *Then, the* Φ_3^3*-measure is not normalizable in the following sense.*
Fix $\delta > 0$. *Given* $N \in \mathbb{N}$, *let* $\nu_{N,\delta}$ *be the following tamed version of the truncated* Φ_3^3*-measure*:

$$d\nu_{N,\delta}(u) = Z_{N,\delta}^{-1} \exp\Big(-\delta \|\pi_N u\|_{B_{3,\infty}^{-\frac{3}{4}}}^{20} - R_N^\diamond(u)\Big) d\mu(u). \tag{1.2.15}$$

Then, $\{\nu_{N,\delta}\}_{N\in\mathbb{N}}$ *converges weakly to some limiting probability measure* ν_δ *and the following* σ*-finite version of the* Φ_3^3*-measure*:

$$\begin{aligned} d\overline{\rho}_\delta &= \exp\Big(\delta\|u\|_{B_{3,\infty}^{-\frac{3}{4}}}^{20}\Big) d\nu_\delta \\ &= \lim_{N\to\infty} Z_{N,\delta}^{-1} \exp\Big(\delta\|u\|_{B_{3,\infty}^{-\frac{3}{4}}}^{20}\Big) \exp\Big(-\delta\|\pi_N u\|_{B_{3,\infty}^{-\frac{3}{4}}}^{20} - R_N^\diamond(u)\Big) d\mu(u) \end{aligned}$$

is a well-defined measure on $\mathcal{C}^{-100}(\mathbb{T}^3)$. *Furthermore, this* σ*-finite version* $\overline{\rho}_\delta$ *of the* Φ_3^3*-measure is not normalizable*:

$$\int 1 d\overline{\rho}_\delta = \infty.$$

Under the same assumption, the sequence $\{\rho_N\}_{N\in\mathbb{N}}$ *of the truncated* Φ_3^3*-measures in* (1.2.11) *does not converge to any weak limit, even up to a subsequence, as measures on the Besov space* $B_{3,\infty}^{-\frac{3}{4}}(\mathbb{T}^3) \supset \mathcal{C}^{-\frac{3}{4}}(\mathbb{T}^3)$.

In the weakly nonlinear regime, we also prove that the Φ_3^3-measure ρ is absolutely continuous with respect to the shifted measure $\text{Law}(Y(1) + \sigma\mathfrak{Z}(1) + \mathcal{W}(1))$, where $\text{Law}(Y(1)) = \mu$, $\mathfrak{Z} = \mathfrak{Z}(Y)$ is the limit of the quadratic process $\mathfrak{Z}^N$ defined in (3.2.3), and the auxiliary quintic process $\mathcal{W} = \mathcal{W}(Y)$ is defined in (A.1.1). While we do not use this property in this paper, we present the proof in Appendix A for completeness.

As in case of the Φ_3^4-measure in [3], we can prove uniform exponential integrability of the truncated density $e^{-R_N^\diamond(u)}$ in $L^p(\mu)$ only for $p = 1$ due to the second renormalization introduced in (1.2.10). See also [13, 54] for a similar phenomenon in the case of the defocusing Hartree Φ_3^4-measure. We point out that the renormalized potential energy $R_N^\diamond(u)$ in (1.2.10) does *not* converge to any limit and neither does the density $e^{-R_N^\diamond(u)}$, which is essentially the source of the singularity of the Φ_3^3-measure with respect to the massive Gaussian free field μ.

As in [54], following the variational approach introduced by Barashkov and Gubinelli [3], we use the Boué–Dupuis variational formula (Lemma 3.1.1) to prove Theorem 1.2.1. In fact, we make use of the Boué–Dupuis variational formula in almost every single step of the proof. In proving Theorem 1.2.1 (i), we first use the variational formula to establish the uniform exponential integrability (1.2.13) of the truncated density $e^{-R_N^\diamond(u)}$, from which tightness of the truncated Φ^3_3-measure ρ_N in (1.2.11) follows. See Section 3.2. Due to the singularity of the Φ^3_3-measure, we need to apply a change of variables (see (3.2.4)) in the variational formulation and thus we need to treat the taming part more carefully than that for the focusing Hartree Φ^4_3-measure studied in [54]. See Lemma 3.2.3 below. This lemma also reflects the critical nature of the Φ^3_3-measure.

In Section 3.3, we prove uniqueness of the limiting Φ^3_3-measure. Our main strategy is to follow the approach introduced in our previous work [54] and compare two (arbitrary) subsequences $\rho_{N_{k_1}}$ and $\rho_{N_{k_2}}$, using the variational formula. We point out, however, that, due to the critical nature of the Φ^3_3-measure, our uniqueness argument becomes more involved than that in [54, Section 6.3] for the subcritical defocusing Hartree Φ^4_3-measure. In particular, we need to make use of a certain orthogonality property to eliminate a problematic term. See Remark 3.3.2. See also Section 1.4.

In proving the singularity of the Φ^3_3-measure, we once again follow the direct approach introduced in [54], making use of the variational formula. We point out that the proof of the singularity of the Φ^4_3-measure by Barashkov and Gubinelli [4] goes through the shifted measure. On the other hand, as in [54], our proof is based on a direct argument without referring to shifted measures. See Section 3.4.

Let us now turn to the strongly nonlinear regime considered in Theorem 1.2.1 (ii). As mentioned above, due to the singularity of the Φ^3_3-measure, our formulation of the non-normalizability result in Theorem 1.2.1 (ii) is rather subtle. In the situation where the truncated density $e^{-R_N^\diamond(u)}$ converges to the limiting density (as in [54,61]), it would suffice to prove

$$\sup_{N\in\mathbb{N}} \mathbb{E}_\mu\big[e^{-R_N^\diamond(u)}\big] = \infty, \tag{1.2.16}$$

since (1.2.16) would imply that there is no normalization constant which would make the limit of the measure $e^{-R_N^\diamond(u)}d\mu(u)$ into a probability measure. In the current problem, however, the potential energy $R_N^\diamond(u)$ in (1.2.10) (and the corresponding density $e^{-R_N^\diamond(u)}$) does *not* converge to any limit. Thus, even if we prove a statement of the form (1.2.16), we may still choose a sequence of constants $\widehat{Z}_N$ such that the measures $\widehat{Z}_N^{-1}e^{-R_N^\diamond(u)}d\mu$ have a weak limit. A similar phenomenon happens for the Φ^4_3-measure, where one needs to introduce the second order renormalization; see [3]. The non-convergence of the truncated Φ^3_3-measures claimed in Theorem 1.2.1 (ii) tells us that this can not happen for the Φ^3_3-measure. See also Remark 1.2.3 below.

Our strategy is to first construct a σ-finite version of the Φ^3_3-measure and then prove its non-normalizability. As stated in Theorem 1.2.1 (ii), we first introduce a tamed version $\nu_{N,\delta}$ of the truncated Φ^3_3-measure, by introducing an appropriate taming function F; see (4.1.6) below. The first step is to show that this tamed truncated Φ^3_3-measure $\nu_{N,\delta}$ converges weakly to some limit ν_δ (Proposition 4.1.1). We then define a σ-finite version $\overline{\rho}_\delta$ of the Φ^3_3-measure by setting

$$d\overline{\rho}_\delta = e^{\delta F(u)} d\nu_\delta$$

and prove that $\overline{\rho}_\delta$ is not normalizable (Proposition 4.1.2). Here, the σ-finite version $\overline{\rho}_\delta$ of the Φ^3_3-measure clearly depends on the choice of a taming function F. Our choice is quite natural since the σ-finite version $\overline{\rho}_\delta$ of the Φ^3_3-measure is absolutely continuous with respect to the shifted measure $\mathrm{Law}(Y(1) + \sigma\mathfrak{Z}(1) + \mathcal{W}(1))$, just like the (normalizable) Φ^3_3-measure in the weakly nonlinear regime discussed above. See Remark A.3.1.

Once we construct the σ-finite version $\overline{\rho}_\delta$ of the Φ^3_3-measure, our argument follows closely the strategy introduced in [54, 61] for establishing non-normalizability, using the Boué–Dupuis variational formula. For this approach, we need to construct a drift achieving the desired divergence, where (the antiderivative of) the drift is designed to look like "$-Y(1)$ + a perturbation", where $\mathrm{Law}(Y(1)) = \mu$; see (4.3.14) below. Here, the perturbation term is bounded in $L^2(\mathbb{T}^3)$ but has a large L^3-norm, thus having a highly concentrated profile, such as a soliton or a finite time blowup profile. As compared to our previous works [54, 61], there is an additional difficulty in proving the non-normalizability claim in Theorem 1.2.1 (ii) due to the singularity of the Φ^3_3-measure, which forces us to use a change of variables (see (3.2.4)) in the variational formulation. See Remark 4.3.1. The non-convergence of the truncated Φ^3_3-measures ρ_N stated in Theorem 1.2.1 (ii) follows as a corollary to the non-normalizability of the σ-finite version $\overline{\rho}_\delta$ of the Φ^3_3-measure; see Proposition 4.1.4 and Section 4.4. If the Φ^3_3-measure existed as a probability measure in the strongly nonlinear regime, then we would expect its support to be contained in $\mathcal{C}^{-\frac{1}{2}-\varepsilon}(\mathbb{T}^3)$ for any $\varepsilon > 0$, just as in the weakly nonlinear regime (and the Φ^4_3-measure). For this reason, the Besov space $B^{-\frac{3}{4}}_{3,\infty}(\mathbb{T}^3) \supset \mathcal{C}^{-\frac{3}{4}}(\mathbb{T}^3)$ is a quite natural space to consider. The restriction $\gamma \geq 3$ in Theorem 1.2.1 (ii) comes from the construction of the tamed version ν_δ of the Φ^3_3-measure; see (4.2.3) below. For $\gamma < 3$, the taming by the Wick-ordered L^2-norm in (1.1.6) becomes weaker and thus we expect an analogous non-normalizability result to hold.

Remark 1.2.2. We prove Theorem 1.2.1 for the cube frequency projector $\pi_N = \pi_N^{\text{cube}}$ defined in (1.2.5). If we instead consider the ball frequency projector π_N^{ball} defined in (1.4.1) below, then our argument for the non-convergence claim in the strongly nonlinear regime (Proposition 4.1.4) breaks down, while the other claims in Theorem 1.2.1 remain true for the ball frequency projector π_N^{ball}. If we consider the smooth frequency projector π_N^{smooth} defined in (1.4.2) below, then our argument for the uniqueness of the limiting Φ^3_3-measure in the weakly nonlinear regime (Proposition 3.3.1) breaks down. In particular, the latter issue is closely related to the critical nature of the Φ^3_3-model and, while we believe that uniqueness of the limiting Φ^3_3-measure holds even in the case of the smooth frequency projector π_N^{smooth}, it seems non-trivial to prove this claim by a modification of our argument. We point out that the same issue also appears in showing uniqueness of the limit ν_δ of the tamed version $\nu_{N,\delta}$ of the truncated Φ^3_3-measure in (1.2.15) in the strongly nonlinear regime (Proposition 4.1.1) and in the dynamical part (Proposition 6.3.3). See Section 1.4 for a further discussion. See also Remarks 3.3.2 and 4.4.2.

Remark 1.2.3. In the strongly nonlinear regime, Theorem 1.2.1 (ii) tells us that the truncated Φ^3_3-measures ρ_N do not converge weakly to any limit as measures on

$$B^{-\frac{3}{4}}_{3,\infty}(\mathbb{T}^3) \supset \mathcal{C}^{-\frac{3}{4}}(\mathbb{T}^3).$$

It is, however, possible that the truncated Φ^3_3-measures converges weakly to some limit (say, the Dirac delta measure δ_0 on the trivial function) as measures on some space with a very weak topology, say $\mathcal{C}^{-100}(\mathbb{T}^3)$. Theorem 1.2.1 (ii) shows that if such weak convergence takes place, it must do so in a very pathological manner.

Remark 1.2.4. The second renormalization in (1.2.10) (i.e. the cancellation of the diverging constant α_N) appears only at the level of the measure. The associated equation (see (1.3.6) below) does not see this additional renormalization.

Remark 1.2.5. It is of interest to investigate a threshold value $\sigma_* > 0$ such that the construction of the Φ^3_3-measure (Theorem 1.2.1 (i)) holds for $0 < |\sigma| < \sigma_*$, while the non-normalizability of the Φ^3_3-measure (Theorem 1.2.1 (ii)) holds for $|\sigma| > \sigma_*$. If such a threshold value σ_* could be determined, it would also be of interest to determine whether the Φ^3_3-measure is normalizable at the threshold $|\sigma| = \sigma_*$. Such a problem, however, requires optimizing all the estimates in the proof of Theorem 1.2.1 and is out of reach at this point. See a recent work [62] by Sosoe and the first and third authors for such analysis in the one-dimensional case.

Remark 1.2.6. Consider the truncated Gibbs measure $\vec{\rho}_N = \rho_N \otimes \mu_0$ for the hyperbolic Φ^3_3-model (1.1.10) with the density:

$$d\vec{\rho}_N(u, v) = Z_N^{-1} e^{-R_N^\diamond(u)} d\vec{\mu}(u, v), \tag{1.2.17}$$

where $R_N^\diamond(u)$ and $\vec{\mu}$ are as in (1.2.10) and (1.2.2), respectively. Since the potential energy $R_N^\diamond(u)$ is independent of the second component v, Theorem 1.2.1 directly applies to the truncated Gibbs measure $\vec{\rho}_N$. In particular, in the weakly nonlinear regime ($0 < |\sigma| < \sigma_0$), the truncated Gibbs measure $\vec{\rho}_N$ converges weakly to the limiting Gibbs measure

$$\vec{\rho} = \rho \otimes \mu_0, \tag{1.2.18}$$

where ρ is the limiting Φ^3_3-measure constructed in Theorem 1.2.1 (i). Moreover, the limiting Gibbs measure $\vec{\rho}$ and the base Gaussian measure $\vec{\mu} = \mu \otimes \mu_0$ are mutually singular.

1.3 Hyperbolic Φ_3^3-model

In this section, we provide a precise meaning to the hyperbolic Φ_3^3-model (1.1.10) and make Theorem 1.1.4 more precise. By considering the Langevin equation for the Gibbs measure $\vec{\rho} = \rho \otimes \mu_0$ constructed in Remark 1.2.6, we formally obtain the following quadratic SdNLW (= the hyperbolic Φ_3^3-model):

$$\partial_t^2 u + \partial_t u + (1-\Delta)u - \sigma :u^2: + M(:u^2:)u = \sqrt{2}\xi, \tag{1.3.1}$$

where M is defined by

$$M(w) = 6A\left|\int_{\mathbb{T}^3} w dx\right| \int_{\mathbb{T}^3} w dx. \tag{1.3.2}$$

Here, the term $M(:u^2:)u$ in (1.3.1) comes from the taming by the Wick-ordered L^2-norm appearing in (1.2.14). The term $:u^2:$ denotes the Wick renormalization[11] of u^2, formally given by $:u^2: = u^2 - \infty$. Namely, the equation (1.3.1) is just a formal expression at this point. In the following, we provide the meaning of the process u in (1.3.1) by a limiting procedure. In Chapter 5, we use the paracontrolled calculus to give a more precise meaning to (1.3.1) by rewriting it into a system for three unknowns. See (5.2.27) below.

Given $N \in \mathbb{N}$, we consider the following quadratic SdNLW with a truncated noise:

$$\partial_t^2 u_N + \partial_t u_N + (1-\Delta)u_N - \sigma :u_N^2: + M(:u_N^2:)u_N = \sqrt{2}\pi_N\xi, \tag{1.3.3}$$

where π_N is as in (1.2.5) and the renormalized nonlinearity is defined by

$$:u_N^2: = u_N^2 - \sigma_N \tag{1.3.4}$$

with σ_N as in (1.2.8). See also (5.2.9). In Chapter 5, we study SdNLW (1.3.3) with the truncated noise and prove the following local well-posedness statement for the hyperbolic Φ_3^3-model.

Theorem 1.3.1. *Given $s > \frac{1}{2}$, let $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^3)$. Let $(\phi_0^\omega, \phi_1^\omega)$ be a pair of the Gaussian random distributions with* $\mathrm{Law}(\phi_0^\omega, \phi_1^\omega) = \vec{\mu} = \mu \otimes \mu_0$. *Then, the solution $(u_N, \partial_t u_N)$ to the quadratic SdNLW* (1.3.3) *with the truncated noise and the initial data*

$$(u_N, \partial_t u_N)|_{t=0} = (u_0, u_1) + \pi_N(\phi_0^\omega, \phi_1^\omega) \tag{1.3.5}$$

converges to a stochastic process $(u, \partial_t u) \in C([0,T]; \mathcal{H}^{-\frac{1}{2}-\varepsilon}(\mathbb{T}^3))$ almost surely, where $T = T(\omega)$ is an almost surely positive stopping time.

[11] In order to give a proper meaning to $:u^2:$, we need to assume a structure on u. We postpone this discussion to Chapter 5.

The limit $(u, \partial_t u)$ formally satisfies the equation (1.3.1). Here, we took the initial data of the form (1.3.5) for simplicity of the presentation. A slight modification of the proof yields an analogue of Theorem 1.3.1 with deterministic initial data $(u_N, \partial_t u_N)|_{t=0} = (u_0, u_1)$. In this case, we need to choose a diverging constant σ_N, depending on t. See [35,36] for such an argument.

We follow the paracontrolled approach in [36], where the quadratic SNLW on $\mathbb{T}^3$ was studied. However, the additional term M in (1.3.1) and (1.3.3) contains an ill-defined product $:u^2:$ (or $:u_N^2:$ in the limiting sense). In order to treat this term, the analysis in [36] is not sufficient and thus we also need to adapt the paracontrolled analysis in our previous work [54] and rewrite the equation into a system for three unknowns. (Note that in [36], the resulting system was for two unknowns.) We also point out that, unlike [36] (see also [47] in the context of the parabolic Φ^4_3-model), the equation for a less regular, paracontrolled component in our system (see (5.2.27) below) is nonlinear in the unknowns. We then construct a continuous map from the space of enhanced data sets to solutions. While the proof of Theorem 1.3.1 follows from a slight modification of the arguments in [36,54], we present details in Chapter 5 for readers' convenience.

In order to establish our main goal in the dynamical part of the program (Theorem 1.1.4), we need to study the hyperbolic Φ^3_3-model with the Gibbs measure initial data. Since the Gibbs measure $\vec{\rho} = \rho \otimes \mu_0$ in (1.2.18) and the Gaussian field $\vec{\mu} = \mu \otimes \mu_0$ are mutually singular as shown in Theorem 1.2.1, it may seem that the local well-posedness in Theorem 1.3.1 with the Gaussian initial data (plus smoother deterministic initial data) is irrelevant. However, as we see in Chapter 6, the analysis for proving Theorem 1.3.1 provides us with a good intuition of the well-posedness problem for the hyperbolic Φ^3_3-model with the Gibbs measure initial data. Furthermore, one of advantages of considering the Gaussian initial data (as in (1.3.5)) is that it provides a clear reason why σ_N appears in the renormalization in (1.3.4), since σ_N is nothing but the variance of the first order approximation (= the stochastic convolution defined in (5.2.3)) to the solution to (1.3.3); see (5.2.9). This is the main reason for considering the local-in-time problem with the Gaussian initial data.

Next, we turn our attention to the globalization problem. For this purpose, we need to consider a different approximating equation. Given $N \in \mathbb{N}$, we consider the truncated hyperbolic Φ^3_3-model:

$$\begin{aligned} &\partial_t^2 u_N + \partial_t u_N + (1 - \Delta) u_N \\ &\quad - \sigma \pi_N\big(:(\pi_N u_N)^2:\big) + M\big(:(\pi_N u_N)^2:\big)\pi_N u_N = \sqrt{2}\xi, \end{aligned} \tag{1.3.6}$$

where $:(\pi_N u_N)^2: = (\pi_N u_N)^2 - \sigma_N$. A slight modification of the proof of Theorem 1.3.1 yields uniform (in N) local well-posedness of the truncated equation (1.3.6) (with the same limiting process $(u, \partial_t u)$ as in Theorem 1.3.1) for the initial data of the form (1.3.5). By exploiting (formal) invariance of the truncated Gibbs measure $\vec{\rho}_N$ in (1.2.17),[12] we see that the truncated hyperbolic Φ^3_3-model (1.3.6) is almost surely globally well-posed with respect to the truncated Gibbs measure $\vec{\rho}_N$ and, moreover, $\vec{\rho}_N$ is invariant under the resulting dynamics; see Lemma 6.2.3.

We now state almost sure global well-posedness of the hyperbolic Φ^3_3-model.

Theorem 1.3.2. *Let* $0 < |\sigma| < \sigma_0$ *and* $A = A(\sigma) > 0$ *is sufficiently large as in Theorem* 1.2.1 (i). *Then, there exists a non-trivial stochastic process* $(u, \partial_t u) \in C(\mathbb{R}_+; \mathcal{H}^{-\frac{1}{2}-\varepsilon}(\mathbb{T}^3))$ *for any* $\varepsilon > 0$ *such that, given any* $T > 0$, *the solution* $(u_N, \partial_t u_N)$ *to the truncated hyperbolic* Φ^3_3*-model* (1.3.6) *with the random initial data distributed by the truncated Gibbs measure* $\vec{\rho}_N = \rho_N \otimes \mu_0$ *in* (1.2.17) *converges to* $(u, \partial_t u)$ *in* $C([0, T]; \mathcal{H}^{-\frac{1}{2}-\varepsilon}(\mathbb{T}^3))$. *Furthermore, we have* $\mathrm{Law}((u(t), \partial_t u(t))) = \vec{\rho}$ *for any* $t \in \mathbb{R}_+$.

The main difficulty in proving Theorem 1.3.2 comes from the mutual singularity of the Gibbs measure $\vec{\rho}$ and the base Gaussian measure $\vec{\mu}$ (and the fact that the truncated Gibbs measure $\vec{\rho}_N$ converges to $\vec{\rho}$ only weakly) such that Bourgain's invariant measure argument [9, 10] is not directly applicable. In the context of the defocusing Hartree NLW on $\mathbb{T}^3$, Bringmann [14] encountered the same issue, and introduced a new globalization argument, where a large time stability theory (in the paracontrolled setting) plays a crucial role. Bourgain's invariant measure argument is often described (see [14]) as "the probabilistic version of a deterministic global theory using a (sub-critical) conservation law". In [14], Bringmann considers the quantity $\vec{\rho}_M((u_N, \partial_t u_N)(t) \in A)$, where $(u_N, \partial_t u_N)$ is the solution to the truncated equation with a cutoff parameter N. While such an expression is not conserved for $M \neq N$, it should be close to being constant in time when $M, N \gg 1$. For this reason, he describes his new globalization argument as "the probabilistic version of a deterministic global theory using almost conservation laws". We also point out that Bringmann's analysis relies on the fact that the (truncated) Gibbs measure is absolutely continuous with respect to a shifted measure [13,54] (as in Appendix A below).

While it is possible to follow Bringmann's approach, we instead introduce a new simple alternative argument to prove almost sure global well-posedness. Our approach consists of the following four steps:

Step 1. We first establish a uniform (in N) exponential integrability of the truncated enhanced data set (see (6.1.10) below) with respect to the truncated measure (Proposition 6.2.4). We directly achieve this by combining the variational approach with space-time estimates *without* any reference to (the truncated version of) the shifted measure constructed in Appendix A.

[12] This is essentially Bourgain's invariant measure argument [9] applied to the truncated hyperbolic Φ^3_3-model (1.3.6), whose nonlinear part is finite dimensional.

Step 2. Next, by a slight modification of the local well-posedness argument, we prove a stability result (Proposition 6.3.1). This is done by a simple contraction argument, with an exponentially decaying weight in time.

Step 3. Then, using the invariance of the truncated Gibbs measure, we establish a uniform (in N) control on the solution to the truncated system (see (6.3.2) below) with a large probability. The argument relies on a discrete Gronwall argument but is very straightforward.

Step 4. In the last step, we study the convergence property of the distributions of the truncated enhanced data sets, emanating from the truncated Gibbs measures. In particular, we study the Wasserstein-1 distance of such a distribution with the limiting distribution, using ideas from theory of optimal transport (the Kantorovich duality). See Proposition 6.3.3 below.

Once we establish these four steps, Theorem 1.3.2 follows in a straightforward manner. We believe that our new globalization argument is very simple, at least at a conceptual level, and is easy to implement. See Chapter 6 for further details.

Remark 1.3.3. (i) In this paper, we treated the hyperbolic Φ_3^3-model. In the three-dimensional case, it is possible to consider the defocusing quartic interaction potential, namely the Φ_3^4-measure. This leads to the following hyperbolic Φ_3^4-model on $\mathbb{T}^3$:

$$\partial_t^2 u + \partial_t u + (1 - \Delta)u + u^3 = \sqrt{2}\xi. \tag{1.3.7}$$

Over the last ten years, the parabolic Φ_3^4-model:

$$\partial_t u + (1 - \Delta)u + u^3 = \sqrt{2}\xi, \tag{1.3.8}$$

has been studied extensively by many authors. See [1, 18, 32, 34, 39, 43, 47, 49] and references therein. Up to date, the well-posedness issue of the hyperbolic Φ_3^4-model (1.3.7) remains as an important open problem.[13] In [65], using Bringmann's analysis [14], Y. Wang, Zine, and the first author recently proved local well-posedness of the cubic stochastic NLW[14] on $\mathbb{T}^3$ with an almost space-time white noise forcing (i.e. replacing ξ by $\langle \nabla \rangle^{-\alpha}\xi$ for any $\alpha > 0$ in (1.3.7)).

(ii) In the parabolic setting (1.1.14), there is no issue is applying Bourgain's invariant measure argument in the usual manner since it is possible to prove local well-posedness with deterministic initial data at the regularity of the Φ_3^3-measure. See [40] in the case of the parabolic Φ_3^4-model (1.3.8).

[13] In a recent paper [15], Bringmann, Deng, Nahmod, and Yue resolved this open problem in the case of the Gibbsian initial data with no stochastic forcing.

[14] In [65], the authors considered the undamped SNLW but the same analysis applies to the damped SNLW.

1.4 On frequency projectors

We conclude this introduction by discussing different frequency projectors. Given $N \in \mathbb{N}$, define the ball frequency projector π_N^{ball} onto the frequencies $\{n \in \mathbb{Z}^3 : |n| \leq N\}$ by setting

$$\pi_N^{\text{ball}} f = \sum_{n \in \mathbb{Z}^3} \chi_N^{\text{ball}}(n) \hat{f}(n) e_n, \tag{1.4.1}$$

associated with a Fourier multiplier

$$\chi_N^{\text{ball}}(n) = \mathbf{1}_B\big(N^{-1} n\big),$$

where B denotes the unit ball in $\mathbb{R}^3$ centered at the origin:

$$B = \big\{\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : |\xi| \leq 1\big\}.$$

We also define the smooth frequency projector π_N^{smooth} onto the frequencies $\{n \in \mathbb{Z}^3 : |n| \leq N\}$ by setting

$$\pi_N^{\text{smooth}} f = \sum_{n \in \mathbb{Z}^3} \chi_N^{\text{smooth}}(n) \hat{f}(n) e_n, \tag{1.4.2}$$

associated with a Fourier multiplier

$$\chi_N^{\text{smooth}}(n) = \chi\big(N^{-1} n\big)$$

for some fixed even function $\chi \in C_c^\infty(\mathbb{R}^3; [0,1])$ with $\operatorname{supp} \chi \subset \{\xi \in \mathbb{R}^3 : |\xi| \leq 1\}$ and $\chi \equiv 1$ on $\{\xi \in \mathbb{R}^3 : |\xi| \leq \frac{1}{2}\}$.

In Sections 1.2 and 1.3, we stated the (non-)construction of the Φ_3^3-measure (Theorem 1.2.1) and the dynamical results for the hyperbolic Φ_3^3-model (Theorems 1.3.1 and 1.3.2), using the cube frequency projector $\pi_N = \pi_N^{\text{cube}}$ defined in (1.2.5). In comparison with the ball frequency projector π_N^{ball} and the smooth frequency projector π_N^{smooth}, there are two important properties that the cube frequency projector π_N^{cube} possesses simultaneously.

(i) As a composition of (modulated) Hilbert transforms in different coordinate directions, the cube frequency projector π_N^{cube} is uniformly (in N) bounded in $L^p(\mathbb{T}^3)$ for any $1 < p < \infty$.

(ii) The cube frequency projector is indeed a projection, in particular satisfying $(\mathrm{Id} - \pi_N^{\text{cube}})\pi_N^{\text{cube}} = 0$.

We make use of both of these properties in a crucial manner. Note that while the ball frequency projector π_N^{ball} satisfies the property (ii), it is bounded in $L^p(\mathbb{T}^3)$ only for $p = 2$ [27] and thus the property (i) is not satisfied. On the other hand, by Young's inequality, the smooth frequency projector π_N^{smooth} is bounded on $L^p(\mathbb{T}^3)$ for any $1 \leq p \leq \infty$ but it does not satisfy the property (ii).

Roughly speaking, Theorem 1.2.1 on the (non-)construction of the Φ_3^3-measure consists of the following five results:

(1) the uniform exponential integrability (1.2.13) and tightness of the truncated Φ_3^3-measures ρ_N in the weakly nonlinear regime,

(2) uniqueness of the limiting Φ_3^3-measure in the weakly nonlinear regime,

(3) mutual singularity of the Φ_3^3-measure and the base Gaussian free field in the weakly nonlinear regime,

(4) non-normalizability of the Φ_3^3-measure in the strongly nonlinear regime,

(5) non-convergence of the truncated Φ_3^3-measures ρ_N in the strongly nonlinear regime.

Starting with the truncated Φ_3^3-measures ρ_N in (1.2.11) defined in terms of the cube frequency projector π_N^{cube} in (1.2.5), we establish (1)–(5) in Chapters 3 and 4. In proving (5), the property (i) above plays an important role and thus our argument does not apply to the ball frequency projector π_N^{ball}. See Remark 4.4.2.

In establishing (2), uniqueness of the limiting Φ_3^3-measure (Proposition 3.3.1), we crucially make use of the property (ii) to show that a certain problematic term vanishes; see I_2 in (3.3.12). It turns out that this problematic term reflects the critical nature of the problem, where there is no room to spare, not even logarithmically. In the case of the cube frequency projector π_N^{cube}, the property (ii) allows us to conclude that this term in fact vanishes. In the case of the smooth projector π_N^{smooth}, the property (ii) does not hold and thus we need to show by hand that this problematic term tends to 0. As mentioned above, however, there is no room to spare and it seems rather non-trivial to prove such a convergence result by a modification of our argument. See Remark 3.3.2. In establishing (4) and (5), we first construct a reference measure ν_δ as a limit of the tamed version $\nu_{N,\delta}$ of the truncated Φ_3^3-measure in (1.2.15) (Proposition 4.1.1). With the smooth projector π_N^{smooth}, the same issue also appears in showing uniqueness of the limit ν_δ.

While we believe that Theorem 1.2.1 holds for both the ball frequency projector π_N^{ball} (in particular (5) above) and the smooth frequency projector π_N^{smooth} (in particular (2) above), we do not pursue these issues further in this paper in order to keep the paper length under control.

Let us now turn to the dynamical part. As for the smooth frequency projector π_N^{smooth}, there is no modification needed for the local well-posedness part. However, as mentioned above, there is no uniqueness of the limiting Φ_3^3-measure in this case. Furthermore, we point out that the proof of Proposition 6.3.3 also breaks down for the smooth frequency projector π_N^{smooth} since part of the argument relies on the proof of Proposition 3.3.1; see (6.3.64). On the other hand, as for the ball frequency projector π_N^{ball}, both Theorems 1.3.1 and 1.3.2 hold as they are stated. However, the proof of the local well-posedness part needs to be modified in view of the unboundedness of the ball frequency projector π_N^{ball} in the Strichartz spaces (see (5.5.1)). Note that this issue can be easily remedied by using the Fourier restriction norm method via the (L^2-based) $X^{s,b}$-spaces as in [14, 64, 65].