## Chapter 2

# Notations and basic lemmas

In describing regularities of functions and distributions, we use $\varepsilon > 0$ to denote a small constant. We usually suppress the dependence on such $\varepsilon > 0$ in an estimate. For $a, b > 0$, we use $a \lesssim b$ to mean that there exists $C > 0$ such that $a \le Cb$. By $a \sim b$, we mean that $a \lesssim b$ and $b \lesssim a$.

In dealing with space-time functions, we use the following shorthand notation $L^q_T L^r_x = L^q([0, T]; L^r(\mathbb{T}^3))$, etc.

## 2.1 Sobolev and Besov spaces

Let $s \in \mathbb{R}$ and $1 \le p \le \infty$. We define the $L^2$-based Sobolev space $H^s(\mathbb{T}^d)$ by the norm:

$$\|f\|_{H^s} = \big\| \langle n \rangle^s \hat{f}(n) \big\|_{\ell^2_n}.$$

We also define the $L^p$-based Sobolev space $W^{s,p}(\mathbb{T}^d)$ by the norm:

$$\|f\|_{W^{s,p}} = \big\| \mathcal{F}^{-1}[\langle n \rangle^s \hat{f}(n)] \big\|_{L^p}.$$

When $p = 2$, we have $H^s(\mathbb{T}^d) = W^{s,2}(\mathbb{T}^d)$.

Let $\phi : \mathbb{R} \to [0, 1]$ be a smooth bump function supported on $[-\frac{8}{5}, \frac{8}{5}]$ and $\phi \equiv 1$ on $[-\frac{5}{4}, \frac{5}{4}]$. For $\xi \in \mathbb{R}^d$, we set $\varphi_0(\xi) = \phi(|\xi|)$ and

$$\varphi_j(\xi) = \phi\Big(\frac{|\xi|}{2^j}\Big) - \phi\Big(\frac{|\xi|}{2^{j-1}}\Big) \tag{2.1.1}$$

for $j \in \mathbb{N}$. Then, for $j \in \mathbb{Z}_{\ge 0} := \mathbb{N} \cup \{0\}$, we define the Littlewood–Paley projector $\mathbf{P}_j$ as the Fourier multiplier operator with a symbol $\varphi_j$. Note that we have

$$\sum_{j=0}^{\infty} \varphi_j(\xi) = 1$$

for each $\xi \in \mathbb{R}^d$. Thus, we have

$$f = \sum_{j=0}^{\infty} \mathbf{P}_j f.$$

Let us now recall the definition and basic properties of paraproducts introduced by Bony [7]. See [2, 34] for further details. Given two functions $f$ and $g$ on $\mathbb{T}^3$ of regularities $s_1$ and $s_2$, we write the product $fg$ as

$$\begin{aligned} fg &= f \olessthan g + f \ominus g + f \ogreaterthan g \\ &:= \sum_{j<k-2} \mathbf{P}_j f \mathbf{P}_k g + \sum_{|j-k|\le 2} \mathbf{P}_j f \mathbf{P}_k g + \sum_{k<j-2} \mathbf{P}_j f \mathbf{P}_k g. \end{aligned} \tag{2.1.2}$$

The first term $f \olessthan g$ (and the third term $f \ogreaterthan g$) is called the paraproduct of $g$ by $f$ (the paraproduct of $f$ by $g$, respectively) and it is always well defined as a distribution of regularity $\min(s_2, s_1 + s_2)$. On the other hand, the resonant product $f \ominus g$ is well defined in general only if $s_1 + s_2 > 0$. See Lemma 2.1.2 below. In the following, we also use the notation $f \circledcirc\!\!\!\geq g := f \ogreaterthan g + f \ominus g$. In studying a nonlinear problem, main difficulty usually arises in making sense of a product. Since paraproducts are always well defined, such a problem comes from a resonant product. In particular, when the sum of regularities is negative, we need to impose an extra structure to make sense of a (seemingly) ill-defined resonant product. See Chapter 5 for a further discussion on the paracontrolled approach in this direction.

Next, we recall the basic properties of the Besov spaces $B^s_{p,q}(\mathbb{T}^d)$ defined by the norm:

$$\|u\|_{B^s_{p,q}} = \big\| 2^{sj} \|\mathbf{P}_j u\|_{L^p_x} \big\|_{\ell^q_j(\mathbb{Z}_{\ge 0})}.$$

We denote the Hölder–Besov space by $\mathcal{C}^s(\mathbb{T}^d) = B^s_{\infty,\infty}(\mathbb{T}^d)$. Note that (i) the parameter $s$ measures differentiability and $p$ measures integrability, (ii) $H^s(\mathbb{T}^d) = B^s_{2,2}(\mathbb{T}^d)$, and (iii) for $s > 0$ and not an integer, $\mathcal{C}^s(\mathbb{T}^d)$ coincides with the classical Hölder spaces $C^s(\mathbb{T}^d)$; see [31].

We recall the basic estimates in Besov spaces. See [2, 38] for example.

**Lemma 2.1.1.** *The following estimates hold.*

(i) (interpolation) *Let $s, s_1, s_2 \in \mathbb{R}$ and $p, p_1, p_2 \in (1, \infty)$ such that $s = \theta s_1 + (1-\theta)s_2$ and $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$ for some $0 < \theta < 1$. Then, we have*

$$\|u\|_{W^{s,p}} \lesssim \|u\|^{\theta}_{W^{s_1,p_1}} \|u\|^{1-\theta}_{W^{s_2,p_2}}. \tag{2.1.3}$$

(ii) (immediate embeddings) *Let $s_1, s_2 \in \mathbb{R}$ and $p_1, p_2, q_1, q_2 \in [1, \infty]$. Then, we have*

$$\begin{aligned} &\|u\|_{B^{s_1}_{p_1,q_1}} \lesssim \|u\|_{B^{s_2}_{p_2,q_2}} \quad \text{for } s_1 \le s_2,\ p_1 \le p_2, \text{ and } q_1 \ge q_2, \\ &\|u\|_{B^{s_1}_{p_1,q_1}} \lesssim \|u\|_{B^{s_2}_{p_1,\infty}} \quad \text{for } s_1 < s_2, \\ &\|u\|_{B^{0}_{p_1,\infty}} \lesssim \|u\|_{L^{p_1}} \lesssim \|u\|_{B^{0}_{p_1,1}}. \end{aligned} \tag{2.1.4}$$

(iii) (Besov embedding) *Let $1 \le p_2 \le p_1 \le \infty$, $q \in [1, \infty]$, and $s_2 \ge s_1 + d\big(\frac{1}{p_2} - \frac{1}{p_1}\big)$. Then, we have*

$$\|u\|_{B^{s_1}_{p_1,q}} \lesssim \|u\|_{B^{s_2}_{p_2,q}}.$$

(iv) (duality) *Let* $s \in \mathbb{R}$ *and* $p, p', q, q' \in [1, \infty]$ *such that* $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$. *Then, we have*

$$\left| \int_{\mathbb{T}^d} uv dx \right| \le \|u\|_{B^s_{p,q}} \|v\|_{B^{-s}_{p',q'}}, \tag{2.1.5}$$

*where* $\int_{\mathbb{T}^d} uv dx$ *denotes the duality pairing between* $B^s_{p,q}(\mathbb{T}^d)$ *and* $B^{-s}_{p',q'}(\mathbb{T}^d)$.

(v) (fractional Leibniz rule) *Let* $p, p_1, p_2, p_3, p_4 \in [1, \infty]$ *such that* $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4} = \frac{1}{p}$. *Then, for every* $s > 0$, *we have*

$$\|uv\|_{B^s_{p,q}} \lesssim \|u\|_{B^s_{p_1,q}} \|v\|_{L^{p_2}} + \|u\|_{L^{p_3}} \|v\|_{B^s_{p_4,q}}. \tag{2.1.6}$$

The interpolation (2.1.3) follows from the Littlewood–Paley characterization of Sobolev norms via the square function and Hölder's inequality.

**Lemma 2.1.2** (Paraproduct and resonant product estimates). *Let* $s_1, s_2 \in \mathbb{R}$ *and* $1 \le p, p_1, p_2, q \le \infty$ *such that* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. *Then, we have*

$$\| f \otimes g\|_{B^{s_2}_{p,q}} \lesssim \|f\|_{L^{p_1}} \|g\|_{B^{s_2}_{p_2,q}}. \tag{2.1.7}$$

*When* $s_1 < 0$, *we have*

$$\| f \otimes g\|_{B^{s_1+s_2}_{p,q}} \lesssim \|f\|_{B^{s_1}_{p_1,q}} \|g\|_{B^{s_2}_{p_2,q}}. \tag{2.1.8}$$

*When* $s_1 + s_2 > 0$, *we have*

$$\| f \ominus g\|_{B^{s_1+s_2}_{p,q}} \lesssim \|f\|_{B^{s_1}_{p_1,q}} \|g\|_{B^{s_2}_{p_2,q}}. \tag{2.1.9}$$

The product estimates (2.1.7), (2.1.8), and (2.1.9) follow easily from the definition (2.1.2) of the paraproduct and the resonant product. See [2, 48] for details of the proofs in the non-periodic case (which can be easily extended to the current periodic setting).

We also recall the following product estimate from [6, 35].

**Lemma 2.1.3.** *Let* $s > 0$.

(i) *Let* $1 < p_j, q_j, r \le \infty$, $j = 1, 2$ *such that* $\frac{1}{r} = \frac{1}{p_j} + \frac{1}{q_j}$. *Then, we have*

$$\|\langle \nabla \rangle^s (fg)\|_{L^r(\mathbb{T}^3)} \lesssim \|\langle \nabla \rangle^s f\|_{L^{p_1}(\mathbb{T}^3)} \|g\|_{L^{q_1}(\mathbb{T}^3)} + \|f\|_{L^{p_2}(\mathbb{T}^3)} \|\langle \nabla \rangle^s g\|_{L^{q_2}(\mathbb{T}^3)}.$$

(ii) *Let* $1 < p \le \infty$ *and* $1 < q, r < \infty$ *such that* $s \ge 3(\frac{1}{p} + \frac{1}{q} - \frac{1}{r})$ *and* $q, r' \ge p'$. *Then, we have*

$$\|\langle \nabla \rangle^{-s} (fg)\|_{L^r(\mathbb{T}^3)} \lesssim \|\langle \nabla \rangle^{-s} f\|_{L^p(\mathbb{T}^3)} \|\langle \nabla \rangle^s g\|_{L^q(\mathbb{T}^3)}.$$

## 2.2 On discrete convolutions

Next, we recall the following basic lemma on a discrete convolution.

**Lemma 2.2.1.** *Let $d \geq 1$ and $\alpha, \beta \in \mathbb{R}$ satisfy*

$$\alpha + \beta > d \quad \text{and} \quad \alpha < d.$$

*Then, we have*

$$\sum_{n=n_1+n_2} \frac{1}{\langle n_1 \rangle^\alpha \langle n_2 \rangle^\beta} \lesssim \langle n \rangle^{-\alpha+\lambda}$$

*for any $n \in \mathbb{Z}^d$, where $\lambda = \max(d-\beta, 0)$ when $\beta \neq d$ and $\lambda = \varepsilon$ when $\beta = d$ for any $\varepsilon > 0$.*

Lemma 2.2.1 follows from elementary computations. See, for example, [29, Lemma 4.2] and [49, Lemma 4.1].

## 2.3 Tools from stochastic analysis

We conclude this chapter by recalling useful lemmas from stochastic analysis. See [51, 69] for basic definitions. Let $(H, B, \mu)$ be an abstract Wiener space. Namely, $\mu$ is a Gaussian measure on a separable Banach space $B$ with $H \subset B$ as its Cameron–Martin space. Given a complete orthonormal system $\{e_j\}_{j\in\mathbb{N}} \subset B^*$ of $H^* = H$, we define a polynomial chaos of order $k$ to be an element of the form $\prod_{j=1}^\infty H_{k_j}(\langle x, e_j \rangle)$, where $x \in B$, $k_j \neq 0$ for only finitely many $j$'s, $k = \sum_{j=1}^\infty k_j$, $H_{k_j}$ is the Hermite polynomial of degree $k_j$, and $\langle \cdot, \cdot \rangle = {}_B\langle \cdot, \cdot \rangle_{B^*}$ denotes the $B$-$B^*$ duality pairing. We then denote the closure of polynomial chaoses of order $k$ under $L^2(B, \mu)$ by $\mathcal{H}_k$. The elements in $\mathcal{H}_k$ are called homogeneous Wiener chaoses of order $k$. We also set

$$\mathcal{H}_{\leq k} = \bigoplus_{j=0}^{k} \mathcal{H}_j$$

for $k \in \mathbb{N}$.

As a consequence of the hypercontractivity of the Ornstein–Uhlenbeck semigroup due to Nelson [50], we have the following Wiener chaos estimate [70, Theorem I.22]. See also [71, Proposition 2.4].

**Lemma 2.3.1.** *Let $k \in \mathbb{N}$. Then, we have*

$$\|X\|_{L^p(\Omega)} \leq (p-1)^{\frac{k}{2}} \|X\|_{L^2(\Omega)}$$

*for any finite $p \geq 2$ and any $X \in \mathcal{H}_{\leq k}$.*

Lastly, we recall the following orthogonality relation for the Hermite polynomials. See [51, Lemma 1.1.1].

**Lemma 2.3.2.** *Let $f$ and $g$ be jointly Gaussian random variables with mean zero and variances $\sigma_f$ and $\sigma_g$. Then, we have*

$$\mathbb{E}\big[H_k(f;\sigma_f)H_\ell(g;\sigma_g)\big] = \delta_{k\ell}k!\big\{\mathbb{E}[fg]\big\}^k,$$

*where $H_k(x,\sigma)$ denotes the Hermite polynomial of degree $k$ with variance parameter $\sigma$.*