Chapter 3

Construction of the Φ_3^3-measure in the weakly nonlinear regime

In this chapter, we present the construction of the Φ_3^3-measure in the weakly nonlinear regime (Theorem 1.2.1 (i)). Our proof is based on the variational approach introduced by Barashkov and Gubinelli [3]. See the Boué–Dupuis variational formula (Lemma 3.1.1) below. In Section 3.1, we briefly go over the setup of the variational formulation for a partition function. In Section 3.2, we first establish the uniform exponential integrability (1.2.13) and then prove tightness of the truncated Φ_3^3-measures ρ_N in (1.2.11), which implies weak convergence of a subsequence. In Section 3.3, we follow the approach introduced in our previous work [54] and prove uniqueness of the limiting Φ_3^3-measure, thus establishing weak convergence of the entire sequence $\{\rho_N\}_{N\in\mathbb{N}}$. Finally, in Section 3.4, we show that the Φ_3^3-measure and the base Gaussian free field μ in (1.2.2) are mutually singular. While our proof of singularity of the Φ_3^3-measure is inspired by the discussion in [4, Section 4], we directly prove singularity without referring to a shifted measure. In Appendix A, we show that the Φ_3^3-measure is indeed absolutely continuous with respect to the shifted measure $\operatorname{Law}(Y(1) + \sigma\mathfrak{Z}(1) + \mathcal{W}(1))$, where $\operatorname{Law}(Y(1)) = \mu$, $\mathfrak{Z} = \mathfrak{Z}(Y)$ is the limit of the quadratic process $\mathfrak{Z}^N$ defined in (3.2.3), and the auxiliary quintic process $\mathcal{W} = \mathcal{W}(Y)$ is defined in (A.1.1).

3.1 Boué–Dupuis variational formula

Let $W(t)$ be the cylindrical Wiener process on $L^2(\mathbb{T}^3)$ (with respect to the underlying probability measure $\mathbb{P}$):

$$W(t) = \sum_{n\in\mathbb{Z}^3} B_n(t) e_n, \tag{3.1.1}$$

where $\{B_n\}_{n\in\mathbb{Z}^3}$ is defined by $B_n(t) = \langle \xi, \mathbf{1}_{[0,t]} \cdot e_n\rangle_{x,t}$. Here, $\langle\cdot,\cdot\rangle_{x,t}$ denotes the duality pairing on $\mathbb{T}^3\times\mathbb{R}$. Note that we have, for any $n\in\mathbb{Z}^3$,

$$\operatorname{Var}(B_n(t)) = \mathbb{E}\Big[\langle \xi, \mathbf{1}_{[0,t]} \cdot e_n\rangle_{x,t}\overline{\langle \xi, \mathbf{1}_{[0,t]} \cdot e_n\rangle_{x,t}}\Big] = \big\|\mathbf{1}_{[0,t]}\cdot e_n\big\|^2_{L^2_{x,t}} = t.$$

As a result, we see that $\{B_n\}_{n\in\Lambda_0}$ is a family of mutually independent complex-valued Brownian motions conditioned so that $B_{-n} = \overline{B_n}$, $n\in\mathbb{Z}^3$.[1] We then define a centered

[1] In particular, B_0 is a standard real-valued Brownian motion.

Gaussian process $Y(t)$ by

$$Y(t) = \langle \nabla \rangle^{-1} W(t). \tag{3.1.2}$$

Then, we have $\mathrm{Law}(Y(1)) = \mu$. By setting $Y_N = \pi_N Y$, we have $\mathrm{Law}(Y_N(1)) = (\pi_N)_\# \mu$. In particular, we have $\mathbb{E}[Y_N(1)^2] = \sigma_N$, where σ_N is as in (1.2.8).

Next, let $\mathbb{H}_a$ denote the space of drifts, which are the progressively measurable processes belonging to $L^2([0,1]; L^2(\mathbb{T}^3))$, $\mathbb{P}$-almost surely. For later use, we also define $\mathbb{H}_a^1$ to be the space of drifts, which are the progressively measurable processes belonging to $L^2([0,1]; H^1(\mathbb{T}^3))$, $\mathbb{P}$-almost surely. Namely, we have

$$\mathbb{H}_a^1 = \langle \nabla \rangle^{-1} \mathbb{H}_a. \tag{3.1.3}$$

We now state the Boué–Dupuis variational formula [8, 77]; in particular, see [77, Theorem 7]. See also [3, Theorem 2].

Lemma 3.1.1. *Let $Y(t) = \langle \nabla \rangle^{-1} W(t)$ be as in (3.1.2). Fix $N \in \mathbb{N}$. Suppose that $F : C^\infty(\mathbb{T}^3) \to \mathbb{R}$ is measurable such that $\mathbb{E}[|F(Y_N(1))|^p] < \infty$ and $\mathbb{E}[|e^{-F(Y_N(1))}|^q] < \infty$ for some $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then, we have*

$$-\log \mathbb{E}\big[e^{-F(Y_N(1))}\big] = \inf_{\theta \in \mathbb{H}_a} \mathbb{E}\bigg[F(Y_N(1) + \pi_N I(\theta)(1)) + \frac{1}{2}\int_0^1 \|\theta(t)\|^2_{L^2_x} dt\bigg], \tag{3.1.4}$$

where $I(\theta)$ is defined by

$$I(\theta)(t) = \int_0^t \langle \nabla \rangle^{-1} \theta(t') dt'. \tag{3.1.5}$$

Lemma 3.1.1 plays a fundamental role in almost every step of the argument presented in this chapter and Chapter 4.

We state a useful lemma on the pathwise regularity estimates of $:Y^k(t):$ and $I(\theta)(1)$.

Lemma 3.1.2. (i) *For $k = 1, 2$, any finite $p \geq 2$, and $\varepsilon > 0$, $:Y_N^k(t):$ converges to $:Y^k(t):$ in $L^p(\Omega; \mathcal{C}^{-\frac{k}{2}-\varepsilon}(\mathbb{T}^3))$ and also almost surely in $\mathcal{C}^{-\frac{k}{2}-\varepsilon}(\mathbb{T}^3)$. Moreover, we have*

$$\mathbb{E}\big[\| :Y_N^k(t): \|^p_{\mathcal{C}^{-\frac{k}{2}-\varepsilon}}\big] \lesssim p^{\frac{k}{2}} < \infty, \tag{3.1.6}$$

uniformly in $N \in \mathbb{N}$ and $t \in [0,1]$. We also have

$$\mathbb{E}\big[\| :Y_N^2(t): \|^2_{H^{-1}}\big] \sim t^2 \log N \tag{3.1.7}$$

for any $t \in [0,1]$.

(ii) *For any $N \in \mathbb{N}$, we have*

$$\mathbb{E}\bigg[\int_{\mathbb{T}^3} :Y_N^3(1): \, dx\bigg] = 0.$$

(iii) *For any* $\theta \in \mathbb{H}_a$, *we have*

$$\|I(\theta)(1)\|_{H^1}^2 \le \int_0^1 \|\theta(t)\|_{L^2}^2 dt.$$

Proof. The bound (3.1.6) for $\varepsilon > 0$ follows immediately from the Wiener chaos estimate (Lemma 2.3.1), Lemma 2.3.2, and then carrying out summations, using Lemma 2.2.1. See, for example, [35, 36]. As for (3.1.7), proceeding as in the proof of [63, Lemma 2.5] with Lemma 2.3.2, we have

$$\begin{aligned}
&\mathbb{E}\big[\| :Y_N^2(t): \|_{H^{-1}}^2\big] \\
&= \sum_{n\in\mathbb{Z}^3} \frac{1}{\langle n\rangle^2} \int_{\mathbb{T}_x^3\times\mathbb{T}_y^3} \mathbb{E}\big[H_2(Y_N(x,t);t\sigma_N)H_2(Y_N(y,t);t\sigma_N)\big]e_n(y-x)dxdy \\
&= \sum_{n\in\mathbb{Z}^3} \frac{t^2}{\langle n\rangle^2} \sum_{n_1,n_2\in\mathbb{Z}^3} \frac{\chi_N^2(n_1)\chi_N^2(n_2)}{\langle n_1\rangle^2\langle n_2\rangle^2} \int_{\mathbb{T}_x^3\times\mathbb{T}_y^3} e_{n_1+n_2-n}(x-y)dxdy \\
&= \sum_{n\in\mathbb{Z}^3} \frac{t^2}{\langle n\rangle^2} \sum_{n=n_1+n_2} \frac{\chi_N^2(n_1)\chi_N^2(n_2)}{\langle n_1\rangle^2\langle n_2\rangle^2},
\end{aligned} \tag{3.1.8}$$

where $\chi_N(n_j)$ is as in (1.2.6). The upper bound in (3.1.7) follows from applying Lemma 2.2.1 to (3.1.8). As for the lower bound, we consider the contribution from $|n| \le \frac{2}{3}N$ and $\frac{1}{4}|n| \le |n_1| \le \frac{1}{2}|n|$ (which implies $|n_2| \sim |n|$ and $|n_j| \le N$, $j = 1,2$). Then, from (3.1.8), we obtain

$$\mathbb{E}\big[\| :Y_N^2(t): \|_{H^{-1}}^2\big] \gtrsim \sum_{\substack{n\in\mathbb{Z}^3 \\ |n|\le\frac{2}{3}N}} \frac{t^2}{\langle n\rangle^3} \sim t^2 \log N,$$

which proves the lower bound in (3.1.7). As for (ii), it follows from recalling the definition $:Y_N^3(1): = H_3(Y_N(1);\sigma_N)$ (with σ_N as in (1.2.8)) and the orthogonality relation of the Hermite polynomials (Lemma 2.3.2 with $k = 3$ and $\ell = 0$). Lastly, the claim in (iii) follows from Minkowski's integral inequality and Cauchy–Schwarz inequality; see [38, Lemma 4.7]. ∎

Remark 3.1.3. In [38, 57], a slightly different (and weaker) variational formula was used. See also [3, Lemma 1]. Given a drift $\theta \in \mathbb{H}_a$, we define the measure $\mathbb{Q}_\theta$ whose Radon–Nikodym derivative with respect to $\mathbb{P}$ is given by the following stochastic exponential:

$$\frac{d\mathbb{Q}_\theta}{d\mathbb{P}} = e^{\int_0^1 \langle\theta(t),dW(t)\rangle - \frac{1}{2}\int_0^1 \|\theta(t)\|_{L_x^2}^2 dt},$$

where $\langle\cdot,\cdot\rangle$ stands for the usual inner product on $L^2(\mathbb{T}^3)$. Let $\mathbb{H}_c$ denote the subspace of $\mathbb{H}_a$ consisting of drifts such that $\mathbb{Q}_\theta(\Omega) = 1$. Then, the (weaker) variational formula used in [38, 57] is given by (3.1.4), where the infimum is taken over $\mathbb{H}_c \subset \mathbb{H}_a$

and we replace Y and $\mathbb{E} = \mathbb{E}_{\mathbb{P}}$ by $Y_\theta = Y - I(\theta)$ and $\mathbb{E}_{\mathbb{Q}_\theta}$. Here, $\mathbb{E} = \mathbb{E}_{\mathbb{P}}$ and $\mathbb{E}_{\mathbb{Q}_\theta}$ denote expectations with respect to the underlying probability measure $\mathbb{P}$ and the measure $\mathbb{Q}_\theta$, respectively. In such a formulation, Y_θ and the measure $\mathbb{Q}_\theta$ depend on a drift θ. This, however, is not suitable for our purpose, since we construct a drift θ in (3.1.4) depending on Y.

3.2 Uniform exponential integrability and tightness

In this section, we first prove the uniform exponential integrability (1.2.13) via the Boué–Dupuis variational formula (Lemma 3.1.1). Then, we establish tightness of the truncated Φ_3^3-measures $\{\rho_N\}_{N\in\mathbb{N}}$.

As in the case of the Φ_3^4-measure studied in [3] (see also [54, Section 6]), we need to introduce a further renormalization than the standard Wick renormalization (see (1.2.10)). As a result, the resulting Φ_3^3-measure is singular with respect to the base Gaussian free field μ; see Section 3.4. We point out that this extra renormalization appears only at the level of the measure and thus does not affect the dynamical problem, at least locally in time.[2] In the following, we use the following shorthand notations: $Y_N(t) = \pi_N Y(t)$, $\Theta(t) = I(\theta)(t)$, and $\Theta_N(t) = \pi_N \Theta(t)$ with $Y_N = Y_N(1)$ and $\Theta_N = \Theta_N(1)$. We also use $Y = Y(1)$ and $\Theta = \Theta(1)$.

Let us first explain the second renormalization introduced in (1.2.10). Let R_N be as in (1.2.9) and set

$$\widetilde{Z}_N = \int e^{-R_N(u)} d\mu(u).$$

By Lemma 3.1.1, we can express the partition function $\widetilde{Z}_N$ as

$$-\log \widetilde{Z}_N = \inf_{\theta\in\mathbb{H}_a} \mathbb{E}\bigg[R_N(Y+\Theta) + \frac{1}{2}\int_0^1 \|\theta(t)\|_{L_x^2}^2 dt\bigg].$$

By expanding the cubic Wick power, we have

$$\begin{aligned} -\frac{\sigma}{3}\int_{\mathbb{T}^3} :(Y_N+\Theta_N)^3: \, dx = -\frac{\sigma}{3}\int_{\mathbb{T}^3} :Y_N^3: \, dx - \sigma \int_{\mathbb{T}^3} :Y_N^2: \Theta_N dx \\ - \sigma \int_{\mathbb{T}^3} Y_N \Theta_N^2 dx - \frac{\sigma}{3}\int_{\mathbb{T}^3} \Theta_N^3 dx. \end{aligned} \tag{3.2.1}$$

In view of Lemma 3.1.2, the first term on the right-hand side vanishes under an expectation, while we can estimate the third and fourth terms on the right-hand side

[2] As mentioned in Chapter 1, this singularity of the Φ_3^3-measure causes an additional difficulty for the globalization problem.

of (3.2.1) (see Lemma 3.2.2). As we see below, the second term turns out to be divergent (and does not vanish under an expectation). From the Ito product formula, we have

$$\mathbb{E}\bigg[\int_{\mathbb{T}^3} :Y_N^2: \Theta_N dx\bigg] = \mathbb{E}\bigg[\int_0^1 \int_{\mathbb{T}^3} :Y_N^2(t): \dot{\Theta}_N(t) dx dt\bigg], \tag{3.2.2}$$

where we have $\dot{\Theta}_N(t) = \langle \nabla \rangle^{-1} \pi_N \theta(t)$ in view of (3.1.5). Define $\mathfrak{Z}^N$ with $\mathfrak{Z}^N(0) = 0$ by its time derivative:

$$\dot{\mathfrak{Z}}^N(t) = (1-\Delta)^{-1} :Y_N^2(t): \tag{3.2.3}$$

and set $\mathfrak{Z}_N = \pi_N \mathfrak{Z}^N$. Then, we perform a change of variables:

$$\dot{\Upsilon}^N(t) = \dot{\Theta}(t) - \sigma \dot{\mathfrak{Z}}_N(t) \tag{3.2.4}$$

and set $\Upsilon_N = \pi_N \Upsilon^N$. From (3.2.2), (3.2.3), and (3.2.4), we have

$$\begin{aligned} &\mathbb{E}\bigg[-\sigma \int_{\mathbb{T}^3} :Y_N^2: \Theta_N dx + \frac{1}{2}\int_0^1 \|\theta(t)\|_{L_x^2}^2 dt\bigg] \\ &\quad = \frac{1}{2}\mathbb{E}\bigg[\int_0^1 \|\dot{\Upsilon}^N(t)\|_{H_x^1}^2 dt\bigg] - \alpha_N, \end{aligned} \tag{3.2.5}$$

where the divergent constant α_N is given by

$$\alpha_N = \frac{\sigma^2}{2}\mathbb{E}\bigg[\int_0^1 \|\dot{\mathfrak{Z}}_N(t)\|_{H_x^1}^2 dt\bigg] \to \infty, \tag{3.2.6}$$

as $N \to \infty$. The divergence in (3.2.6) can be easily seen from the spatial regularity $1 - \varepsilon$ of $\dot{\mathfrak{Z}}_N(t) = (1-\Delta)^{-1} :Y_N^2(t):$ (with a uniform bound in $N \in \mathbb{N}$). See Lemma 3.1.2.

In view of the discussion above, we define $R_N^\diamond$ as in (1.2.10), which removes the divergent constant α_N in (3.2.5). Then, from (1.2.12) and the Boué–Dupuis variational formula (Lemma 3.1.1), we have

$$-\log Z_N = \inf_{\theta \in \mathbb{H}_a} \mathbb{E}\bigg[R_N^\diamond(Y + \Theta) + \frac{1}{2}\int_0^1 \|\theta(t)\|_{L_x^2}^2 dt\bigg] \tag{3.2.7}$$

for any $N \in \mathbb{N}$. By setting

$$\mathcal{W}_N(\theta) = \mathbb{E}\bigg[R_N^\diamond(Y + \Theta) + \frac{1}{2}\int_0^1 \|\theta(t)\|_{L_x^2}^2 dt\bigg], \tag{3.2.8}$$

it follows from (1.2.9) with $\gamma = 3$, (1.2.10), (3.2.1), (3.2.5), and Lemma 3.1.2 (ii) that

$$\begin{aligned} \mathcal{W}_N(\theta) = \mathbb{E}\bigg[&-\sigma \int_{\mathbb{T}^3} Y_N \Theta_N^2 dx - \frac{\sigma}{3}\int_{\mathbb{T}^3} \Theta_N^3 dx \\ &+ A\bigg|\int_{\mathbb{T}^3} \big(:Y_N^2: + 2Y_N\Theta_N + \Theta_N^2\big) dx\bigg|^3 + \frac{1}{2}\int_0^1 \|\dot{\Upsilon}^N(t)\|_{H_x^1}^2 dt\bigg]. \end{aligned} \tag{3.2.9}$$

We also set

$$\Upsilon_N = \Upsilon_N(1) = \pi_N \Upsilon^N(1) \quad \text{and} \quad \mathfrak{Z}_N = \mathfrak{Z}_N(1) = \pi_N \mathfrak{Z}^N(1). \tag{3.2.10}$$

In view of the change of variables (3.2.4), we have

$$\Theta_N = \Upsilon_N + \sigma\pi_N \mathfrak{Z}_N =: \Upsilon_N + \sigma\widetilde{\mathfrak{Z}}_N, \quad \text{i.e. } \widetilde{\mathfrak{Z}}_N := \pi_N \mathfrak{Z}_N. \tag{3.2.11}$$

Namely, the original drift θ in (3.2.7) depends on Y. By the definition (3.2.3) and (3.2.10), $\mathfrak{Z}_N$ is determined by Y_N. Hence, in the following, we view $\dot{\Upsilon}^N$ as a drift and study the minimization problem (3.2.7) by first studying each term in (3.2.9) (where we now view $\mathcal{W}_N$ as a function of $\dot{\Upsilon}^N$) and then taking an infimum in $\dot{\Upsilon}^N \in \mathbb{H}^1_a$, where $\mathbb{H}^1_a$ is as in (3.1.3). Our main goal is to show that $\mathcal{W}_N(\dot{\Upsilon}^N)$ in (3.2.9) is bounded away from $-\infty$, uniformly in $N \in \mathbb{N}$ and $\dot{\Upsilon}^N \in \mathbb{H}^1_a$.

Remark 3.2.1. In this paper, we work with the cube frequency projector $\pi_N = \pi_N^{\text{cube}}$ defined in (1.2.5), satisfying $\pi_N^2 = \pi_N$. In view of (3.2.10) and (3.2.11), we have $\widetilde{\mathfrak{Z}}_N = \mathfrak{Z}_N$. Nonetheless, we introduce the notation $\widetilde{\mathfrak{Z}}_N$ in (3.2.11) to indicate the modifications necessary to consider the case of the smooth frequency projector π_N^{smooth} defined in (1.4.2), which does not satisfy $(\pi_N^{\text{smooth}})^2 = \pi_N^{\text{smooth}}$. This comment applies to the remaining part of the paper.

We first state two lemmas whose proofs are presented at the end of this section. While the first lemma is elementary, the second lemma (Lemma 3.2.3) requires much more careful analysis, reflecting the critical nature of the Φ^3_3-measure.

Lemma 3.2.2. *Let $A > 0$ and $0 < |\sigma| < 1$. Then, there exist small $\varepsilon > 0$ and a constant $c > 0$ such that, for any $\delta > 0$, there exists $C_\delta > 0$ such that*

$$\left|\int_{\mathbb{T}^3} Y_N \Theta_N^2 dx\right| \lesssim 1 + C_\delta \|Y_N\|^c_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}} + \delta\|\Upsilon_N\|^6_{L^2} + \delta\|\Upsilon_N\|^2_{H^1} + \|\mathfrak{Z}_N\|^c_{\mathcal{C}^{1-\varepsilon}}, \tag{3.2.12}$$

$$\left|\int_{\mathbb{T}^3} \Theta_N^3 dx\right| \lesssim 1 + \|\Upsilon_N\|^6_{L^2} + \|\Upsilon_N\|^2_{H^1} + \|\mathfrak{Z}_N\|^3_{\mathcal{C}^{1-\varepsilon}}, \tag{3.2.13}$$

and

$$\begin{aligned} A&\left|\int_{\mathbb{T}^3} (:Y_N^2: + 2Y_N\Theta_N + \Theta_N^2) dx\right|^3 \\ &\geq \frac{A}{2}\left|\int_{\mathbb{T}^3} (2Y_N\Upsilon_N + \Upsilon_N^2) dx\right|^3 - \delta\|\Upsilon_N\|^6_{L^2} \\ &\quad - C_{\delta,\sigma}\left\{\left|\int_{\mathbb{T}^3} :Y_N^2:\, dx\right|^3 + \|Y_N\|^6_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}} + \|\mathfrak{Z}_N\|^6_{\mathcal{C}^{1-\varepsilon}}\right\}, \end{aligned} \tag{3.2.14}$$

uniformly in $N \in \mathbb{N}$, where $\Theta_N = \Upsilon_N + \sigma\widetilde{\mathfrak{Z}}_N$ as in (3.2.11).

The next lemma allows us to control the term $\|\Upsilon_N\|_{L^2}^6$ appearing in Lemma 3.2.2.

Lemma 3.2.3. *There exists a non-negative random variable $B(\omega)$ with $\mathbb{E}[B^p] \le C_p < \infty$ for any finite $p \ge 1$ such that*

$$\|\Upsilon_N\|_{L^2}^6 \lesssim \left| \int_{\mathbb{T}^3} (2Y_N \Upsilon_N + \Upsilon_N^2) dx \right|^3 + \|\Upsilon_N\|_{H^1}^2 + B(\omega), \tag{3.2.15}$$

uniformly in $N \in \mathbb{N}$.

By assuming Lemmas 3.2.2 and 3.2.3, we now prove the uniform exponential integrability (1.2.13) and tightness of the truncated Φ_3^3-measures ρ_N.

Uniform exponential integrability. In view of (3.2.9) and Lemma 3.2.3, define the positive part $\mathcal{U}_N$ of $\mathcal{W}_N$ by

$$\mathcal{U}_N(\dot{\Upsilon}^N) = \mathbb{E}\left[\frac{A}{2} \left| \int_{\mathbb{T}^3} (2Y_N \Upsilon_N + \Upsilon_N^2) dx \right|^3 + \frac{1}{2} \int_0^1 \|\dot{\Upsilon}^N(t)\|_{H_x^1}^2 dt \right]. \tag{3.2.16}$$

As a corollary to Lemma 3.1.2 (i) with (3.2.3), we have, for any finite $p \ge 1$,

$$\mathbb{E}\big[\|\mathfrak{Z}_N\|_{\mathcal{C}^{1-\varepsilon}}^p\big] \le \int_0^1 \mathbb{E}\big[\| :Y_N^2(t): \|_{\mathcal{C}^{-1-\varepsilon}}^p\big] dt \lesssim p < \infty, \tag{3.2.17}$$

uniformly in $N \in \mathbb{N}$. Then, by applying Lemmas 3.2.2 and 3.2.3 to (3.2.9) together with Lemma 3.1.2 and (3.2.17), we obtain

$$\begin{aligned} \mathcal{W}_N(\dot{\Upsilon}^N) &\ge -C_0 + \mathbb{E}\bigg[\left(\frac{A}{2} - c|\sigma| \right) \left| \int_{\mathbb{T}^3} (2Y_N \Upsilon_N + \Upsilon_N^2) dx \right|^3 \\ &\qquad\qquad + \left(\frac{1}{2} - c|\sigma| \right) \int_0^1 \|\dot{\Upsilon}^N(t)\|_{H_x^1}^2 dt \bigg] \\ &\ge -C_0' + \frac{1}{10} \mathcal{U}_N(\dot{\Upsilon}^N), \end{aligned} \tag{3.2.18}$$

for any $0 < |\sigma| < \sigma_0$, provided $A = A(\sigma_0) > 0$ is sufficiently large. Noting that the estimate (3.2.18) is uniform in $N \in \mathbb{N}$ and $\dot{\Upsilon}^N \in \mathbb{H}_a^1$, we conclude that

$$\inf_{N \in \mathbb{N}} \inf_{\dot{\Upsilon}^N \in \mathbb{H}_a^1} \mathcal{W}_N(\dot{\Upsilon}^N) \ge \inf_{N \in \mathbb{N}} \inf_{\dot{\Upsilon}^N \in \mathbb{H}_a^1} \left\{ -C_0' + \frac{1}{10} \mathcal{U}_N(\dot{\Upsilon}^N) \right\} \ge -C_0' > -\infty. \tag{3.2.19}$$

Therefore, the uniform exponential integrability (1.2.13) follows from (3.2.7), (3.2.8), and (3.2.19).

Tightness. Next, we prove tightness of the truncated Φ_3^3-measures $\{\rho_N\}_{N \in \mathbb{N}}$. Although it follows from a slight modification of the argument in our previous work [54, Section 6.2], we present a proof here for readers' convenience.

As a preliminary step, we first prove that Z_N in (1.2.12) is uniformly bounded away from 0:

$$\inf_{N\in\mathbb{N}} Z_N > 0. \tag{3.2.20}$$

In view of (3.2.7) and (3.2.8), it suffices to establish an upper bound on $\mathcal{W}_N$ in (3.2.9). By Lemma 2.1.1 and (3.2.11), we have

$$\begin{aligned}\bigg|\int_{\mathbb{T}^3} 2Y_N\Theta_N dx\bigg|^3 &\lesssim \|Y_N\|^3_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}}\|\Theta_N\|^3_{H^{\frac{1}{2}+2\varepsilon}}\\ &\lesssim 1 + \|Y_N\|^c_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}} + \|\mathfrak{Z}_N\|^c_{\mathcal{C}^{1-\varepsilon}} + \|\Upsilon_N\|^c_{H^1}.\end{aligned}$$

Thus, we have

$$\begin{aligned}A\bigg|&\int_{\mathbb{T}^3}(:Y_N^2: + 2Y_N\Theta_N + \Theta_N^2)dx\bigg|^3\\ &\lesssim 1 + \|:Y_N^2:\|^3_{\mathcal{C}^{-1-\varepsilon}} + \|Y_N\|^c_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}} + \|\mathfrak{Z}_N\|^c_{\mathcal{C}^{1-\varepsilon}} + \|\Upsilon_N\|^c_{H^1}.\end{aligned} \tag{3.2.21}$$

Then, from (3.2.9), Lemma 3.2.2, and (3.2.21) with Lemma 3.1.2 and (3.2.17), we obtain

$$\inf_{\dot{\Upsilon}^N\in\mathbb{H}^1_a}\mathcal{W}_N \lesssim 1 + \inf_{\dot{\Upsilon}^N\in\mathbb{H}^1_a}\mathbb{E}\bigg[\bigg(\int_0^1\|\dot{\Upsilon}^N(t)\|^2_{H^1_x}dt\bigg)^c\bigg] \lesssim 1$$

by taking $\dot{\Upsilon}^N \equiv 0$, for example. This proves (3.2.20).

We now prove tightness of the truncated Φ^3_3-measures. Fix small $\varepsilon > 0$ and let $B_R \subset H^{-\frac{1}{2}-\varepsilon}(\mathbb{T}^3)$ be the closed ball of radius $R > 0$ centered at the origin. Then, by Rellich's compactness lemma, we see that B_R is compact in $H^{-\frac{1}{2}-2\varepsilon}(\mathbb{T}^3)$. In the following, we show that given any small $\delta > 0$, there exists $R = R(\delta) \gg 1$ such that

$$\sup_{N\in\mathbb{N}}\rho_N(B_R^c) < \delta. \tag{3.2.22}$$

Given $M \gg 1$, let F be a bounded smooth non-negative function such that

$$F(u) = \begin{cases} M, & \text{if } \|u\|_{H^{-\frac{1}{2}-\varepsilon}} \le \frac{R}{2},\\ 0, & \text{if } \|u\|_{H^{-\frac{1}{2}-\varepsilon}} > R.\end{cases} \tag{3.2.23}$$

Then, from (3.2.20), we have

$$\begin{aligned}\rho_N(B_R^c) &\le Z_N^{-1}\int e^{-F(u)-R_N^\diamond(u)}d\mu\\ &\lesssim \int e^{-F(u)-R_N^\diamond(u)}d\mu =: \widehat{Z}_N,\end{aligned} \tag{3.2.24}$$

uniformly in $N \gg 1$. Under the change of variables (3.2.4) (see also (3.2.5)), define $\widehat{R}_N^{\diamond}(Y + \Upsilon^N + \sigma \mathfrak{Z}_N)$ by

$$\widehat{R}_N^{\diamond}(Y + \Upsilon^N + \sigma \mathfrak{Z}_N) = -\frac{\sigma}{3}\int_{\mathbb{T}^3} :Y_N^3: \, dx - \sigma \int_{\mathbb{T}^3} Y_N \Theta_N^2 dx - \frac{\sigma}{3}\int_{\mathbb{T}^3} \Theta_N^3 dx + A\bigg|\int_{\mathbb{T}^3} \big(:Y_N^2: + 2Y_N\Theta_N + \Theta_N^2\big)dx\bigg|^3, \tag{3.2.25}$$

where $\Theta_N = \Upsilon_N + \sigma \widetilde{\mathfrak{Z}}_N$ with $\widetilde{\mathfrak{Z}}_N = \pi_N \mathfrak{Z}_N$ as in (3.2.11). Then, by (3.2.24) and the Boué–Dupuis variational formula (Lemma 3.1.1), we have

$$-\log \widehat{Z}_N = \inf_{\dot{\Upsilon}^N \in \mathbb{H}_a^1} \mathbb{E}\bigg[F(Y + \Upsilon^N + \sigma \mathfrak{Z}_N) + \widehat{R}_N^{\diamond}(Y + \Upsilon^N + \sigma \mathfrak{Z}_N) + \frac{1}{2}\int_0^1 \|\dot{\Upsilon}^N(t)\|_{H_x^1}^2 dt\bigg]. \tag{3.2.26}$$

Since $Y + \sigma \mathfrak{Z}_N \in \mathcal{H}_{\leq 2}$, it follows from Lemma 3.1.2, (3.2.17), Chebyshev's inequality, and choosing $R \gg 1$ that

$$\begin{aligned}
\mathbb{P}\bigg(\|Y + \Upsilon^N + \sigma \mathfrak{Z}_N\|_{H^{-\frac{1}{2}-\varepsilon}} > \frac{R}{2}\bigg)
&\leq \mathbb{P}\bigg(\|Y + \sigma \mathfrak{Z}_N\|_{H^{-\frac{1}{2}-\varepsilon}} > \frac{R}{4}\bigg) + \mathbb{P}\bigg(\|\Upsilon^N\|_{H^1} > \frac{R}{4}\bigg)\\
&\leq \frac{1}{2} + \frac{16}{R^2}\mathbb{E}\big[\|\Upsilon^N\|_{H_x^1}^2\big],
\end{aligned} \tag{3.2.27}$$

uniformly in $N \in \mathbb{N}$ and $R \gg 1$. Then, from (3.2.23), (3.2.27), and Lemma 3.1.2, we obtain

$$\begin{aligned}
\mathbb{E}\big[F(Y + \Upsilon^N + \sigma \mathfrak{Z}_N)\big] &\geq M\mathbb{E}\Big[\mathbf{1}_{\{\|Y + \Upsilon^N + \sigma \mathfrak{Z}_N\|_{H^{-\frac{1}{2}-\varepsilon}} \leq \frac{R}{2}\}}\Big]\\
&\geq \frac{M}{2} - \frac{16M}{R^2}\mathbb{E}\big[\|\Upsilon^N\|_{H_x^1}^2\big]\\
&\geq \frac{M}{2} - \frac{1}{4}\mathbb{E}\bigg[\int_0^1 \|\dot{\Upsilon}^N(t)\|_{H_x^1}^2 dt\bigg],
\end{aligned} \tag{3.2.28}$$

where we set $M = \frac{1}{64}R^2$ in the last step. Hence, from (3.2.26), (3.2.28), and repeating the computation leading to (3.2.19) (by possibly making σ_0 smaller), we obtain

$$\begin{aligned}
-\log \widehat{Z}_N &\geq \frac{M}{2} + \inf_{\dot{\Upsilon}^N \in \mathbb{H}_a^1} \mathbb{E}\bigg[\widehat{R}_N^{\diamond}(Y + \Upsilon^N + \sigma \mathfrak{Z}_N) + \frac{1}{4}\int_0^1 \|\dot{\Upsilon}^N(t)\|_{H_x^1}^2 dt\bigg]\\
&\geq \frac{M}{4},
\end{aligned} \tag{3.2.29}$$

uniformly $N \in \mathbb{N}$ and $M = \frac{1}{64}R^2 \gg 1$. Therefore, given any small $\delta > 0$, by choosing $R = R(\delta) \gg 1$ and setting $M = \frac{1}{64}R^2 \gg 1$, the desired bound (3.2.22) follows from (3.2.24) and (3.2.29). This proves tightness of the truncated Φ^3_3-measures $\{\rho_N\}_{N\in\mathbb{N}}$.

We conclude this section by presenting the proofs of Lemmas 3.2.2 and 3.2.3.

Proof of Lemma 3.2.2. From (2.1.5), (2.1.6), (2.1.4), and (2.1.3) in Lemma 2.1.1 followed by Young's inequality, we have

$$\begin{aligned}
&\bigg|\int_{\mathbb{T}^3} Y_N \Theta_N^2 dx\bigg| \lesssim \|Y_N\|_{\mathcal{C}^{-\frac12-\varepsilon}} \|\Theta_N\|_{H^{\frac12+2\varepsilon}} \|\Theta_N\|_{L^2}\\
&\lesssim \|Y_N\|_{\mathcal{C}^{-\frac12-\varepsilon}} \Big(\|\Upsilon_N\|_{H^{\frac12+2\varepsilon}} (\|\Upsilon_N\|_{L^2} + \|\mathfrak{Z}_N\|_{\mathcal{C}^{1-\varepsilon}}) + \|\mathfrak{Z}_N\|^2_{\mathcal{C}^{1-\varepsilon}}\Big)\\
&\lesssim \|Y_N\|_{\mathcal{C}^{-\frac12-\varepsilon}} \Big(\|\Upsilon_N\|_{L^2}^{\frac12-2\varepsilon} \|\Upsilon_N\|_{H^1}^{\frac12+2\varepsilon} (\|\Upsilon_N\|_{L^2} + \|\mathfrak{Z}_N\|_{\mathcal{C}^{1-\varepsilon}}) + \|\mathfrak{Z}_N\|^2_{\mathcal{C}^{1-\varepsilon}}\Big)\\
&\lesssim 1 + C_\delta \|Y_N\|^c_{\mathcal{C}^{-\frac12-\varepsilon}} + \delta\|\Upsilon_N\|^6_{L^2} + \delta \|\Upsilon_N\|^2_{H^1} + \|\mathfrak{Z}_N\|^c_{\mathcal{C}^{1-\varepsilon}},
\end{aligned} \tag{3.2.30}$$

which yields (3.2.12). As for the second estimate (3.2.13), it follows from Sobolev's inequality, the interpolation (2.1.3), and Young's inequality that

$$\bigg|\int_{\mathbb{T}^3} \Upsilon_N^3 dx\bigg| \lesssim \|\Upsilon_N\|^3_{H^{\frac12}} \lesssim \|\Upsilon_N\|^{\frac32}_{L^2} \|\Upsilon_N\|^{\frac32}_{H^1} \lesssim \|\Upsilon_N\|^6_{L^2} + \|\Upsilon_N\|^2_{H^1}, \tag{3.2.31}$$

while Hölder's inequality with (2.1.4) shows

$$\bigg|\int_{\mathbb{T}^3} \Upsilon_N^2 \widetilde{\mathfrak{Z}}_N dx\bigg| + \bigg|\int_{\mathbb{T}^3} \Upsilon_N \widetilde{\mathfrak{Z}}_N^2 dx\bigg| + \bigg|\int_{\mathbb{T}^3} \widetilde{\mathfrak{Z}}_N^3 dx\bigg| \lesssim 1 + \|\Upsilon_N\|^6_{L^2} + \|\mathfrak{Z}_N\|^3_{\mathcal{C}^{1-\varepsilon}}.$$

Note that, given any $\gamma > 0$, there exists a constant $C = C(J) > 0$ such that

$$\bigg|\sum_{j=1}^J a_j\bigg|^\gamma \geq \frac12 |a_1|^\gamma - C\bigg(\sum_{j=2}^J |a_j|^\gamma\bigg) \quad \text{for any } a_j \in \mathbb{R}. \tag{3.2.32}$$

See [54, Section 5]. Then, from (3.2.32) and Cauchy's inequality, we have

$$\begin{aligned}
&A\bigg|\int_{\mathbb{T}^3} (:Y_N^2: + 2Y_N\Theta_N + \Theta_N^2) dx\bigg|^3\\
&\geq \frac{A}{2}\bigg|\int_{\mathbb{T}^3} (2Y_N\Upsilon_N + \Upsilon_N^2) dx\bigg|^3 - CA\bigg\{\bigg|\int_{\mathbb{T}^3} :Y_N^2:\, dx\bigg|^3 + |\sigma|^3 \bigg|\int_{\mathbb{T}^3} Y_N \widetilde{\mathfrak{Z}}_N dx\bigg|^3\\
&\qquad + |\sigma|^3 \bigg|\int_{\mathbb{T}^3} \Upsilon_N \widetilde{\mathfrak{Z}}_N dx\bigg|^3 + \sigma^6 \bigg|\int_{\mathbb{T}^3} \widetilde{\mathfrak{Z}}_N^2 dx\bigg|^3\bigg\}\\
&\geq \frac{A}{2}\bigg|\int_{\mathbb{T}^3} (2Y_N\Upsilon_N + \Upsilon_N^2) dx\bigg|^3 - \delta \|\Upsilon_N\|^6_{L^2}\\
&\quad - C_{\delta,\sigma}\bigg\{\bigg|\int_{\mathbb{T}^3} :Y_N^2:\, dx\bigg|^3 + \|Y_N\|^6_{\mathcal{C}^{-\frac12-\varepsilon}} + \|\mathfrak{Z}_N\|^6_{\mathcal{C}^{1-\varepsilon}}\bigg\}.
\end{aligned}$$

This proves (3.2.14). This completes the proof of Lemma 3.2.2. ■

Next, we present the proof of Lemma 3.2.3.

Proof of Lemma 3.2.3. If we have

$$\|\Upsilon_N\|_{L^2}^2 \gg \left|\int_{\mathbb{T}^3} Y_N \Upsilon_N dx\right|, \tag{3.2.33}$$

then, we have

$$\|\Upsilon_N\|_{L^2}^6 = \left(\int_{\mathbb{T}^3} \Upsilon_N^2 dx\right)^3 \sim \left|\int_{\mathbb{T}^3} (2Y_N \Upsilon_N + \Upsilon_N^2) dx\right|^3, \tag{3.2.34}$$

which shows (3.2.15). Hence, we assume that

$$\|\Upsilon_N\|_{L^2}^2 \lesssim \left|\int_{\mathbb{T}^3} Y_N \Upsilon_N dx\right| \tag{3.2.35}$$

in the following.

Given $j \in \mathbb{N}$, define the sharp frequency projections Π_j with a Fourier multiplier $\mathbf{1}_{\{|n|\le 2\}}$ when $j = 1$ and $\mathbf{1}_{\{2^{j-1}<|n|\le 2^j\}}$ when $j \geq 2$. We also set

$$\Pi_{\le j} = \sum_{k=1}^{j} \Pi_k \quad \text{and} \quad \Pi_{>j} = \mathrm{Id} - \Pi_{\le j}.$$

Then, write Υ_N as

$$\Upsilon_N = \sum_{j=1}^{\infty} \Pi_j \Upsilon_N = \sum_{j=1}^{\infty} (\lambda_j \Pi_j Y_N + w_j), \tag{3.2.36}$$

where λ_j and w_j are given by

$$\lambda_j := \begin{cases} \frac{\langle \Upsilon_N, \Pi_j Y_N\rangle}{\|\Pi_j Y_N\|_{L^2}^2}, & \text{if } \|\Pi_j Y_N\|_{L^2} \neq 0, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad w_j := \Pi_j \Upsilon_N - \lambda_j \Pi_j Y_N. \tag{3.2.37}$$

By definition, $w_j = \Pi_j w_j$ is orthogonal to $\Pi_j Y_N$ (and also to Y_N) in $L^2(\mathbb{T}^3)$. Thus, we have

$$\|\Upsilon_N\|_{L^2}^2 = \sum_{j=1}^{\infty} \left(\lambda_j^2 \|\Pi_j Y_N\|_{L^2}^2 + \|w_j\|_{L^2}^2\right), \tag{3.2.38}$$

$$\int_{\mathbb{T}^3} Y_N \Upsilon_N dx = \sum_{j=1}^{\infty} \lambda_j \|\Pi_j Y_N\|_{L^2}^2. \tag{3.2.39}$$

Hence, from (3.2.35), (3.2.38), and (3.2.39), we have

$$\sum_{j=1}^{\infty} \lambda_j^2 \|\Pi_j Y_N\|_{L^2}^2 \lesssim \bigg|\sum_{j=1}^{\infty} \lambda_j \|\Pi_j Y_N\|_{L^2}^2\bigg|. \tag{3.2.40}$$

Fix $j_0 = j_0(\omega) \in \mathbb{N}$ (to be chosen later). By Cauchy–Schwarz's inequality and (3.2.37), we have

$$\begin{aligned} \bigg|\sum_{j=j_0+1}^{\infty} \lambda_j \|\Pi_j Y_N\|_{L^2}^2\bigg| &\le \bigg(\sum_{j=1}^{\infty} \lambda_j^2 2^{2j} \|\Pi_j Y_N\|_{L^2}^2\bigg)^{\frac{1}{2}} \bigg(\sum_{j=j_0+1}^{\infty} 2^{-2j} \|\Pi_j Y_N\|_{L^2}^2\bigg)^{\frac{1}{2}} \\ &\le \bigg(\sum_{j=1}^{\infty} 2^{2j} \|\Pi_j \Upsilon_N\|_{L^2}^2\bigg)^{\frac{1}{2}} \bigg(\sum_{j=j_0+1}^{\infty} 2^{-2j} \|\Pi_j Y_N\|_{L^2}^2\bigg)^{\frac{1}{2}} \\ &\sim \|\Upsilon_N\|_{H^1} \|\Pi_{>j_0} Y_N\|_{H^{-1}}. \end{aligned} \tag{3.2.41}$$

On the other hand, it follows from Cauchy–Schwarz's inequality, (3.2.40), and Cauchy's inequality that

$$\begin{aligned} \bigg|\sum_{j=1}^{j_0} \lambda_j \|\Pi_j Y_N\|_{L^2}^2\bigg| &\le \bigg(\sum_{j=1}^{\infty} \lambda_j^2 \|\Pi_j Y_N\|_{L^2}^2\bigg)^{\frac{1}{2}} \bigg(\sum_{j=1}^{j_0} \|\Pi_j Y_N\|_{L^2}^2\bigg)^{\frac{1}{2}} \\ &\le C \bigg|\sum_{j=1}^{\infty} \lambda_j \|\Pi_j Y_N\|_{L^2}^2\bigg|^{\frac{1}{2}} \bigg(\sum_{j=1}^{j_0} \|\Pi_j Y_N\|_{L^2}^2\bigg)^{\frac{1}{2}} \\ &\le \frac{1}{2} \bigg|\sum_{j=1}^{\infty} \lambda_j \|\Pi_j Y_N\|_{L^2}^2\bigg| + C' \|\Pi_{\le j_0} Y_N\|_{L^2}^2. \end{aligned} \tag{3.2.42}$$

Hence, from (3.2.41) and (3.2.42), we obtain

$$\bigg|\sum_{j=1}^{\infty} \lambda_j \|\Pi_j Y_N\|_{L^2}^2\bigg| \lesssim \|\Upsilon_N\|_{H^1} \|\Pi_{>j_0} Y_N\|_{H^{-1}} + \|\Pi_{\le j_0} Y_N\|_{L^2}^2. \tag{3.2.43}$$

Since Y_N is spatially homogeneous, we have

$$\|\Pi_{>j_0} Y_N\|_{H^{-1}}^2 = \int_{\mathbb{T}^3} :(\langle\nabla\rangle^{-1} \Pi_{>j_0} Y_N)^2:\, dx + \mathbb{E}\big[(\langle\nabla\rangle^{-1} \Pi_{>j_0} Y_N)^2\big]. \tag{3.2.44}$$

Recalling (3.1.2), we can bound the second term by

$$\widetilde{\sigma}_{j_0} := \mathbb{E}\big[(\langle\nabla\rangle^{-1} \Pi_{>j_0} Y_N)^2\big] = \sum_{\substack{n\in\mathbb{Z}^3 \\ |n|>2^{j_0}}} \frac{\chi_N^2(n)}{\langle n\rangle^4} \lesssim 2^{-j_0}. \tag{3.2.45}$$

Let $Z_{N,j_0} = \langle \nabla \rangle^{-1} \Pi_{>j_0} Y_N$. Proceeding as in the proof of [63, Lemma 2.5] with Lemma 2.3.2, we have

$$
\begin{aligned}
\mathbb{E}\Big[\Big(\int_{\mathbb{T}^3} :Z_{N,j_0}^2:\, dx\Big)^2\Big]
&= \int_{\mathbb{T}^3_x \times \mathbb{T}^3_y} \mathbb{E}\big[H_2(Z_{N,j_0}(x);\widetilde{\sigma}_{j_0}) H_2(Z_{N,j_0}(y);\widetilde{\sigma}_{j_0})\big] dx dy \\
&= 2 \sum_{\substack{n_1,n_2 \in \mathbb{Z}^3 \\ |n_j| > 2^{j_0}}} \frac{\chi_N^2(n_1)\chi_N^2(n_2)}{\langle n_1 \rangle^4 \langle n_2 \rangle^4} \int_{\mathbb{T}^3_x \times \mathbb{T}^3_y} e_{n_1+n_2}(x-y) dx dy \\
&= 2 \sum_{\substack{n \in \mathbb{Z}^3 \\ |n| > 2^{j_0}}} \frac{\chi_N^4(n)}{\langle n \rangle^8} \sim 2^{-5j_0}.
\end{aligned}
\tag{3.2.46}
$$

Now, define a non-negative random variable $B_1(\omega)$ by

$$
B_1(\omega) = \bigg(\sum_{j=1}^{\infty} 2^{4j} \Big(\int_{\mathbb{T}^3} :Z_{N,j}^2:\, dx\Big)^2\bigg)^{\frac{1}{2}}. \tag{3.2.47}
$$

By Minkowski's integral inequality, the Wiener chaos estimate (Lemma 2.3.1), and (3.2.46), we have

$$
\mathbb{E}\big[B_1^p\big] \le p^p \bigg(\sum_{j=1}^{\infty} 2^{4j} \Big\| \int_{\mathbb{T}^3} :Z_{N,j}^2:\, dx \Big\|_{L^2(\Omega)}^2\bigg)^{\frac{p}{2}} \lesssim p^p < \infty \tag{3.2.48}
$$

for any finite $p \ge 2$ (and hence for any finite $p \ge 1$). Hence, from (3.2.44), (3.2.45), and (3.2.47), we obtain

$$
\|\Pi_{>j_0} Y_N\|_{H^{-1}}^2 \lesssim 2^{-2j_0} B_1(\omega) + 2^{-j_0}. \tag{3.2.49}
$$

Next, define a non-negative random variable $B_2(\omega)$ by

$$
B_2(\omega) = \sum_{j=1}^{\infty} \bigg| \int_{\mathbb{T}^3} :(\Pi_j Y_N)^2:\, dx \bigg|.
$$

Then, a similar computation shows

$$
\begin{aligned}
\|\Pi_{\le j_0} Y_N\|_{L^2}^2 &= \int_{\mathbb{T}^3} :(\Pi_{\le j_0} Y_N)^2:\, dx + \mathbb{E}\big[(\Pi_{\le j_0} Y_N)^2\big] \\
&\lesssim B_2(\omega) + 2^{j_0}
\end{aligned}
\tag{3.2.50}
$$

and $\mathbb{E}[B_2^p] \le C_p < \infty$ for any finite $p \ge 1$.

Therefore, putting (3.2.35), (3.2.39), (3.2.43), (3.2.49), and (3.2.50) together, choosing $2^{j_0} \sim 1 + \|\Upsilon_N\|_{H^1}^{\frac{2}{3}}$, and applying Cauchy's inequality, we obtain

$$
\begin{aligned}
\|\Upsilon_N\|_{L^2}^6 &\lesssim \left| \int_{\mathbb{T}^3} Y_N \Upsilon_N dx \right|^3 = \left| \sum_{j=0}^{\infty} \lambda_j \|\Pi_j Y_N\|_{L^2}^2 \right|^3 \\
&\lesssim \big(2^{-3j_0} B_1(\omega)^{\frac{3}{2}} + 2^{-\frac{3}{2}j_0}\big)\|\Upsilon_N\|_{H^1}^3 + B_2^3(\omega) + 2^{3j_0} \\
&\lesssim \|\Upsilon_N\|_{H^1}^2 + B_1^3(\omega) + B_2^3(\omega) + 1,
\end{aligned} \tag{3.2.51}
$$

where the implicit constant is independent of $N \in \mathbb{N}$. This proves (3.2.15) in the case (3.2.35) holds. This concludes the proof of Lemma 3.2.3. ■

Remark 3.2.4. From the proof of Lemma 3.2.3 (see (3.2.33) and (3.2.51)) with Lemma 3.2.3, we also have

$$
\begin{aligned}
\mathbb{E}\Big[\Big|\int_{\mathbb{T}^3} Y_N \Upsilon_N dx\Big|^3\Big] &\lesssim \mathbb{E}\big[\|\Upsilon_N\|_{L^2}^6 + \|\Upsilon_N\|_{H^1}^2\big] + 1 \\
&\lesssim \mathcal{U}_N + 1,
\end{aligned} \tag{3.2.52}
$$

where $\mathcal{U}_N$ is as in (3.2.16).

3.3 Uniqueness of the limiting Φ_3^3-measure

The tightness of the truncated Gibbs measures $\{\rho_N\}_{N\in\mathbb{N}}$, proven in the previous section, together with Prokhorov's theorem implies existence of a weakly convergent subsequence. In this section, we prove uniqueness of the limiting Φ_3^3-measure, which allows us to conclude the weak convergence of the entire sequence $\{\rho_N\}_{N\in\mathbb{N}}$. While we follow the uniqueness argument in our previous work [54, Section 6.3], there are extra terms to control due to the focusing nature of the problem under consideration.

Proposition 3.3.1. *Let $\{\rho_{N_k^1}\}_{k=1}^{\infty}$ and $\{\rho_{N_k^2}\}_{k=1}^{\infty}$ be two weakly convergent subsequences of the truncated Φ_3^3-measures $\{\rho_N\}_{N\in\mathbb{N}}$ defined in (1.2.11), converging weakly to $\rho^{(1)}$ and $\rho^{(2)}$ as $k \to \infty$, respectively. Then, we have $\rho^{(1)} = \rho^{(2)}$.*

Proof. We break the proof into two steps.

Step 1. We first show that

$$
\lim_{k\to\infty} Z_{N_k^1} = \lim_{k\to\infty} Z_{N_k^2}, \tag{3.3.1}
$$

where Z_N is as in (1.2.12). By taking a further subsequence, we may assume that $N_k^1 \geq N_k^2$, $k \in \mathbb{N}$. Recall the change of variables (3.2.4) and let $\widehat{R}_N^{\diamond}(Y + \Upsilon^N + \sigma \mathfrak{Z}_N)$

be as in (3.2.25). Then, by the Boué–Dupuis variational formula (Lemma 3.1.1), we have

$$-\log Z_{N_k^j} = \inf_{\dot{\Upsilon}^{N_k^j} \in \mathbb{H}_a^1} \mathbb{E}\bigg[\widehat{R}^{\diamond}_{N_k^j}(Y + \Upsilon^{N_k^j} + \sigma \mathfrak{Z}_{N_k^j}) + \frac{1}{2}\int_0^1 \|\dot{\Upsilon}^{N_k^j}(t)\|_{H_x^1}^2 dt \bigg] \tag{3.3.2}$$

for $j = 1, 2$ and $k \in \mathbb{N}$. We point out that Y and $\mathfrak{Z}_N$ do not depend on the drift $\dot{\Upsilon}^N$ in (3.3.2).

Given $\delta > 0$, let $\underline{\Upsilon}^{N_k^2}$ be an almost optimizer for (3.3.2) with $j = 2$:

$$-\log Z_{N_k^2} \geq \mathbb{E}\bigg[\widehat{R}^{\diamond}_{N_k^2}(Y + \underline{\Upsilon}^{N_k^2} + \sigma \mathfrak{Z}_{N_k^2}) + \frac{1}{2}\int_0^1 \|\underline{\dot{\Upsilon}}^{N_k^2}(t)\|_{H_x^1}^2 dt \bigg] - \delta. \tag{3.3.3}$$

By setting $\underline{\Upsilon}_{N_k^2} := \pi_{N_k^2} \underline{\Upsilon}^{N_k^2}$, we have

$$\pi_{N_k^1} \underline{\Upsilon}_{N_k^2} = \underline{\Upsilon}_{N_k^2} \tag{3.3.4}$$

since $N_k^1 \geq N_k^2$. Then, by choosing $\Upsilon^{N_k^1} = \underline{\Upsilon}_{N_k^2}$, it follows from (3.3.3) and (3.3.4) that

$$\begin{aligned}
&-\log Z_{N_k^1} + \log Z_{N_k^2} \\
&\quad \leq \inf_{\dot{\Upsilon}^{N_k^1} \in \mathbb{H}_a^1} \mathbb{E}\bigg[\widehat{R}^{\diamond}_{N_k^1}(Y + \Upsilon^{N_k^1} + \sigma \mathfrak{Z}_{N_k^1}) + \frac{1}{2}\int_0^1 \|\dot{\Upsilon}^{N_k^1}(t)\|_{H_x^1}^2 dt \bigg] \\
&\qquad - \mathbb{E}\bigg[\widehat{R}^{\diamond}_{N_k^2}(Y + \underline{\Upsilon}^{N_k^2} + \sigma \mathfrak{Z}_{N_k^2}) + \frac{1}{2}\int_0^1 \|\underline{\dot{\Upsilon}}^{N_k^2}(t)\|_{H_x^1}^2 dt \bigg] + \delta \\
&\quad \leq \mathbb{E}\bigg[\widehat{R}^{\diamond}_{N_k^1}(Y + \underline{\Upsilon}_{N_k^2} + \sigma \mathfrak{Z}_{N_k^1}) + \frac{1}{2}\int_0^1 \|\underline{\dot{\Upsilon}}_{N_k^2}(t)\|_{H_x^1}^2 dt \bigg] \\
&\qquad - \mathbb{E}\bigg[\widehat{R}^{\diamond}_{N_k^2}(Y + \underline{\Upsilon}^{N_k^2} + \sigma \mathfrak{Z}_{N_k^2}) + \frac{1}{2}\int_0^1 \|\underline{\dot{\Upsilon}}^{N_k^2}(t)\|_{H_x^1}^2 dt \bigg] + \delta \\
&\quad \leq \mathbb{E}\Big[\widehat{R}^{\diamond}(Y_{N_k^1} + \underline{\Upsilon}_{N_k^2} + \sigma \widetilde{\mathfrak{Z}}_{N_k^1}) - \widehat{R}^{\diamond}(Y_{N_k^2} + \underline{\Upsilon}_{N_k^2} + \sigma \widetilde{\mathfrak{Z}}_{N_k^2}) \Big] + \delta,
\end{aligned} \tag{3.3.5}$$

where $\widetilde{\mathfrak{Z}}_{N_k^j} = \pi_{N_k^j} \mathfrak{Z}_{N_k^j}$ is as in (3.2.11). Here, $\widehat{R}^{\diamond}$ is defined by

$$\begin{aligned}
\widehat{R}^{\diamond}(Y + \Upsilon + \sigma \mathfrak{Z}) = -\sigma \int_{\mathbb{T}^3} Y \Theta^2 dx - \frac{\sigma}{3} \int_{\mathbb{T}^3} \Theta^3 dx \\
+ A \bigg| \int_{\mathbb{T}^3} (:Y^2: + 2Y\Theta + \Theta^2) dx \bigg|^3,
\end{aligned} \tag{3.3.6}$$

where $\Theta = \Upsilon + \sigma \mathfrak{Z}$.

We now estimate the right-hand side of (3.3.5). The main point is that in the difference

$$\mathbb{E}\Big[\widehat{R}^\diamond(Y_{N_k^1} + \underline{\Upsilon}_{N_k^2} + \sigma\widetilde{\mathfrak{Z}}_{N_k^1}) - \widehat{R}^\diamond(Y_{N_k^2} + \underline{\Upsilon}_{N_k^2} + \sigma\widetilde{\mathfrak{Z}}_{N_k^2})\Big], \tag{3.3.7}$$

we only have differences in Y-terms and $\mathfrak{Z}$-terms, which allows us to gain a negative power of N_k^2. The contribution from the first term on the right-hand side in (3.3.6) is given by

$$\begin{aligned} &-\sigma\mathbb{E}\Big[\int_{\mathbb{T}^3}(Y_{N_k^1} - Y_{N_k^2})\underline{\Upsilon}^2_{N_k^2}dx\Big] \\ &\quad -\sigma^2\mathbb{E}\Big[\int_{\mathbb{T}^3}(Y_{N_k^1} - Y_{N_k^2})(2\underline{\Upsilon}_{N_k^2} + \sigma\widetilde{\mathfrak{Z}}_{N_k^1})\widetilde{\mathfrak{Z}}_{N_k^1}dx\Big] \\ &\quad -\sigma^2\mathbb{E}\Big[\int_{\mathbb{T}^3}Y_{N_k^2}(\widetilde{\mathfrak{Z}}_{N_k^1} - \widetilde{\mathfrak{Z}}_{N_k^2})(2\underline{\Upsilon}_{N_k^2} + \sigma\widetilde{\mathfrak{Z}}_{N_k^1} + \sigma\widetilde{\mathfrak{Z}}_{N_k^2})dx\Big]. \end{aligned} \tag{3.3.8}$$

Let $\mathcal{U}_{N_k^2} = \mathcal{U}_{N_k^2}(\dot{\underline{\Upsilon}}^{N_k^2})$ be as in (3.2.16) with $\Upsilon_N = \underline{\Upsilon}_{N_k^2}$ and $\Upsilon^N = \underline{\Upsilon}^{N_k^2}$. Then, from Lemmas 3.1.2 and 3.2.3, we have

$$\mathbb{E}\Big[\|\underline{\Upsilon}_{N_k^2}\|^2_{H^1} + \|\underline{\Upsilon}_{N_k^2}\|^6_{L^2}\Big] \lesssim 1 + \mathcal{U}_{N_k^2}.$$

Now, proceeding as in (3.2.30) together with Hölder's inequality in ω and Young's inequality, we bound the first term in (3.3.8) by

$$\begin{aligned} &\mathbb{E}\Big[\|Y_{N_k^1} - Y_{N_k^2}\|_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}}\|\underline{\Upsilon}_{N_k^2}\|^{\frac{3}{2}-2\varepsilon}_{L^2}\|\underline{\Upsilon}_{N_k^2}\|^{\frac{1}{2}+2\varepsilon}_{H^1}\Big] \\ &\quad \le \|Y_{N_k^1} - Y_{N_k^2}\|_{L_\omega^{\frac{6}{3-4\varepsilon}}\mathcal{C}_x^{-\frac{1}{2}-\varepsilon}}\|\underline{\Upsilon}_{N_k^2}\|^{\frac{3}{2}-2\varepsilon}_{L^6_\omega L^2_x}\|\underline{\Upsilon}_{N_k^2}\|^{\frac{1}{2}+2\varepsilon}_{L^2_\omega H^1_x} \\ &\quad \lesssim (N_k^2)^{-a}\|\underline{\Upsilon}_{N_k^2}\|^{\frac{3}{2}-2\varepsilon}_{L^6_\omega L^2_x}\|\underline{\Upsilon}_{N_k^2}\|^{\frac{1}{2}+2\varepsilon}_{L^2_\omega H^1_x} \\ &\quad \lesssim (N_k^2)^{-a}(1 + \mathcal{U}_{N_k^2}), \end{aligned} \tag{3.3.9}$$

where the implicit constant is independent of N_k, $k \in \mathbb{N}$. Here, the second inequality follows from a modification of the proof of Lemma 3.1.2 (i) and noting that the Fourier transform of $Y_{N_k^1} - Y_{N_k^2}$ is supported on the frequencies $\{|n| \gtrsim N_k^2\}$, which allows us to gain a small negative power of N_k^2. Note that the implicit constants in (3.3.9) depend on $A > 0$ and σ. However, the sizes of A and $|\sigma|$ do not play any role in the subsequent analysis and thus we suppress the dependence on A and σ in the following. The same comment applies to Sections 3.3 and 3.4.

The second and third terms in (3.3.8) and the second term on the right-hand side of (3.3.6) can be handled in a similar manner (with (3.2.17) to control the $\widetilde{\mathfrak{Z}}_{N_k^j}$-terms). As a result, we can bound the first two terms on the right-hand side of (3.3.6) by

$$(N_k^2)^{-a}\Big(C(Y_{N_k^1}, Y_{N_k^2}, \mathfrak{Z}_{N_k^1}, \mathfrak{Z}_{N_k^2}) + \mathcal{U}_{N_k^2}\Big) \lesssim (N_k^2)^{-a}(1 + \mathcal{U}_{N_k^2}) \tag{3.3.10}$$

for some small $a > 0$, where $C(Y_{N_k^1}, Y_{N_k^2}, \mathfrak{Z}_{N_k^1}, \mathfrak{Z}_{N_k^2})$ denotes certain high moments of various stochastic terms involving $Y_{N_k^j}$ and $\mathfrak{Z}_{N_k^j}$, $j = 1, 2$, which are bounded by some constant, independent of N_k^j, $j = 1, 2$, in view of Lemma 3.1.2 and (3.2.17).

It remains to treat the difference coming from the last term in (3.3.6). By Young's and Hölder's inequalities, we have

$$
\begin{aligned}
&\mathbb{E}\bigg[\bigg|\int_{\mathbb{T}^3}\Big(:Y_{N_k^1}^2: + 2Y_{N_k^1}(\underline{\Upsilon}_{N_k^2} + \sigma\widetilde{\mathfrak{Z}}_{N_k^1}) + (\underline{\Upsilon}_{N_k^2} + \sigma\widetilde{\mathfrak{Z}}_{N_k^1})^2\Big)dx\bigg|^3 \\
&\quad - \bigg|\int_{\mathbb{T}^3}\Big(:Y_{N_k^2}^2: + 2Y_{N_k^2}(\underline{\Upsilon}_{N_k^2} + \sigma\widetilde{\mathfrak{Z}}_{N_k^2}) + (\underline{\Upsilon}_{N_k^2} + \sigma\widetilde{\mathfrak{Z}}_{N_k^2})^2\Big)dx\bigg|^3\bigg] \\
&\quad \lesssim \bigg\{\bigg\|\int_{\mathbb{T}^3}(:Y_{N_k^1}^2: - :Y_{N_k^2}^2:)dx\bigg\|_{L^3_\omega} + \bigg\|\int_{\mathbb{T}^3}(Y_{N_k^1} - Y_{N_k^2})\underline{\Upsilon}_{N_k^2}dx\bigg\|_{L^3_\omega} \\
&\qquad + \bigg\|\int_{\mathbb{T}^3}(Y_{N_k^1} - Y_{N_k^2})\widetilde{\mathfrak{Z}}_{N_k^1}dx\bigg\|_{L^3_\omega} + \bigg\|\int_{\mathbb{T}^3}Y_{N_k^2}(\widetilde{\mathfrak{Z}}_{N_k^1} - \widetilde{\mathfrak{Z}}_{N_k^2})dx\bigg\|_{L^3_\omega} \\
&\qquad + \bigg\|\int_{\mathbb{T}^3}(\widetilde{\mathfrak{Z}}_{N_k^1} - \widetilde{\mathfrak{Z}}_{N_k^2})(2\underline{\Upsilon}_{N_k^2} + \sigma\widetilde{\mathfrak{Z}}_{N_k^1} + \sigma\widetilde{\mathfrak{Z}}_{N_k^2})dx\bigg\|_{L^3_\omega}\bigg\} \\
&\qquad \times \bigg\{\bigg\|\int_{\mathbb{T}^3}\Big(:Y_{N_k^1}^2: + 2Y_{N_k^1}(\underline{\Upsilon}_{N_k^2} + \sigma\widetilde{\mathfrak{Z}}_{N_k^1}) + (\underline{\Upsilon}_{N_k^2} + \sigma\widetilde{\mathfrak{Z}}_{N_k^1})^2\Big)dx\bigg\|_{L^3_\omega}^2 \\
&\qquad\quad + \bigg\|\int_{\mathbb{T}^3}\Big(:Y_{N_k^2}^2: + 2Y_{N_k^2}(\underline{\Upsilon}_{N_k^2} + \sigma\widetilde{\mathfrak{Z}}_{N_k^2}) + (\underline{\Upsilon}_{N_k^2} + \sigma\widetilde{\mathfrak{Z}}_{N_k^2})^2\Big)dx\bigg\|_{L^3_\omega}^2\bigg\} \\
&\quad =: \mathrm{I} \times \mathrm{II}. \qquad (3.3.11)
\end{aligned}
$$

We divide I into two groups:

$$
\begin{aligned}
\mathrm{I} &= \bigg(\mathrm{I} - \bigg\|\int_{\mathbb{T}^3}(Y_{N_k^1} - Y_{N_k^2})\underline{\Upsilon}_{N_k^2}dx\bigg\|_{L^3_\omega}\bigg) + \bigg\|\int_{\mathbb{T}^3}(Y_{N_k^1} - Y_{N_k^2})\underline{\Upsilon}_{N_k^2}dx\bigg\|_{L^3_\omega} \\
&=: \mathrm{I}_1 + \mathrm{I}_2. \qquad (3.3.12)
\end{aligned}
$$

By the definition (1.2.5) of the cube frequency projector $\pi_N = \pi_N^{\text{cube}}$, we have

$$
\int_{\mathbb{T}^3}(Y_{N_k^1} - Y_{N_k^2})\underline{\Upsilon}_{N_k^2}dx \int_{\mathbb{T}^3}\pi_{N_k^2}(Y_{N_k^1} - Y_{N_k^2}) \cdot \underline{\Upsilon}_{N_k^2}dx = 0 \qquad (3.3.13)
$$

and thus $\mathrm{I}_2 = 0$.

By Lemma 2.1.1, Hölder's inequality in ω, and Young's inequality, followed by Lemma 3.2.3 with (3.2.16), we can estimate I_1 in (3.3.12) by

$$
\begin{aligned}
\mathrm{I}_1 &\lesssim \| :\! Y^2_{N^1_k} \!: - :\! Y^2_{N^2_k} \!: \|_{L^3_\omega \mathcal{C}^{-1-\varepsilon}_x} \\
&\quad + \| Y_{N^1_k} - Y_{N^2_k} \|_{L^6_\omega \mathcal{C}^{-\frac{1}{2}-\varepsilon}_x} \| \widetilde{\mathfrak{Z}}_{N^1_k} \|_{L^6_\omega \mathcal{C}^{1-\varepsilon}_x} \\
&\quad + \| Y_{N^2_k} \|_{L^6_\omega \mathcal{C}^{-\frac{1}{2}-\varepsilon}_x} \| \widetilde{\mathfrak{Z}}_{N^1_k} - \widetilde{\mathfrak{Z}}_{N^2_k} \|_{L^6_\omega \mathcal{C}^{1-\varepsilon}_x} \\
&\quad + \| \widetilde{\mathfrak{Z}}_{N^1_k} - \widetilde{\mathfrak{Z}}_{N^2_k} \|_{L^6_\omega \mathcal{C}^{1-\varepsilon}_x} \\
&\quad \times \Big(\| \underline{\Upsilon}_{N^2_k} \|_{L^6_\omega L^2_x} + \| \widetilde{\mathfrak{Z}}_{N^1_k} \|_{L^6_\omega \mathcal{C}^{1-\varepsilon}_x} + \| \widetilde{\mathfrak{Z}}_{N^2_k} \|_{L^6_\omega \mathcal{C}^{1-\varepsilon}_x} \Big) \\
&\lesssim (N^2_k)^{-a} \big(1 + \mathcal{U}_{N^2_k}\big)^{\frac{1}{6}},
\end{aligned}
\tag{3.3.14}
$$

where we used Lemma 3.1.2 and (3.2.17) in bounding the terms involving $Y_{N^j_k}$ and $\widetilde{\mathfrak{Z}}_{N^j_k} = \pi_{N^j_k} \mathfrak{Z}_{N^j_k}$. As for II in (3.3.11), it follows from (3.2.52), Lemma 3.2.3, and (3.2.16) that

$$
\begin{aligned}
&\bigg\| \int_{\mathbb{T}^3} \Big(:\! Y^2_{N^j_k} \!: + 2 Y_{N^j_k} \big(\underline{\Upsilon}_{N^2_k} + \sigma \widetilde{\mathfrak{Z}}_{N^j_k} \big) + \big(\underline{\Upsilon}_{N^2_k} + \sigma \widetilde{\mathfrak{Z}}_{N^j_k} \big)^2 \Big) dx \bigg\|_{L^3_\omega} \\
&\quad \lesssim 1 + \bigg\| \int_{\mathbb{T}^3} Y_{N^j_k} \underline{\Upsilon}_{N^2_k} dx \bigg\|_{L^3_\omega} + \| \underline{\Upsilon}_{N^2_k} \|^2_{L^6_\omega L^2_x} \\
&\quad \lesssim 1 + \mathcal{U}^{\frac{1}{3}}_{N^2_k}.
\end{aligned}
\tag{3.3.15}
$$

From (3.2.18), (3.2.8), (3.2.9), (3.2.25), and replacing $\underline{\Upsilon}^{N^2_k}$ by 0 in view of (3.3.2), we have

$$
\begin{aligned}
&\sup_{k \in \mathbb{N}} \mathcal{U}_{N^2_k} \big(\dot{\underline{\Upsilon}}^{N^2_k} \big) \\
&\quad \le 10 C'_0 + 10 \sup_{k \in \mathbb{N}} \mathbb{E} \bigg[\widehat{R}^\diamond_{N^2_k} \big(Y + \underline{\Upsilon}^{N^2_k} + \sigma \mathfrak{Z}_{N^2_k} \big) + \frac{1}{2} \int_0^1 \| \dot{\underline{\Upsilon}}^{N^2_k}(t) \|^2_{H^1_x} dt \bigg] \\
&\quad \lesssim 1 + \delta + \sup_{k \in \mathbb{N}} \mathbb{E} \Big[\widehat{R}^\diamond_{N^2_k} \big(Y + 0 + \sigma \mathfrak{Z}_{N^2_k} \big) \Big] \\
&\quad \lesssim 1.
\end{aligned}
\tag{3.3.16}
$$

Hence, from (3.3.13), (3.3.14), (3.3.15), and (3.3.16), we obtain that

$$
\mathrm{I} \cdot \mathrm{II} \lesssim (N^2_k)^{-a} \to 0,
\tag{3.3.17}
$$

as $k \to \infty$. Therefore, from (3.3.10) and (3.3.17), we conclude that

$$
\mathbb{E} \Big[\widehat{R}^\diamond \big(Y_{N^1_k} + \underline{\Upsilon}_{N^2_k} + \sigma \widetilde{\mathfrak{Z}}_{N^1_k} \big) - \widehat{R}^\diamond \big(Y_{N^2_k} + \underline{\Upsilon}_{N^2_k} + \sigma \widetilde{\mathfrak{Z}}_{N^2_k} \big) \Big] \to 0,
\tag{3.3.18}
$$

as $k \to \infty$. Since the choice of $\delta > 0$ was arbitrary, it follows from (3.3.5) and (3.3.18) that

$$\lim_{k\to\infty} Z_{N_k^1} \geq \lim_{k\to\infty} Z_{N_k^2}. \tag{3.3.19}$$

By taking a subsequence of $\{N_k^2\}_{k\in\mathbb{N}}$, still denoted by $\{N_k^2\}_{k\in\mathbb{N}}$, we may assume that $N_k^1 \leq N_k^2$. By repeating the computation above, we then obtain

$$\lim_{k\to\infty} Z_{N_k^1} \leq \lim_{k\to\infty} Z_{N_k^2}. \tag{3.3.20}$$

Therefore, (3.3.1) follows from (3.3.19) and (3.3.20).

Step 2. Next, we prove $\rho^{(1)} = \rho^{(2)}$. This claim follows from a small modification of Step 1. For this purpose, we need to prove that for every bounded Lipschitz continuous function $F : \mathcal{C}^{-100}(\mathbb{T}^3) \to \mathbb{R}$, we have

$$\lim_{k\to\infty} \int \exp(F(u))d\rho_{N_k^1} \geq \lim_{k\to\infty} \int \exp(F(u))d\rho_{N_k^2}$$

under the condition $N_k^1 \geq N_k^2$, $k \in \mathbb{N}$ (which can be always satisfied by taking a subsequence of $\{N_k^1\}_{k\in\mathbb{N}}$). In view of (1.2.12) and (3.3.1), it suffices to show

$$\begin{aligned} \limsup_{k\to\infty} \bigg[&- \log\bigg(\int \exp\big(F(u) - R^{\diamond}_{N_k^1}(u)\big) d\mu \bigg) \\ &+ \log\bigg(\int \exp\big(F(u) - R^{\diamond}_{N_k^2}(u)\big) d\mu \bigg) \bigg] \leq 0. \end{aligned} \tag{3.3.21}$$

By the Boué–Dupuis variational formula (Lemma 3.1.1), we have

$$\begin{aligned} &- \log\bigg(\int \exp\big(F(u) - R^{\diamond}_{N_k^j}(u)\big) d\mu \bigg) \\ &= \inf_{\dot{\Upsilon}^{N_k^j} \in \mathbb{H}_a^1} \mathbb{E}\bigg[-F\big(Y + \Upsilon^{N_k^j} + \sigma \mathfrak{Z}_{N_k^j}\big) \\ &\qquad + \widehat{R}^{\diamond}_{N_k^j}\big(Y + \Upsilon^{N_k^j} + \sigma \mathfrak{Z}_{N_k^j}\big) + \frac{1}{2} \int_0^1 \| \dot{\Upsilon}^{N_k^j}(t) \|^2_{H_x^1} dt \bigg], \end{aligned} \tag{3.3.22}$$

where $\widehat{R}^{\diamond}_{N_k^j}(Y + \Upsilon^{N_k^j} + \sigma \mathfrak{Z}_{N_k^j})$ is as in (3.2.25). Given $\delta > 0$, let $\underline{\Upsilon}^{N_k^2}$ be an almost optimizer for (3.3.22) with $j = 2$:

$$\begin{aligned} &- \log\bigg(\int \exp\big(F(u) - R^{\diamond}_{N_k^2}(u)\big) d\mu \bigg) \\ &\geq \mathbb{E}\bigg[-F\big(Y + \underline{\Upsilon}^{N_k^2} + \sigma \mathfrak{Z}_{N_k^2}\big) \\ &\qquad + \widehat{R}^{\diamond}_{N_k^2}\big(Y + \underline{\Upsilon}^{N_k^2} + \sigma \mathfrak{Z}_{N_k^2}\big) + \frac{1}{2} \int_0^1 \| \underline{\dot{\Upsilon}}^{N_k^2}(t) \|^2_{H_x^1} dt \bigg] - \delta. \end{aligned}$$

Then, by choosing $\Upsilon^{N^1_k} = \underline{\Upsilon}_{N^2_k} = \pi_{N^2_k}\underline{\Upsilon}^{N^2_k}$ and proceeding as in (3.3.5), we have

$$\begin{aligned}
&-\log\Big(\int \exp\big(F(u) - R^\diamond_{N^1_k}(u)\big)d\mu\Big) + \log\Big(\int \exp\big(F(u) - R^\diamond_{N^2_k}(u)\big)d\mu\Big)\\
&\quad\le \mathbb{E}\Big[-F\big(Y + \underline{\Upsilon}_{N^2_k} + \sigma \mathfrak{Z}_{N^1_k}\big)\\
&\qquad\quad + \widehat{R}^\diamond_{N^1_k}\big(Y + \underline{\Upsilon}_{N^2_k} + \sigma\mathfrak{Z}_{N^1_k}\big) + \frac{1}{2}\int_0^1 \|\dot{\underline{\Upsilon}}_{N^2_k}(t)\|^2_{H^1_x}dt\Big]\\
&\qquad - \mathbb{E}\Big[-F\big(Y + \underline{\Upsilon}^{N^2_k} + \sigma\mathfrak{Z}_{N^2_k}\big)\\
&\qquad\quad + \widehat{R}^\diamond_{N^2_k}\big(Y + \underline{\Upsilon}^{N^2_k} + \sigma\mathfrak{Z}_{N^2_k}\big) + \frac{1}{2}\int_0^1 \|\dot{\underline{\Upsilon}}^{N^2_k}(t)\|^2_{H^1_x}dt\Big] + \delta\\
&\quad\le \mathrm{Lip}(F)\cdot \mathbb{E}\Big[\big\|\pi^\perp_{N^2_k}\underline{\Upsilon}^{N^2_k} - \sigma\big(\mathfrak{Z}_{N^1_k} - \mathfrak{Z}_{N^2_k}\big)\big\|_{\mathcal{C}^{-100}}\Big]\\
&\qquad + \mathbb{E}\Big[\widehat{R}^\diamond\big(Y_{N^1_k} + \underline{\Upsilon}_{N^2_k} + \sigma\widetilde{\mathfrak{Z}}_{N^1_k}\big) - \widehat{R}^\diamond\big(Y_{N^2_k} + \underline{\Upsilon}_{N^2_k} + \sigma\widetilde{\mathfrak{Z}}_{N^2_k}\big)\Big] + \delta,
\end{aligned} \tag{3.3.23}$$

where $\pi^\perp_N = \mathrm{Id} - \pi_N$ and $\widehat{R}^\diamond$ is as in (3.3.6). We can proceed as in Step 1 to show that the second term on the right-hand side of (3.3.23) satisfies (3.3.18). Here, we need to use the boundedness of F in showing an analogue of (3.3.16) in the current context (with an almost optimizer $\underline{\Upsilon}^{N^2_k}$ for (3.3.22)).

Finally, we estimate the first term on the right-hand side of (3.3.23). Write

$$\begin{aligned}
&\mathbb{E}\Big[\big\|\pi^\perp_{N^2_k}\underline{\Upsilon}^{N^2_k} - \sigma\big(\mathfrak{Z}_{N^1_k} - \mathfrak{Z}_{N^2_k}\big)\big\|_{\mathcal{C}^{-100}}\Big]\\
&\qquad \lesssim \mathbb{E}\Big[\big\|\pi^\perp_{N^2_k}\Upsilon^{N^2_k}\big\|_{\mathcal{C}^{-100}}\Big] + \mathbb{E}\Big[\big\|\mathfrak{Z}_{N^1_k} - \mathfrak{Z}_{N^2_k}\big\|_{\mathcal{C}^{-100}}\Big].
\end{aligned}$$

A standard computation with (3.2.3) shows that the second term on the right-hand side tends to 0 as $k \to \infty$. As for the first term, from Lemma 3.1.2 and (an analogue of) (3.3.16), we obtain

$$\mathbb{E}\Big[\big\|\pi^\perp_{N^2_k}\underline{\Upsilon}^{N^2_k}\big\|_{\mathcal{C}^{-100}}\Big] \lesssim (N^2_k)^{-a}\big\|\underline{\Upsilon}^{N^2_k}\big\|_{L^2_\omega H^1_x} \lesssim (N^2_k)^{-a}\Big(\sup_{k\in\mathbb{N}} \mathcal{U}_{N^2_k}\Big)^{\frac{1}{2}} \to 0,$$

as $k \to \infty$. Since the choice of $\delta > 0$ was arbitrary, we conclude (3.3.21) and hence $\rho^{(1)} = \rho^{(2)}$. This completes the proof of Proposition 3.3.1. ∎

Remark 3.3.2. In the proof of Proposition 3.3.1, we used the orthogonality relation (3.3.13) to conclude that $\mathrm{I}_2 = 0$. While the same orthogonality holds for the ball frequency projector π^{ball}_N in (1.4.1), such an orthogonality relation is false for the smooth frequency projector π^{smooth}_N in (1.4.2). As seen from the proof of Lemma 3.2.3

and the uniform bound (3.3.16) on $\mathcal{U}_{N_k^2}(\dot{\underline{\Upsilon}}^{N_k^2})$, the quantity I_2 in (3.3.12) is critical (with respect to the spatial regularity/integrability and also with respect to the ω-integrability). From Remark 3.2.4 and (3.3.16), we see that the quantity I_2 is bounded, uniformly in $k \in \mathbb{N}$. In the absence of the orthogonality (3.3.13), however, we do not know how to show that this term tends to 0 as $k \to \infty$ in the case of the smooth frequency projector π_N^{smooth}. We point out that the same issue also appears in the proofs of Propositions 4.1.1 and 6.3.3 in the case of the smooth frequency projector π_N^{smooth}.

3.4 Singularity of the Φ_3^3-measure

We conclude this chapter by proving mutual singularity of the Φ_3^3-measure ρ, constructed in the previous sections, and the base Gaussian free field μ in (1.2.2). In [4, Section 4], Barashkov and Gubinelli proved the singularity of the Φ_3^4-measure by making use of the shifted measure. In the following, we follow our previous work [54] and present a direct proof of singularity of the Φ_3^3-measure without referring to a shifted measure. See also Appendix A, where we construct a shifted measure with respect to which the Φ_3^3-measure is absolutely continuous.

Proposition 3.4.1. *Let R_N be as in (1.2.9) with $\gamma = 3$, and $\varepsilon > 0$. Then, there exists a strictly increasing sequence $\{N_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ such that the set*

$$S := \Big\{u \in H^{-\frac{1}{2}-\varepsilon}(\mathbb{T}^3) : \lim_{k\to\infty} (\log N_k)^{-\frac{3}{4}} R_{N_k}(u) = 0\Big\}$$

satisfies

$$\mu(S) = 1 \quad \textit{but} \quad \rho(S) = 0. \tag{3.4.1}$$

In particular, the Φ_3^3-measure ρ and the massive Gaussian free field μ in (1.2.2) are mutually singular.

Proof. From (1.2.9) with $\gamma = 3$, the Wiener chaos estimate (Lemma 2.3.1), Lemmas 2.3.2 and 2.2.1, we have

$$\begin{aligned}
\|R_N(u)\|_{L^2(\mu)}^2 &\lesssim \bigg\| \int_{\mathbb{T}^3} :u_N^3: \, dx \bigg\|_{L^2(\mu)}^2 + \bigg\| \int_{\mathbb{T}^3} :u_N^2: \, dx \bigg\|_{L^6(\mu)}^6 \\
&\lesssim \bigg\| \int_{\mathbb{T}^3} :u_N^3: \, dx \bigg\|_{L^2(\mu)}^2 + \bigg\| \int_{\mathbb{T}^3} :u_N^2: \, dx \bigg\|_{L^2(\mu)}^6 \\
&\lesssim \sum_{\substack{n_1+n_2+n_3=0 \\ n_j \in NQ}} \langle n_1\rangle^{-2}\langle n_2\rangle^{-2}\langle n_3\rangle^{-2} + \bigg(\sum_{\substack{n_1+n_2=0 \\ n_j \in NQ}} \langle n_1\rangle^{-2}\langle n_2\rangle^{-2} \bigg)^3 \\
&\lesssim \sum_{|n_1|,|n-n_1|\lesssim N} \langle n_1\rangle^{-2}\langle n-n_1\rangle^{-1} + 1 \lesssim \log N,
\end{aligned}$$

where Q denotes the cube of side length 2 in $\mathbb{R}^3$ centered at the origin as in (1.2.7). Thus, we have

$$\lim_{N\to\infty} (\log N)^{-\frac{3}{4}} \|R_N(u)\|_{L^2(\mu)} \lesssim \lim_{N\to\infty} (\log N)^{-\frac{1}{4}} = 0.$$

Hence, there exists a subsequence such that

$$\lim_{k\to\infty} (\log N_k)^{-\frac{3}{4}} R_{N_k}(u) = 0,$$

almost surely with respect to μ. This proves $\mu(S) = 1$ in (3.4.1).

Given $k \in \mathbb{N}$, define $G_k(u)$ by

$$G_k(u) = (\log N_k)^{-\frac{3}{4}} R_{N_k}(u). \tag{3.4.2}$$

In the following, we show that $e^{G_k(u)}$ tends to 0 in $L^1(\rho)$. This will imply that there exists a subsequence of $G_k(u)$ tending to $-\infty$, almost surely with respect to the Φ^3_3-measure ρ, which in turn yields the second claim in (3.4.1): $\rho(S) = 0$.

Let ϕ be a smooth bump function as in Section 2.1. By Fatou's lemma, the weak convergence of ρ_M to ρ, the boundedness of ϕ, and (1.2.11), we have

$$\begin{aligned}
\int e^{G_k(u)} d\rho(u) &\le \liminf_{K\to\infty} \int \phi\bigg(\frac{G_k(u)}{K}\bigg) e^{G_k(u)} d\rho(u) \\
&= \liminf_{K\to\infty} \lim_{M\to\infty} \int \phi\bigg(\frac{G_k(u)}{K}\bigg) e^{G_k(u)} d\rho_M(u) \\
&\le \lim_{M\to\infty} \int e^{G_k(u)} d\rho_M(u) = Z^{-1} \lim_{M\to\infty} \int e^{G_k(u) - R^{\diamond}_M(u)} d\mu(u) \\
&=: Z^{-1} \lim_{M\to\infty} C_{M,k},
\end{aligned} \tag{3.4.3}$$

provided that $\lim_{M\to\infty} C_{M,k}$ exists. Here, $Z = \lim_{M\to\infty} Z_M$ denotes the partition function for ρ.

Our main goal is to show that the right-hand side of (3.4.3) tends to 0 as $k \to \infty$. As in the previous sections, we proceed with the change of variables (3.2.4):

$$\dot{\Upsilon}^M(t) = \dot{\Theta}(t) - \sigma \dot{\mathfrak{Z}}_M(t).$$

Then, by the Boué–Dupuis variational formula (Lemma 3.1.1) and (3.4.2), we have

$$\begin{aligned}
-\log C_{M,k} = \inf_{\dot{\Upsilon}^M \in \mathbb{H}^1_a} \mathbb{E}\bigg[&-(\log N_k)^{-\frac{3}{4}} R_{N_k}(Y + \Upsilon^M + \sigma \mathfrak{Z}_M) \\
&+ \widehat{R}^{\diamond}_M(Y + \Upsilon^M + \sigma \mathfrak{Z}_M) + \frac{1}{2} \int_0^1 \|\dot{\Upsilon}^M(t)\|^2_{H^1_x} dt \bigg] \\
=: \inf_{\dot{\Upsilon}^M \in \mathbb{H}^1_a} &\widehat{W}_{M,k}(\dot{\Upsilon}^M),
\end{aligned} \tag{3.4.4}$$

where $\widehat{R}^{\diamond}_N$ is as in (3.2.25). In the following, we prove that the right-hand side (and hence the left-hand side) of (3.4.4) diverges to ∞ as $k \to \infty$.

Proceeding as in Section 3.2 (see (3.2.18)), we bound the last two terms on the right-hand side of (3.4.4) as

$$\mathbb{E}\bigg[\widehat{R}^{\diamond}_M(Y + \Upsilon^M + \sigma\mathfrak{Z}_M) + \frac{1}{2}\int_0^1 \|\dot{\Upsilon}^M(t)\|^2_{H^1_x} dt\bigg] \geq -C_0 + \frac{1}{10}\mathcal{U}_M, \qquad (3.4.5)$$

where $\mathcal{U}_M = \mathcal{U}_M(\dot{\Upsilon}^M)$ is given by (3.2.16) with $\Upsilon_N = \pi_M \Upsilon^M$ and $\Upsilon^N = \Upsilon^M$:

$$\mathcal{U}_M = \mathbb{E}\bigg[\frac{A}{2}\bigg|\int_{\mathbb{T}^3}(2Y_M \pi_M \Upsilon^M + (\pi_M\Upsilon^M)^2)dx\bigg|^3 + \frac{1}{2}\int_0^1 \|\dot{\Upsilon}^M(t)\|^2_{H^1_x} dt\bigg]. \qquad (3.4.6)$$

Next, we study the first term on the right-hand side of (3.4.4), which gives the main (divergent) contribution. From (1.2.9) with $\gamma = 3$, we have

$$\begin{aligned} R_{N_k}(Y + \Upsilon^M + \sigma\mathfrak{Z}_M) &= -\frac{\sigma}{3}\int_{\mathbb{T}^3} :Y^3_{N_k}:\, dx - \sigma\int_{\mathbb{T}^3} :Y^2_{N_k}: \Theta_{N_k} dx \\ &\quad - \sigma\int_{\mathbb{T}^3} Y_{N_k}\Theta^2_{N_k} dx - \frac{\sigma}{3}\int_{\mathbb{T}^3}\Theta^3_{N_k} dx \\ &\quad + A\bigg|\int_{\mathbb{T}^3}(:Y^2_{N_k}: + 2Y_{N_k}\Theta_{N_k} + \Theta^2_{N_k})dx\bigg|^3 \\ &=: \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V} \end{aligned} \qquad (3.4.7)$$

for $N_k \leq M$, where Θ_{N_k} is given by

$$\Theta_{N_k} := \pi_{N_k}\Theta = \pi_{N_k}\Upsilon^M + \sigma\pi_{N_k}\mathfrak{Z}_M. \qquad (3.4.8)$$

As we see below, under an expectation, the second term II on the right-hand side of (3.4.7) (which is precisely the term removed by the second renormalization) gives a divergent contribution; see (3.4.14) below. From Lemma 3.1.2, the first term I on the right-hand side of (3.4.7) gives 0 under an expectation. As for the last three terms, we proceed as in Section 3.2 (see also the proof of Proposition 3.3.1) and obtain

$$\big|\mathbb{E}[\mathrm{III} + \mathrm{IV} + \mathrm{V}]\big| \lesssim C(Y_{N_k}, \pi_{N_k}\mathfrak{Z}_M) + \mathcal{U}_{N_k} \lesssim 1 + \mathcal{U}_{N_k} \qquad (3.4.9)$$

where $C(Y_{N_k}, \pi_{N_k}\mathfrak{Z}_M)$ denotes certain high moments of various stochastic terms involving Y_{N_k} and $\pi_{N_k}\mathfrak{Z}_M$ and $\mathcal{U}_{N_k} = \mathcal{U}_{N_k}(\partial_t \pi_{N_k}\Upsilon^M)$ is given by (3.2.16) with $\Upsilon_N = \Upsilon^N = \pi_{N_k}\Upsilon^M$:

$$\begin{aligned} \mathcal{U}_{N_k} = \mathbb{E}\bigg[&\frac{A}{2}\bigg|\int_{\mathbb{T}^3}(2Y_{N_k}\pi_{N_k}\Upsilon^M + (\pi_{N_k}\Upsilon^M)^2)dx\bigg|^3 \\ &+ \frac{1}{2}\int_0^1 \|\partial_t(\pi_{N_k}\Upsilon^M)(t)\|^2_{H^1_x} dt\bigg]. \end{aligned} \qquad (3.4.10)$$

In view of the smallness of $(\log N_k)^{-\frac{3}{4}}$ in (3.4.4), the second term in (3.4.10) can be controlled by the positive terms $\mathcal{U}_M$ in (3.4.5) (in particular by the second term in (3.4.6)). As for the first term in (3.4.10), it follows from (3.2.52), $\pi_{N_k}\Upsilon^M = \pi_{N_k}\pi_M\Upsilon^M$ for $N_k \le M$, and Lemma 3.2.3 with (3.4.6) that

$$\begin{aligned}
&\mathbb{E}\bigg[\bigg|\int_{\mathbb{T}^3}(2Y_{N_k}\Upsilon^M + (\pi_{N_k}\Upsilon^M)^2)dx\bigg|^3\bigg]\\
&\quad\lesssim \bigg\|\int_{\mathbb{T}^3} Y_{N_k}\pi_{N_k}\Upsilon^M dx\bigg\|^3_{L^3_\omega} + \|\pi_{N_k}\Upsilon^M\|^6_{L^6_\omega L^2_x}\\
&\quad\lesssim 1 + \|\pi_M\Upsilon^M\|^6_{L^6_\omega L^2_x} + \|\Upsilon^M\|^2_{L^2_\omega H^1_x}\\
&\quad\lesssim 1 + \mathcal{U}_M
\end{aligned}$$

for $N_k \le M$. Hence, $\mathcal{U}_{N_k}$ in (3.4.10) can be controlled by $\mathcal{U}_M$ in (3.4.6):

$$\mathcal{U}_{N_k} \lesssim 1 + \mathcal{U}_M. \tag{3.4.11}$$

Hence, from (3.4.4), (3.4.5), (3.4.7), (3.4.9), and (3.4.11), we obtain

$$\widehat{\mathcal{W}}_{M,k}(\dot\Upsilon^M) \ge \sigma(\log N_k)^{-\frac{3}{4}}\mathbb{E}\bigg[\int_{\mathbb{T}^3} :Y^2_{N_k}: \Theta_{N_k} dx\bigg] - C_1 + \frac{1}{20}\mathcal{U}_M \tag{3.4.12}$$

for any $M \ge N_k \gg 1$.

Therefore, it remains to estimate the contribution from the second term on the right-hand side of (3.4.7). Let us first state a lemma whose proof is presented at the end of this section.

Lemma 3.4.2. *We have*

$$\mathbb{E}\bigg[\int_0^1 \langle \dot{\mathfrak{Z}}_N(t), \dot{\mathfrak{Z}}_M(t)\rangle_{H^1_x} dt\bigg] \sim \log N \tag{3.4.13}$$

for any $1 \le N \le M$, *where* $\dot{\mathfrak{Z}}_N = \pi_N \dot{\mathfrak{Z}}^N$.

By assuming Lemma 3.4.2, we complete the proof of Proposition 3.4.1. By (3.2.2), (3.2.3) with $\mathfrak{Z}_{N_k} = \pi_{N_k}\mathfrak{Z}^{N_k}$, (3.4.8), Lemma 3.4.2, Cauchy's inequality (with small $\varepsilon_0 > 0$), and Lemma 3.1.2 (see (3.1.7)), we have

$$\begin{aligned}
\sigma\mathbb{E}\bigg[\int_{\mathbb{T}^3} :Y^2_{N_k}: \Theta_{N_k} dx\bigg] &= \sigma\mathbb{E}\bigg[\int_0^1\int_{\mathbb{T}^3} :Y^2_{N_k}(t): \dot\Theta_{N_k}(t) dt\bigg]\\
&= \sigma^2\mathbb{E}\bigg[\int_0^1 \langle \dot{\mathfrak{Z}}_{N_k}(t), \dot{\mathfrak{Z}}_M(t)\rangle_{H^1_x} dt\bigg] + \sigma\mathbb{E}\bigg[\int_0^1 \langle \dot{\mathfrak{Z}}_{N_k}(t), \dot\Upsilon^M(t)\rangle_{H^1_x} dt\bigg]\\
&\ge c\log N_k - \varepsilon_0\mathbb{E}\bigg[\int_0^1 \| :Y^2_{N_k}(t): \|^2_{H^{-1}_x} dt\bigg] - C_{\varepsilon_0}\mathbb{E}\bigg[\int_0^1 \|\dot\Upsilon^M(t)\|^2_{H^1_x} dt\bigg]\\
&\ge \frac{c}{2}\log N_k - C_{\varepsilon_0}\mathbb{E}\bigg[\int_0^1 \|\dot\Upsilon^M(t)\|^2_{H^1_x} dt\bigg]
\end{aligned} \tag{3.4.14}$$

for $M \geq N_k \gg 1$. Thus, putting (3.4.4), (3.4.12), and (3.4.14) together, we have

$$-\log C_{M,k} \geq \inf_{\dot{\Upsilon}^M \in \mathbb{H}_a^1} \Big\{ c(\log N_k)^{\frac{1}{4}} - C_2 + \frac{1}{40}\mathcal{U}_M \Big\} \geq c(\log N_k)^{\frac{1}{4}} - C_2 \quad (3.4.15)$$

for any sufficiently large $k \gg 1$ (such that $N_k \gg 1$). Hence, from (3.4.15), we obtain

$$C_{M,k} \lesssim \exp\big(-c(\log N_k)^{\frac{1}{4}}\big) \quad (3.4.16)$$

for $M \geq N_k \gg 1$, uniformly in $M \in \mathbb{N}$. Therefore, by taking limits in $M \to \infty$ and then $k \to \infty$, we conclude from (3.4.3) and (3.4.16) that

$$\lim_{k\to\infty} \int e^{G_k(u)} d\rho(u) = 0$$

as desired. This completes the proof of Proposition 3.4.1. ∎

We conclude this chapter by presenting the proof of Lemma 3.4.2.

Proof of Lemma 3.4.2. For simplicity, we suppress the time dependence in the following. From (3.2.3), we have

$$\hat{\dot{\mathfrak{Z}}}_N(n) = \langle n\rangle^{-2} \sum_{\substack{n_1,n_2 \in \mathbb{Z}^3 \\ n = n_1 + n_2 \neq 0}} \widehat{Y}_N(n_1)\widehat{Y}_N(n_2) \quad (3.4.17)$$

for $n \neq 0$. On the other hand, when $n = 0$, it follows from Lemma 2.3.2 that

$$\mathbb{E}\big[|\hat{\dot{\mathfrak{Z}}}_N(0)|^2\big] = \mathbb{E}\Big[\Big(\sum_{\substack{n_1 \in \mathbb{Z}^3 \\ n_1 \in NQ}} (|\widehat{Y}_N(n_1)|^2 - \langle n_1\rangle^{-2})\Big)^2\Big] \lesssim \sum_{n_1 \in \mathbb{Z}^3} \langle n_1\rangle^{-4} \lesssim 1, \quad (3.4.18)$$

where Q is as in (1.2.7). Hence, from (3.4.17) and (3.4.18), we have

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^1 \langle \dot{\mathfrak{Z}}_N(t), \dot{\mathfrak{Z}}_M(t)\rangle_{H_x^1} dt\Big] \\
&= \int_0^1 \mathbb{E}\Big[\sum_{n\in\mathbb{Z}^3} \langle n\rangle^2 \hat{\dot{\mathfrak{Z}}}_N(n,t)\overline{\hat{\dot{\mathfrak{Z}}}_M(n,t)}\Big] dt \\
&= \int_0^1 \mathbb{E}\Big[\sum_{n\in\mathbb{Z}^3\setminus\{0\}} \langle n\rangle^2 \hat{\dot{\mathfrak{Z}}}_N(n,t)\overline{\hat{\dot{\mathfrak{Z}}}_M(n,t)}\Big] dt + O(1).
\end{aligned}$$

We now proceed as in the proof of (3.1.7) in Lemma 3.1.2 (i). By applying (3.2.3) and Lemma 2.3.2, and summing over $\{|n| \leq \frac{2}{3}N, \frac{1}{4}|n| \leq |n_1| \leq \frac{1}{2}|n|\}$ (which implies

$|n_2| \sim |n|$ and $|n_j| \leq N$, $j = 1, 2$), we have

$$\begin{aligned}
&\mathbb{E}\Big[\sum_{n\in\mathbb{Z}^3\setminus\{0\}} \langle n\rangle^2 \widehat{\dot{\mathfrak{Z}}}_N(n,t)\overline{\widehat{\dot{\mathfrak{Z}}}_M(n,t)}\Big] \\
&\quad = \sum_{n\in\mathbb{Z}^3} \frac{\chi_N(n)\chi_M(n)}{\langle n\rangle^2} \\
&\qquad \times \int_{\mathbb{T}^3_x\times\mathbb{T}^3_y} \mathbb{E}\big[H_2(Y_N(x,t);t\sigma_N)H_2(Y_N(y,t);t\sigma_N)\big]e_n(y-x)dxdy \\
&\quad = \sum_{n\in\mathbb{Z}^3} \frac{t^2\chi_N(n)\chi_M(n)}{\langle n\rangle^2} \sum_{n_1,n_2\in\mathbb{Z}^3} \frac{\chi_N^2(n_1)\chi_N^2(n_2)}{\langle n_1\rangle^2\langle n_2\rangle^2} \int_{\mathbb{T}^3_x\times\mathbb{T}^3_y} e_{n_1+n_2-n}(x-y)dxdy \\
&\quad = \sum_{n\in\mathbb{Z}^3} \frac{t^2\chi_N(n)\chi_M(n)}{\langle n\rangle^2} \sum_{n=n_1+n_2} \frac{\chi_N^2(n_1)\chi_N^2(n_2)}{\langle n_1\rangle^2\langle n_2\rangle^2} \sim t^2\log N,
\end{aligned}$$

where $\chi_N(n_j)$ is as in (1.2.6). By integrating on [0, 1], we obtain the desired bound (3.4.13). ∎