Chapter 4

Non-normalizability in the strongly nonlinear regime

4.1 Reference measures and the σ-finite Φ_3^3-measure

In this chapter, we prove non-normalizability of the Φ_3^3-measure in the strongly nonlinear regime (Theorem 1.2.1 (ii)). In [54], we introduced a strategy for establishing non-normalizability in the context of the focusing Hartree Φ_3^4-measures on $\mathbb{T}^3$, using the Boué–Dupuis variational formula. We point out that, in [54], the focusing Hartree Φ_3^4-measures were absolutely continuous with respect to the base Gaussian free field μ. Moreover, the truncated potential energy $R_N^{\text{Hartree}}(u)$ and the corresponding density $e^{-R_N^{\text{Hartree}}(u)}$ of the truncated focusing Hartree Φ_3^4-measures formed convergent sequences. In [54], we proved the following version of the non-normalizability of the focusing Hartree Φ_3^4-measure:

$$\sup_{N\in\mathbb{N}} \mathbb{E}_\mu\big[e^{-R_N^{\text{Hartree}}(u)}\big] = \infty. \tag{4.1.1}$$

Denoting the limiting density by $e^{-R^{\text{Hartree}}(u)}$, this result says that the σ-finite version of the focusing Hartree Φ_3^4-measure:

$$e^{-R^{\text{Hartree}}(u)}d\mu(u)$$

is not normalizable (i.e. there is no normalization constant to make this into a probability measure). See also [61] for an analogous non-normalizability result for the log-correlated focusing Gibbs measures with a quartic interaction potential.

The main new difficulty in our current problem is the singularity of the Φ_3^3-measure. In particular, the potential energy $R_N^\diamond(u)$ in (1.2.10) (and the corresponding density $e^{-R_N^\diamond(u)}$) does *not* converge to any limit. Hence, even if we prove a non-normalizability statement of the form (4.1.1), it might still be possible that by choosing a sequence of constants $\widehat{Z}_N$ appropriately, the measure $\widehat{Z}_N^{-1}e^{-R_N^\diamond(u)}d\mu$ has a weak limit. This is precisely the case for the Φ_3^4-measure; see [3]. The non-convergence claim in Theorem 1.2.1 (ii) for the truncated Φ_3^3-measures (see Proposition 4.1.4 below) tells us that this is not the case for the Φ_3^3-measure.

In order to overcome this issue, we first construct a reference measure ν_δ as a weak limit of the following tamed version of the truncated Φ_3^3-measure (with $\delta > 0$):

$$d\nu_{N,\delta}(u) = Z_{N,\delta}^{-1}\exp\big(-\delta F(\pi_N u) - R_N^\diamond(u)\big)d\mu(u)$$

for some appropriate taming function F; see (4.1.6). See Proposition 4.1.1. We also show that $F(u)$, without the frequency projection π_N on u, is well defined almost surely with respect to the limiting reference measure $\nu_\delta = \lim_{N\to\infty} \nu_{N,\delta}$. This allows us to construct a σ-finite version of the Φ^3_3-measure:

$$d\overline{\rho}_\delta = e^{\delta F(u)} d\nu_\delta = \lim_{N\to\infty} Z_{N,\delta}^{-1} e^{\delta F(u)} e^{-\delta F(\pi_N u) - R_N^\diamond(u)} d\mu(u). \tag{4.1.2}$$

The main point is that while the truncated Φ^3_3-measure ρ_N ($= \nu_{N,\delta}$ with $\delta = 0$) may not be convergent, the tamed version $\nu_{N,\delta}$ of the truncated Φ^3_3-measure converges to the limit ν_δ, thus allowing us to define a σ-finite version of the Φ^3_3-measure. We then show that this σ-finite version $\overline{\rho}_\delta$ of the Φ^3_3-measure in (4.1.2) is not normalizable in the strongly nonlinear regime. See Proposition 4.1.2. Furthermore, as a corollary to this non-normalizability result of the σ-finite version $\overline{\rho}_\delta$ of the Φ^3_3-measure, we also show that the sequence $\{\rho_N\}_{N\in\mathbb{N}}$ of the truncated Φ^3_3-measures defined in (1.2.11) does not converge weakly in a natural space[1] $\mathcal{A}(\mathbb{T}^3)$ (see (4.1.3) below) for the Φ^3_3-measure. See Proposition 4.1.4.

We first state the construction of the reference measure. Let p_t be the kernel of the heat semigroup $e^{t\Delta}$. Then, define the space $\mathcal{A} = \mathcal{A}(\mathbb{T}^3)$ via the norm:

$$\|u\|_{\mathcal{A}} := \sup_{0<t\le 1} \big(t^{\frac{3}{8}} \|p_t * u\|_{L^3(\mathbb{T}^3)}\big). \tag{4.1.3}$$

Recall from [45, Theorem 5.3][2] (see also [76, eq. (2.41)] and [2, Theorem 2.34]) that

$$\mathcal{A} = B^{-\frac{3}{4}}_{3,\infty}(\mathbb{T}^3). \tag{4.1.4}$$

In particular, the space $\mathcal{A}$ contains the support of the massive Gaussian free field μ on $\mathbb{T}^3$ and thus we have $\|u\|_{\mathcal{A}} < \infty$, μ-almost surely. See Lemma 4.2.2 below. In the following, for simplicity of notation, we use $\mathcal{A}$ rather than $B^{-\frac{3}{4}}_{3,\infty}(\mathbb{T}^3)$. Moreover, the notation $\mathcal{A}$ is suitable for our purpose, since we make use of the characterization (4.1.3) extensively via the Schauder estimate, which we recall now (see for example [60]):

$$\|p_t * u\|_{L^q(\mathbb{T}^3)} \le C_{\alpha,p,q} t^{-\frac{\alpha}{2} - \frac{3}{2}(\frac{1}{p} - \frac{1}{q})} \|\langle\nabla\rangle^{-\alpha} u\|_{L^p(\mathbb{T}^3)} \tag{4.1.5}$$

for any $\alpha \ge 0$ and $1 \le p \le q \le \infty$. From the Schauder estimate (4.1.5) (or directly from (4.1.4)), we see that $W^{-\frac{3}{4},3}(\mathbb{T}^3) \subset \mathcal{A}$.

Given $N \in \mathbb{N}$, we set $u_N = \pi_N u$. Then, given $\delta > 0$ and $N \in \mathbb{N}$, we define the measure $\nu_{N,\delta}$ by

$$d\nu_{N,\delta}(u) = Z_{N,\delta}^{-1} \exp\big(-\delta \|u_N\|^{20}_{\mathcal{A}} - R_N^\diamond(u)\big) d\mu(u) \tag{4.1.6}$$

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[1]For example, in the weakly nonlinear regime, the support of the limiting Φ^3_3-measure constructed in Theorem 1.2.1 (i) is contained in the space $\mathcal{A}(\mathbb{T}^3) \supset \mathcal{C}^{-\frac{3}{4}}(\mathbb{T}^3)$.

[2]The discussion in [45] is on $\mathbb{R}^d$, but a slight modification yields the corresponding result on $\mathbb{T}^d$.

for $N \in \mathbb{N}$ and $\delta > 0$, where $R_N^\diamond$ is as in (1.2.10) and

$$Z_{N,\delta} = \int \exp\big(-\delta \|u_N\|_{\mathcal{A}}^{20} - R_N^\diamond(u)\big) d\mu(u). \tag{4.1.7}$$

Namely, $\nu_{N,\delta}$ is a tamed version of the truncated Φ^3_3-measure ρ_N in (1.2.11). We prove that the sequence $\{\nu_{N,\delta}\}_{N\in\mathbb{N}}$ converges weakly to some limiting probability measure ν_δ.

Proposition 4.1.1. *Let $\sigma \neq 0$ and $\gamma \geq 3$. Then, given any $\delta > 0$, the sequence of measures $\{\nu_{N,\delta}\}_{N\in\mathbb{N}}$ defined in (4.1.6) converges weakly to a unique probability measure ν_δ, and similarly $Z_{N,\delta}$ converges to Z_δ. Moreover, $\|u\|_{\mathcal{A}}$ is finite ν_δ-almost surely, and we have*

$$d\nu_\delta(u) = \frac{\exp(-(\delta-\delta')\|u\|_{\mathcal{A}}^{20})}{\int \exp(-(\delta-\delta')\|u\|_{\mathcal{A}}^{20}) d\nu_{\delta'}(u)} d\nu_{\delta'}(u) \tag{4.1.8}$$

for $\delta > \delta' > 0$.

This proposition allows us to define a σ-finite version of the Φ^3_3-measure by

$$d\overline{\rho}_\delta = e^{\delta\|u\|_{\mathcal{A}}^{20}} d\nu_\delta \tag{4.1.9}$$

for any $\delta > 0$. At a very *formal* level, $\delta\|u\|_{\mathcal{A}}^{20}$ in the exponent of (4.1.9) and $-\delta\|u_N\|_{\mathcal{A}}^{20}$ in the exponent of (4.1.6) cancel each other in the limit as $N \to \infty$, and thus the right-hand side of (4.1.8) formally looks like $Z_\delta^{-1} \lim_{N\to\infty} e^{-R_N^\diamond(u)} d\mu$. While this discussion is merely formal, it explains why we refer to the measure $\overline{\rho}_\delta$ as a σ-finite version of the Φ^3_3-measure. The identity (4.1.8) shows how ν_δ's for different values of $\delta > 0$ are related. When $\delta = 0$, the expression $Z_\delta \overline{\rho}_\delta$ would formally correspond to a limit of $e^{-R_N^\diamond(u)} d\mu$, but in order to achieve the weak convergence claimed in Proposition 4.1.1 and construct a σ-finite version of the Φ^3_3-measure, we need to start with a tamed version (i.e. $\delta > 0$) of the truncated Φ^3_3-measure. For the sake of concreteness, we chose a taming via the $\mathcal{A}$-norm but it is possible to consider a different taming (say, based on some other norm) and obtain the same result.

The next proposition shows that the σ-finite version $\overline{\rho}_\delta$ of the Φ^3_3-measure defined in (4.1.9) is not normalizable in the strongly nonlinear regime.

Proposition 4.1.2. *Let $\sigma \gg 1$ and $\gamma \geq 3$. Given $\delta > 0$, let ν_δ be the measure constructed in Proposition 4.1.1 and let $\overline{\rho}_\delta$ be as in (4.1.9). Then, we have*

$$\int 1 d\overline{\rho}_\delta = \int \exp\big(\delta\|u\|_{\mathcal{A}}^{20}\big) d\nu_\delta = \infty. \tag{4.1.10}$$

Remark 4.1.3. (i) A slight modification of the computation in Section 3.4 combined with the analysis in Section 4.2 presented below (Step 1 of the proof of Proposition 4.1.1) shows that the tamed version ν_δ of the Φ^3_3-measure, constructed in Proposition 4.1.1, and the massive Gaussian free field μ are mutually singular, just like the Φ_3^3-measure in the weakly nonlinear regime, constructed in Chapter 3. As a consequence, the σ-finite version $\overline{\rho}_\delta$ of the Φ_3^3-measure defined in (4.1.9) and the massive Gaussian free field μ are mutually singular.

(ii) In Appendix A, we show that the limiting Φ_3^3-measure is absolutely continuous with respect to the shifted measure $\mathrm{Law}(Y(1) + \sigma \mathfrak{Z}(1) + \mathcal{W}(1))$ in the weakly nonlinear regime. A slight modification of the argument in Appendix A also shows that the tamed version ν_δ of the Φ_3^3-measure constructed in Proposition 4.1.1 and the σ-finite version $\overline{\rho}_\delta$ of the Φ_3^3-measure in (4.1.9) are also absolutely continuous with respect to the same shifted measure, even in the strongly nonlinear regime. See Remark A.3.1. This shows that the measure $\overline{\rho}_\delta$ in (4.1.9) is a quite natural candidate to consider as a σ-finite version of the Φ_3^3-measure.

As a corollary to (the proofs of) Propositions 4.1.1 and 4.1.2, we show the following non-convergence result for the truncated Φ_3^3-measure ρ_N in (1.2.11).

Proposition 4.1.4. *Let $\sigma \gg 1$, $\gamma \geq 3$, and $\mathcal{A} = \mathcal{A}(\mathbb{T}^3)$ be as in (4.1.3). Then, the sequence $\{\rho_N\}_{N\in\mathbb{N}}$ of the truncated Φ_3^3-measures defined in (1.2.11) does not converge weakly to any limit as probability measures on $\mathcal{A}$. The same claim holds for any subsequence $\{\rho_{N_k}\}_{k\in\mathbb{N}}$.*

In Section 4.2, we present the proof of Proposition 4.1.1. In Section 4.3, we then prove the non-normalizability (Proposition 4.1.2). Finally, we present the proof of Proposition 4.1.4 in Section 4.4.

4.2 Construction of the reference measure

In this section, we present the proof of Proposition 4.1.1 on the construction of the reference measure ν_δ. We first establish several preliminary lemmas.

Lemma 4.2.1. *Let the $\mathcal{A}$-norm be as in (4.1.3). Then, we have*

$$\|u\|_{\mathcal{A}} \lesssim \|u\|_{H^{-\frac{1}{4}}}.$$

Proof. This is immediate from the Schauder estimate (4.1.5). ∎

Lemma 4.2.2. *We have $W^{-\frac{3}{4},3}(\mathbb{T}^3) \subset \mathcal{A}$ and thus the quantity $\|u\|_{\mathcal{A}}$ is finite μ-almost surely. Moreover, given any $1 \leq p < \infty$, we have*

$$\mathbb{E}_\mu\big[\|\pi_N u\|_{\mathcal{A}}^p\big] \leq C_p < \infty, \tag{4.2.1}$$

uniformly in $N \in \mathbb{N} \cup \{\infty\}$ with the understanding that $\pi_\infty = \mathrm{Id}$.

Proof. As we already mentioned, the first claim follows from the Schauder estimate (4.1.5) (or from (4.1.4)). As for the bound (4.2.1), from the Schauder estimate (4.1.5), Minkowski's integral inequality, and the Wiener chaos estimate (Lemma 2.3.1) with (1.2.4), we have

$$\begin{aligned}\mathbb{E}_\mu\big[\|\pi_N u\|_{\mathcal{A}}^p\big] \lesssim \mathbb{E}_\mu\big[\|u\|_{W^{-\frac{3}{4},3}}^p\big] &\lesssim \Big\| \big\|\langle\nabla\rangle^{-\frac{3}{4}} u(x)\big\|_{L^p(\mu)}\Big\|_{L_x^3}^p \\ &\leq p^{\frac{p}{2}} \Big\| \big\|\langle\nabla\rangle^{-\frac{3}{4}} u(x)\big\|_{L^2(\mu)}\Big\|_{L_x^3}^p \\ &\leq p^{\frac{p}{2}} \bigg(\sum_{n\in\mathbb{Z}^3} \frac{1}{\langle n\rangle^{\frac{7}{2}}}\bigg)^p < \infty.\end{aligned}$$

This proves (4.2.1). ■

We now present the proof of Proposition 4.1.1.

Proof of Proposition 4.1.1. We break the proof into three steps.

Step 1. In this first part, we prove that $Z_{N,\delta}$ in (4.1.7) is uniformly bounded in $N \in \mathbb{N}$. As for the tightness of $\{\nu_{N,\delta}\}_{N\in\mathbb{N}}$ and the uniqueness of ν_δ claimed in the statement, we can repeat arguments analogous to those in Sections 3.2 and 3.3 and thus we omit details.

From (4.1.7) and the Boué–Dupuis variational formula (Lemma 3.1.1) with the change of variables (3.2.4), we have

$$\begin{aligned}-\log Z_{N,\delta} = \inf_{\dot\Upsilon^N \in \mathbb{H}_a^1} \mathbb{E}\bigg[&\delta\|Y_N + \Theta_N\|_{\mathcal{A}}^{20} - \sigma\int_{\mathbb{T}^3} Y_N\Theta_N^2\,dx - \frac{\sigma}{3}\int_{\mathbb{T}^3}\Theta_N^3\,dx \\ &+ A\bigg|\int_{\mathbb{T}^3}(:Y_N^2: + 2Y_N\Theta_N + \Theta_N^2)\,dx\bigg|^\gamma \\ &+ \frac{1}{2}\int_0^1 \|\dot\Upsilon^N(t)\|_{H_x^1}^2\,dt\bigg],\end{aligned} \tag{4.2.2}$$

where $\Theta_N = \Upsilon_N + \sigma\widetilde{\mathfrak{Z}}_N$ with $\widetilde{\mathfrak{Z}}_N = \pi_N\mathfrak{Z}_N$ as in (3.2.11). Our goal is to establish a uniform lower bound on the right-hand side of (4.2.2). Unlike Section 3.2, we do not assume smallness on $|\sigma|$. In this case, a rescue comes from the extra positive term $\delta\|Y_N + \Theta_N\|_{\mathcal{A}}^{20}$ as compared to (3.2.9).

Given any $0 < c_0 < 1$, it follows from Young's inequality (3.2.32) with $\gamma \geq 3$ that

$$\begin{aligned}&\bigg|\int_{\mathbb{T}^3}(:Y_N^2: + 2Y_N\Theta_N + \Theta_N^2)\,dx\bigg|^\gamma \\ &\qquad \geq c_0\bigg|\int_{\mathbb{T}^3}(:Y_N^2: + 2Y_N\Theta_N + \Theta_N^2)\,dx\bigg|^3 - C.\end{aligned} \tag{4.2.3}$$

Then, taking an expectation and applying Lemmas 3.2.2 and 3.2.3 with Lemma 3.1.2 and (3.2.17), we have

$$\mathbb{E}\bigg[A\bigg|\int_{\mathbb{T}^3}(:Y_N^2: + 2Y_N\Theta_N + \Theta_N^2)dx\bigg|^{\gamma}\bigg] \geq C_0\mathbb{E}\big[\|\Upsilon_N\|_{L^2}^6\big] - C_1\mathbb{E}\big[\|\Upsilon_N\|_{H^1}^2\big] - C \tag{4.2.4}$$

for some $C_0 > 0$, $0 < C_1 \leq \frac{1}{4}$. Hence, it follows from (4.2.2), (4.2.4), and Lemma 3.2.2 together with Lemma 3.1.2 and (3.2.17) that there exists $C_2 > 0$ such that

$$-\log Z_{N,\delta} \geq \inf_{\dot{\Upsilon}^N\in\mathbb{H}_a^1}\mathbb{E}\bigg[\delta\|Y_N + \Upsilon_N + \sigma\widetilde{\mathfrak{Z}}_N\|_{\mathcal{A}}^{20} - \frac{\sigma}{3}\int_{\mathbb{T}^3}(\Upsilon_N + \sigma\widetilde{\mathfrak{Z}}_N)^3dx + C_2\|\Upsilon_N\|_{L^2}^6 + C_2\|\Upsilon_N\|_{H^1}^2\bigg] - C. \tag{4.2.5}$$

By Young's inequality, we have

$$\begin{aligned}\bigg|\int_{\mathbb{T}^3}\Upsilon_N^2\widetilde{\mathfrak{Z}}_N dx\bigg| + \bigg|\int_{\mathbb{T}^3}\Upsilon_N\widetilde{\mathfrak{Z}}_N^2 dx\bigg| &\leq \|\Upsilon_N\|_{L^2}^2\|\mathfrak{Z}_N\|_{\mathcal{C}^{1-\varepsilon}} + \|\Upsilon_N\|_{L^2}\|\mathfrak{Z}_N\|_{\mathcal{C}^{1-\varepsilon}}^2\\ &\leq \frac{C_2}{2|\sigma|}\|\Upsilon_N\|_{L^2}^6 + \|\mathfrak{Z}_N\|_{\mathcal{C}^{1-\varepsilon}}^c + C_\sigma.\end{aligned} \tag{4.2.6}$$

Hence, from (4.2.5) and (4.2.6) with (3.2.32) (with $\gamma = 20$) and Lemma 4.2.2, we obtain

$$-\log Z_{N,\delta} \geq \inf_{\dot{\Upsilon}_N\in\mathbb{H}_a^1}\mathbb{E}\bigg[\frac{\delta}{2}\|\Upsilon_N\|_{\mathcal{A}}^{20} - \frac{|\sigma|}{3}\|\Upsilon_N\|_{L^3}^3 + \frac{C_2}{2}\|\Upsilon_N\|_{L^2}^6 + C_2\|\Upsilon_N\|_{H^1}^2\bigg] - C. \tag{4.2.7}$$

Now, we need to estimate the L^3-norm of Υ_N. From (4.1.3), Sobolev's inequality, and the mean value theorem: $|1 - e^{-t|n|^2}| \lesssim (t|n|^2)^\theta$ for any $0 \leq \theta \leq 1$, we have

$$\begin{aligned}\|\Upsilon_N\|_{L^3}^3 &\lesssim t^{-\frac{9}{8}}\|\Upsilon_N\|_{\mathcal{A}}^3 + \|\Upsilon_N - p_t * \Upsilon_N\|_{H^{\frac{1}{2}}}^3\\ &\lesssim t^{-\frac{9}{8}}\|\Upsilon_N\|_{\mathcal{A}}^3 + t^{\frac{3}{4}}\|\Upsilon_N\|_{H^1}^3\end{aligned}$$

for $0 < t \leq 1$. By choosing $t^{\frac{3}{4}} \sim (1 + \frac{|\sigma|}{C_2}\|\Upsilon_N\|_{H^1})^{-1}$ and applying Young's inequality, we obtain

$$\begin{aligned}|\sigma|\|\Upsilon_N\|_{L^3}^3 &\leq C_{C_2,|\sigma|}\|\Upsilon_N\|_{H^1}^{\frac{3}{2}}\|\Upsilon_N\|_{\mathcal{A}}^3 + \frac{C_2}{4}\|\Upsilon_N\|_{H^1}^2 + 1\\ &\leq C_{C_2,|\sigma|,\delta} + \frac{\delta}{4}\|\Upsilon_N\|_{\mathcal{A}}^{20} + \frac{C_2}{2}\|\Upsilon_N\|_{H^1}^2.\end{aligned} \tag{4.2.8}$$

Therefore, from (4.2.7) and (4.2.8), we conclude that

$$Z_{N,\delta} \le C_\delta < \infty,$$

uniformly in $N \in \mathbb{N}$.

Step 2. Next, we show that $\|u\|_{\mathcal{A}}$ is finite ν_δ-almost surely. Let η be a smooth function with compact support with $\int_{\mathbb{R}^3} |\eta(\xi)|^2 d\xi = 1$ and set

$$\widehat{\rho}(\xi) = \int_{\mathbb{R}^3} \eta(\xi - \xi_1)\overline{\eta(-\xi_1)}d\xi_1.$$

Given $\varepsilon > 0$, define ρ_ε by

$$\rho_\varepsilon(x) = \sum_{n\in\mathbb{Z}^3} \widehat{\rho}(\varepsilon n)e^{in\cdot x}. \tag{4.2.9}$$

Since the support of $\widehat{\rho}$ is compact, the sum on the right-hand side is over finitely many frequencies. Thus, given any $\varepsilon > 0$, there exists $N_0(\varepsilon) \in \mathbb{N}$ such that

$$\rho_\varepsilon * u = \rho_\varepsilon * u_N \tag{4.2.10}$$

for any $N \ge N_0(\varepsilon)$. From the Poisson summation formula, we have

$$\rho_\varepsilon(x) = \sum_{n\in\mathbb{Z}^3} \varepsilon^{-3}\big|\mathcal{F}^{-1}_{\mathbb{R}^3}(\eta)(\varepsilon^{-1}x + 2\pi n)\big|^2 \ge 0,$$

where $\mathcal{F}^{-1}_{\mathbb{R}^3}$ denotes the inverse Fourier transform on $\mathbb{R}^3$. Noting that

$$\|\rho_\varepsilon\|_{L^1(\mathbb{T}^3)} = \int_{\mathbb{T}^3} \rho_\varepsilon(x)dx = \widehat{\rho}(0) = \|\eta\|^2_{L^2(\mathbb{R}^3)} = 1,$$

we have, from Young's inequality, that

$$\|\rho_\varepsilon * u\|_{\mathcal{A}} \le \|u\|_{\mathcal{A}}. \tag{4.2.11}$$

Moreover, $\{\rho_\varepsilon\}$ defined above is an approximation to the identity on $\mathbb{T}^3$ and thus for any distribution u on $\mathbb{T}^3$, $\rho_\varepsilon * u \to u$ in the $\mathcal{A}$-norm, as $\varepsilon \to 0$.

Let $\delta > \delta' > 0$. By Fatou's lemma, the weak convergence of $\{\nu_{N,\delta}\}_{N\in\mathbb{N}}$ from Step 1 with (4.2.10), (4.2.11), and the definition (4.1.6) of $\nu_{N,\delta}$, we have

$$\begin{aligned}
\int \exp\big((\delta-\delta')\|u\|^{20}_{\mathcal{A}}\big)d\nu_\delta &\le \liminf_{\varepsilon\to 0}\int \exp\big((\delta-\delta')\|\rho_\varepsilon * u\|^{20}_{\mathcal{A}}\big)d\nu_\delta\\
&= \liminf_{\varepsilon\to 0}\lim_{N\to\infty}\int \exp\big((\delta-\delta')\|\rho_\varepsilon * u_N\|^{20}_{\mathcal{A}}\big)d\nu_{N,\delta}\\
&\le \lim_{N\to\infty}\int \exp\big((\delta-\delta')\|u_N\|^{20}_{\mathcal{A}}\big)d\nu_{N,\delta}\\
&= \lim_{N\to\infty}\frac{Z_{N,\delta'}}{Z_{N,\delta}}\int 1 d\nu_{N,\delta'} = \frac{Z_{\delta'}}{Z_\delta}.
\end{aligned}$$

Hence, we have

$$\int \exp\big((\delta - \delta')\|u\|_{\mathcal{A}}^{20}\big) d\nu_\delta < \infty$$

for any $\delta > \delta' > 0$. By choosing $\delta' = \frac{\delta}{2}$, we obtain

$$\int \exp\bigg(\frac{\delta}{2}\|u\|_{\mathcal{A}}^{20}\bigg) d\nu_\delta < \infty,$$

which shows that $\|u\|_{\mathcal{A}}$ is finite almost surely with respect to ν_δ.

Step 3. Finally, we prove the relation (4.1.8). We first note that it suffices to show that

$$\frac{Z_\delta}{Z_{\delta'}} d\nu_\delta = \exp\big(-(\delta - \delta')\|u\|_{\mathcal{A}}^{20}\big) d\nu_{\delta'} \tag{4.2.12}$$

for any $\delta > \delta' > 0$. In fact, once we have (4.2.12), by integration, we obtain

$$\frac{Z_\delta}{Z_{\delta'}} = \int \exp\big(-(\delta - \delta')\|u\|_{\mathcal{A}}^{20}\big) d\nu_{\delta'} \tag{4.2.13}$$

and thus (4.1.8) follows from (4.2.12) and (4.2.13).

Let $F : \mathcal{C}^{-100}(\mathbb{T}^3) \to \mathbb{R}$ be a bounded Lipschitz function with $F \geq 0$. The dominated convergence theorem, the weak convergence of $\{\nu_{N,\delta}\}_{N\in\mathbb{N}}$ from Step 1, and (4.1.6) yield that

$$\begin{aligned}
&\frac{Z_\delta}{Z_{\delta'}} \int F(u) d\nu_\delta - \int F(u) \exp\big(-(\delta - \delta')\|u\|_{\mathcal{A}}^{20}\big) d\nu_{\delta'} \\
&= \lim_{\varepsilon \to 0} \bigg(\frac{Z_\delta}{Z_{\delta'}} \int F(u) d\nu_\delta \\
&\qquad - \int F(u) \exp\big(-(\delta - \delta')\|\rho_\varepsilon * u\|_{\mathcal{A}}^{20}\big) d\nu_{\delta'}\bigg) \\
&= \lim_{\varepsilon \to 0} \lim_{N\to\infty} \bigg(\frac{Z_{N,\delta}}{Z_{N,\delta'}} \int F(u) d\nu_{N,\delta} \\
&\qquad - \int F(u) \exp\big(-(\delta - \delta')\|\rho_\varepsilon * u_N\|_{\mathcal{A}}^{20}\big) d\nu_{N,\delta'}\bigg) \\
&= \lim_{\varepsilon \to 0} \lim_{N\to\infty} \int F(u)\big[\exp\big(-(\delta - \delta')\|u_N\|_{\mathcal{A}}^{20}\big) \\
&\qquad - \exp\big(-(\delta - \delta')\|\rho_\varepsilon * u_N\|_{\mathcal{A}}^{20}\big)\big] d\nu_{N,\delta'}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\left|\frac{Z_\delta}{Z_{\delta'}}\int F(u)d\nu_\delta - \int F(u)\exp\big(-(\delta-\delta')\|u\|_{\mathcal{A}}^{20}\big)d\nu_{\delta'}\right| \\
&\quad\lesssim \limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\int \Big|\exp\big(-(\delta-\delta')\|u_N\|_{\mathcal{A}}^{20}\big) \\
&\qquad\qquad - \exp\big(-(\delta-\delta')\|\rho_\varepsilon * u_N\|_{\mathcal{A}}^{20}\big)\Big|d\nu_{N,\delta'}(u) \\
&\quad\lesssim \limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\int \Big|\exp\big(-(\delta-\delta')\|\pi_N u^N(\omega)\|_{\mathcal{A}}^{20}\big) \\
&\qquad\qquad - \exp\big(-(\delta-\delta')\|\rho_\varepsilon * \pi_N u^N(\omega)\|_{\mathcal{A}}^{20}\big)\Big|d\mathbb{P}(\omega),
\end{aligned} \tag{4.2.14}$$

where u^N is a random variable with $\mathrm{Law}(u^N) = \nu_{N,\delta'}$. Noting that the integrand is uniformly bounded by 2, it follows from the bounded convergence theorem that the right-hand side of (4.2.14) tends to 0 once we show that $\|\rho_\varepsilon * \pi_N u^N(\omega) - \pi_N u^N(\omega)\|_{\mathcal{A}}$ tends to 0 in measure (with respect to $\mathbb{P}$). Namely, it suffices to show

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\lim_{N\to\infty}\mathbb{P}\Big(\big\{\omega\in\Omega : \|\rho_\varepsilon * \pi_N u^N(\omega) - \pi_N u^N(\omega)\|_{\mathcal{A}} > \alpha\big\}\Big) \\
&\quad = \lim_{\varepsilon\to 0}\lim_{N\to\infty}\nu_{N,\delta'}\Big(\big\{\|u_N - \rho_\varepsilon * u_N\|_{\mathcal{A}} > \alpha\big\}\Big) = 0
\end{aligned}$$

for any $\alpha > 0$.

From (4.1.3) and (4.1.5), we have

$$\|u_N - \rho_\varepsilon * u_N\|_{\mathcal{A}} \lesssim \|u_N - \rho_\varepsilon * u_N\|_{W^{-\frac{3}{4},3}} \lesssim \varepsilon^{\frac{1}{8}}\|u_N\|_{W^{-\frac{5}{8},3}}. \tag{4.2.15}$$

Hence, from Chebyshev's inequality and (4.2.15), it suffices to prove

$$\int \|u_N\|_{W^{-\frac{5}{8},3}}d\nu_{N,\delta'} \lesssim \int \exp\big(\|u_N\|_{W^{-\frac{5}{8},3}}\big)d\nu_{N,\delta'} \le C_{\delta'} < \infty, \tag{4.2.16}$$

uniformly in $N \in \mathbb{N}$. We use the variational formulation as in (4.2.2), and write

$$\begin{aligned}
&-\log\Big(\int \exp\big(\|u_N\|_{W^{-\frac{5}{8},3}}\big)d\nu_{N,\delta'}\Big) \\
&\quad = \inf_{\dot\Upsilon^N\in\mathbb{H}_a^1}\mathbb{E}\bigg[\delta'\|Y_N+\Theta_N\|_{\mathcal{A}}^{20} - \|Y_N+\Theta_N\|_{W^{-\frac{5}{8},3}} - \sigma\int_{\mathbb{T}^3} Y_N\Theta_N^2 dx \\
&\qquad\qquad - \frac{\sigma}{3}\int_{\mathbb{T}^3}\Theta_N^3 dx + A\bigg|\int_{\mathbb{T}^3}(:Y_N^2: + 2Y_N\Theta_N + \Theta_N^2)dx\bigg|^\gamma \\
&\qquad\qquad + \frac{1}{2}\int_0^1 \|\dot\Upsilon^N(t)\|_{H_x^1}^2 dt\bigg] \\
&\qquad + \log Z_{N,\delta'},
\end{aligned}$$

where

$$\Theta_N = \Upsilon_N + \sigma \widetilde{\mathfrak{Z}}_N.$$

From Lemma 3.1.2 and (3.2.17), we have, for any finite $p \geq 1$,

$$\mathbb{E}\big[\|Y_N\|^p_{W^{-\frac{5}{8},3}} + \|\mathfrak{Z}_N\|^p_{W^{-\frac{5}{8},3}}\big] < \infty, \tag{4.2.17}$$

uniformly in $N \in \mathbb{N}$. See also the proof of Lemma 4.2.2. Then, arguing as in (4.2.7) and (4.2.8) with Young's inequality, Sobolev's inequality, and (4.2.17), we obtain

$$\begin{aligned}
&-\log\Big(\int \exp(\|u_N\|_{W^{-\frac{5}{8},3}}) d\nu_{N,\delta'}\Big)\\
&\geq \inf_{\dot{\Upsilon}^N \in \mathbb{H}^1_a} \mathbb{E}\Big[-\|\Upsilon_N\|_{W^{-\frac{5}{8},3}}\\
&\qquad + C_0\big(\|\Upsilon_N\|^6_{L^2} + \|\Upsilon_N\|^2_{H^1}\big) + \frac{\delta'}{4}\|\Upsilon_N\|^{20}_{\mathcal{A}}\Big] - C_{C_0,\delta'}\\
&\gtrsim -1.
\end{aligned}$$

This proves (4.2.16) and hence concludes the proof of Proposition 4.1.1. ∎

4.3 Non-normalizability of the σ-finite measure $\overline{\rho}_\delta$

In this section, we present the proof of Proposition 4.1.2 on the non-normalizability of the σ-finite version $\overline{\rho}_\delta$ of the Φ^3_3-measure defined in (4.1.9).

Given $\varepsilon > 0$, let ρ_ε be as in (4.2.9). Then, by (4.2.11), the weak convergence of $\{\nu_{N,\delta}\}_{N\in\mathbb{N}}$ (Proposition 4.1.1), (4.2.10), and (4.1.6), we have

$$\begin{aligned}
&\int \exp(\delta\|u\|^{20}_{\mathcal{A}}) d\nu_\delta \geq \int \exp(\delta\|\rho_\varepsilon * u\|^{20}_{\mathcal{A}}) d\nu_\delta\\
&\geq \limsup_{L\to\infty} \int \exp\big(\delta \min\big(\|\rho_\varepsilon * u\|^{20}_{\mathcal{A}}, L\big)\big) d\nu_\delta\\
&= \limsup_{L\to\infty} \lim_{N\to\infty} \int \exp\big(\delta \min\big(\|\rho_\varepsilon * u_N\|^{20}_{\mathcal{A}}, L\big)\big) d\nu_{N,\delta}\\
&= \limsup_{L\to\infty} \lim_{N\to\infty} Z^{-1}_{N,\delta} \int \exp\big(\delta \min\big(\|\rho_\varepsilon * u_N\|^{20}_{\mathcal{A}}, L\big) - \delta\|u_N\|^{20}_{\mathcal{A}} - R^\diamond_N(u)\big) d\mu(u).
\end{aligned}$$

Hence, (4.1.10) is reduced to showing that

$$\limsup_{L\to\infty} \lim_{N\to\infty} \mathbb{E}_\mu\big[\exp\big(\delta \min\big(\|\rho_\varepsilon * u_N\|^{20}_{\mathcal{A}}, L\big) - \delta\|u_N\|^{20}_{\mathcal{A}} - R^\diamond_N(u)\big)\big] = \infty. \tag{4.3.1}$$

Let $Y = Y(1)$ be as in equation (3.1.2). By the Boué–Dupuis variational formula (Lemma 3.1.1) with the change of variables (3.2.4), we have

$$\begin{aligned}
&- \log \mathbb{E}\Big[\exp\big(\delta \min\big(\|\rho_\varepsilon * u_N\|_{\mathcal{A}}^{20}, L\big) - \delta \|u_N\|_{\mathcal{A}}^{20} - R_N^\diamond(u)\big)\Big] \\
&\quad = \inf_{\dot{\Upsilon}^N \in \mathbb{H}_a^1} \mathbb{E}\Big[-\delta \min\big(\|\rho_\varepsilon * (Y_N + \Upsilon_N + \sigma \widetilde{\mathfrak{Z}}_N)\|_{\mathcal{A}}^{20}, L\big) \\
&\qquad\qquad + \delta \|Y_N + \Upsilon_N + \sigma \widetilde{\mathfrak{Z}}_N\|_{\mathcal{A}}^{20} \\
&\qquad\qquad + \widehat{R}_N^\diamond(Y + \Upsilon^N + \sigma \mathfrak{Z}_N) + \frac{1}{2}\int_0^1 \|\dot{\Upsilon}^N(t)\|_{H_x^1}^2 dt\Big],
\end{aligned} \tag{4.3.2}$$

where $\widehat{R}_N^\diamond$ is as in (3.2.25) with the third power in the last term replaced by the γth power. With $\Theta_N = \Upsilon_N + \sigma \widetilde{\mathfrak{Z}}_N$, a slight modification of (3.2.30) yields

$$\begin{aligned}
\Big|\int_{\mathbb{T}^3} Y_N \Theta_N^2 dx\Big| &= \Big|\int_{\mathbb{T}^3} Y_N \big(\Upsilon_N^2 + 2\sigma \Upsilon_N \widetilde{\mathfrak{Z}}_N + \sigma^2 \widetilde{\mathfrak{Z}}_N^2\big) dx\Big| \\
&\le C_\sigma \big(1 + \|Y_N\|_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}}^c + \|\mathfrak{Z}_N\|_{\mathcal{C}^{1-\varepsilon}}^c\big) \\
&\quad + \frac{1}{100|\sigma|}\big(\|\Upsilon_N\|_{L^2}^3 + \|\Upsilon_N\|_{H^1}^2\big).
\end{aligned} \tag{4.3.3}$$

By Young's inequality, we have

$$\begin{aligned}
\Big|\int_{\mathbb{T}^3} \Theta_N^3 dx - \int_{\mathbb{T}^3} \Upsilon_N^3 dx\Big| &= \Big|\int_{\mathbb{T}^3} \big(3\sigma \Upsilon_N^2 \widetilde{\mathfrak{Z}}_N + 3\sigma^2 \Upsilon_N \widetilde{\mathfrak{Z}}_N^2 + \sigma^3 \widetilde{\mathfrak{Z}}_N^3\big) dx\Big| \\
&\le C_\sigma \|\mathfrak{Z}_N\|_{\mathcal{C}^{1-\varepsilon}}^3 + \frac{1}{100|\sigma|}\|\Upsilon_N\|_{L^2}^3.
\end{aligned} \tag{4.3.4}$$

Then, applying (4.3.3) and (4.3.4) with Lemma 3.1.2 and (3.2.17) to (4.3.2), we obtain

$$\begin{aligned}
&- \log \mathbb{E}\Big[\exp\big(\delta \min\big(\|\rho_\varepsilon * u\|_{\mathcal{A}}^{20}, L\big) - \delta \|u_N\|_{\mathcal{A}}^{20} - R_N^\diamond(u)\big)\Big] \\
&\quad \le \inf_{\dot{\Upsilon}^N \in \mathbb{H}_a^1} \mathbb{E}\Big[-\delta \min\big(\|\rho_\varepsilon * (Y_N + \Upsilon_N + \sigma \widetilde{\mathfrak{Z}}_N)\|_{\mathcal{A}}^{20}, L\big) \\
&\qquad\qquad + \delta \|Y_N + \Upsilon_N + \sigma \widetilde{\mathfrak{Z}}_N\|_{\mathcal{A}}^{20} - \frac{\sigma}{3}\int_{\mathbb{T}^3} \Upsilon_N^3 dx + \|\Upsilon_N\|_{L^2}^3 \\
&\qquad\qquad + A\Big|\int_{\mathbb{T}^3} \big(:Y_N^2: + 2Y_N \Theta_N + \Theta_N^2\big) dx\Big|^\gamma \\
&\qquad\qquad + \frac{3}{4}\int_0^1 \|\dot{\Upsilon}^N(t)\|_{H_x^1}^2 dt\Big] + C_\sigma,
\end{aligned} \tag{4.3.5}$$

where $\Theta_N = \Upsilon_N + \sigma \widetilde{\mathfrak{Z}}_N$.

In the following, we show that the right-hand side of (4.3.5) tends to $-\infty$ as $N, L \to \infty$, provided that $|\sigma| > 0$ is sufficiently large. By following the strategy introduced in our previous works [54, 61], we construct a drift $\dot{\Upsilon}^N$, achieving this goal. The main idea is to construct a drift $\dot{\Upsilon}^N$ such that Υ^N looks like "$-Y(1)+$ a perturbation" (see (4.3.14)), where the perturbation term is bounded in $L^2(\mathbb{T}^3)$ but has a large cubic integral (see (4.3.9) below). While we do not make use of solitons in this paper, one should think of this perturbation as something like a soliton or a finite blowup solution (at a fixed time) with a highly concentrated profile.

Remark 4.3.1. While our construction of the drift follows that in [54], we need to proceed more carefully in our current problem in handling the first two terms under the expectation in (4.3.5). If we simply apply (3.2.32) (with $\gamma = 20$) to separate Υ_N from Y_N and $\sigma\widetilde{\mathfrak{Z}}_N$, we end up with an expression like

$$-\delta \min\bigg(\frac{1}{2}\|\rho_\varepsilon * \Upsilon_N\|_{\mathcal{A}}^{20}, L\bigg) + 2\delta\|\Upsilon_N\|_{\mathcal{A}}^{20}$$

such that the coefficients of $\|\rho_\varepsilon * \Upsilon_N\|_{\mathcal{A}}^{20}$ and $\|\Upsilon_N\|_{\mathcal{A}}^{20}$ no longer agree, which causes a serious trouble. We instead need to keep the same coefficient for the first two terms under the expectation in (4.3.5) and make use of the difference structure. Compare this with the analysis in [54, 61], where no such cancellation was needed.

Fix a parameter $M \gg 1$. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a real-valued Schwartz function such that the Fourier transform $\hat{f}$ is a smooth even non-negative function supported $\{\frac{1}{2} < |\xi| \le 1\}$ such that $\int_{\mathbb{R}^3} |\hat{f}(\xi)|^2 d\xi = 1$. Define a function f_M on $\mathbb{T}^3$ by

$$f_M(x) := M^{-\frac{3}{2}} \sum_{n\in\mathbb{Z}^3} \hat{f}\bigg(\frac{n}{M}\bigg) e_n, \tag{4.3.6}$$

where $\hat{f}$ denotes the Fourier transform on $\mathbb{R}^3$ defined by

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} f(x) e^{-in\cdot x} dx.$$

Then, a direct calculation shows the following lemma.

Lemma 4.3.2. *For any $M \in \mathbb{N}$ and $\alpha > 0$, we have*

$$\int_{\mathbb{T}^3} f_M^2 dx = 1 + O(M^{-\alpha}), \tag{4.3.7}$$

$$\int_{\mathbb{T}^3} (\langle\nabla\rangle^{-1} f_M)^2 dx \lesssim M^{-2}, \tag{4.3.8}$$

$$\int_{\mathbb{T}^3} |f_M|^3 dx \sim \int_{\mathbb{T}^3} f_M^3 dx \sim M^{\frac{3}{2}}. \tag{4.3.9}$$

Proof. As for (4.3.7) and (4.3.8), see the proof of [54, Lemma 5.13]. From (4.3.6) and the fact that $\hat{f}$ is supported on $\{\frac{1}{2} < |\xi| \le 1\}$, we have

$$\int_{\mathbb{T}^3} f_M^3 dx = M^{-\frac{9}{2}} \sum_{n_1,n_2\in\mathbb{Z}^3} \hat{f}\Big(\frac{n_1}{M}\Big)\hat{f}\Big(\frac{n_2}{M}\Big)\hat{f}\Big(-\frac{n_1+n_2}{M}\Big) \sim M^{\frac{3}{2}}. \quad (4.3.10)$$

The bound $\| f_M\|_{L^3}^3 \gtrsim M^{\frac{3}{2}}$ follows from (4.3.10), while $\| f_M\|_{L^3}^3 \lesssim M^{\frac{3}{2}}$ follows from Hausdorff–Young's inequality. This proves (4.3.9). ∎

We define Z_M and α_M by

$$Z_M := \sum_{|n|\le M} \widehat{Y\Big(\frac{1}{2}\Big)}(n)e_n \quad \text{and} \quad \alpha_M := \mathbb{E}\big[Z_M^2(x)\big]. \quad (4.3.11)$$

Note that α_M is independent of $x \in \mathbb{T}^3$ thanks to the spatial translation invariance of Z_M. Then, we have the following lemma. See [54, Lemma 5.14] for the proof.

Lemma 4.3.3. *Let $M \gg 1$ and $1 \le p < \infty$. Then, we have*

$$\alpha_M \sim M, \quad (4.3.12)$$
$$\mathbb{E}\Big[\int_{\mathbb{T}^3} |Z_M|^p dx\Big] \le C(p)M^{\frac{p}{2}},$$
$$\mathbb{E}\Big[\Big(\int_{\mathbb{T}^3} Z_M^2 dx - \alpha_M\Big)^2\Big] + \mathbb{E}\Big[\Big(\int_{\mathbb{T}^3} Y_N Z_M dx - \int_{\mathbb{T}^3} Z_M^2 dx\Big)^2\Big] \lesssim 1,$$
$$\mathbb{E}\Big[\Big(\int_{\mathbb{T}^3} Y_N f_M dx\Big)^2\Big] + \mathbb{E}\Big[\Big(\int_{\mathbb{T}^3} Z_M f_M dx\Big)^2\Big] \lesssim M^{-2}$$

for any $N \ge M$.

We now present the proof of Proposition 4.1.2.

Proof of Proposition 4.1.2. As described above, our main goal is to prove (4.3.1).

Fix $N \in \mathbb{N}$, appearing in (4.3.5). For $M \gg 1$, we set f_M, Z_M, and α_M as in (4.3.6) and (4.3.11). We now choose a drift $\dot{\Upsilon}^N$ for (4.3.5) by setting

$$\dot{\Upsilon}^N(t) = 2\cdot \mathbf{1}_{t>\frac{1}{2}}\langle\nabla\rangle\big(-Z_M + \operatorname{sgn}(\sigma)\sqrt{\alpha_M} f_M\big), \quad (4.3.13)$$

where $\operatorname{sgn}(\sigma)$ is the sign of $\sigma \ne 0$. Then, we have

$$\Upsilon^N := I(\dot{\Upsilon}^N)(1) = \int_0^1 \langle\nabla\rangle^{-1}\dot{\Upsilon}^N(t)dt = -Z_M + \operatorname{sgn}(\sigma)\sqrt{\alpha_M} f_M. \quad (4.3.14)$$

Note that for $N \ge M \ge 1$, we have $\Upsilon_N = \pi_N \Upsilon^N = \Upsilon^N$, since Z_M and f_M are supported on the frequencies $\{|n| \le M\}$.

Let us first make some preliminary computations. We start with the first two terms under the expectation in (4.3.5):

$$\begin{aligned}
&-\delta\min\big(\|\rho_\varepsilon*(Y_N+\Upsilon_N+\sigma\widetilde{\mathfrak{Z}}_N)\|_{\mathcal{A}}^{20},L\big)+\delta\|Y_N+\Upsilon_N+\sigma\widetilde{\mathfrak{Z}}_N\|_{\mathcal{A}}^{20}\\
&\quad=-\delta\min\big(\|\rho_\varepsilon*(Y_N+\Upsilon_N+\sigma\widetilde{\mathfrak{Z}}_N)\|_{\mathcal{A}}^{20}-\|Y_N+\Upsilon_N+\sigma\widetilde{\mathfrak{Z}}_N\|_{\mathcal{A}}^{20},\\
&\qquad\qquad L-\|Y_N+\Upsilon_N+\sigma\widetilde{\mathfrak{Z}}_N\|_{\mathcal{A}}^{20}\big)\\
&\quad=:-\delta\min(\mathrm{I},\mathrm{II}).
\end{aligned}\tag{4.3.15}$$

We first consider II. From Lemma 4.2.1, (2.1.3), and Lemma 4.3.2, we have

$$\|f_M\|_{\mathcal{A}}\lesssim\|f_M\|_{H^{-\frac14}}\lesssim\|f_M\|_{L^2}^{\frac34}\|f_M\|_{H^{-1}}^{\frac14}\lesssim M^{-\frac14}.\tag{4.3.16}$$

From (4.3.14), (4.3.12) in Lemma 4.3.3, and (4.3.16), we have

$$\begin{aligned}
\mathrm{II}&\geq L-2\alpha_M^{10}\|f_M\|_{\mathcal{A}}^{20}-C\big(\|Y_N\|_{\mathcal{A}}^{20}+\|Z_M\|_{\mathcal{A}}^{20}+|\sigma|\|\mathfrak{Z}_N\|_{\mathcal{A}}^{20}\big)\\
&\geq L-C_0M^5-C\big(\|Y_N\|_{\mathcal{A}}^{20}+\|Z_M\|_{\mathcal{A}}^{20}+|\sigma|\|\mathfrak{Z}_N\|_{\mathcal{A}}^{20}\big)\\
&\geq\frac12L-C\big(\|Y_N\|_{\mathcal{A}}^{20}+\|Z_M\|_{\mathcal{A}}^{20}+|\sigma|\|\mathfrak{Z}_N\|_{\mathcal{A}}^{20}\big)
\end{aligned}\tag{4.3.17}$$

for $L\gg M^5$. Note that the second term on the right-hand side is harmless since it is bounded under an expectation. Next, we turn to I in (4.3.15). Let δ_0 denote the Dirac delta on $\mathbb{T}^3$. Then, by applying (4.3.14), Young's inequality, Lemma 4.2.1, (4.3.12), and (4.3.7) in Lemma 4.3.2 and by choosing $\varepsilon=\varepsilon(M)>0$ sufficiently small, we have

$$\begin{aligned}
\mathrm{I}&\geq-\big|\|\rho_\varepsilon*(Y_N+\Upsilon_N+\sigma\widetilde{\mathfrak{Z}}_N)\|_{\mathcal{A}}^{20}-\|Y_N+\Upsilon_N+\sigma\widetilde{\mathfrak{Z}}_N\|_{\mathcal{A}}^{20}\big|\\
&\geq-C\|(\rho_\varepsilon-\delta_0)*(Y_N+\Upsilon_N+\sigma\widetilde{\mathfrak{Z}}_N)\|_{\mathcal{A}}\|Y_N+\Upsilon_N+\sigma\widetilde{\mathfrak{Z}}_N\|_{\mathcal{A}}^{19}\\
&\geq-C\alpha_M^{10}\|(\rho_\varepsilon-\delta_0)*f_M\|_{H^{-\frac14}}^{20}-C\big(\|Y_N\|_{\mathcal{A}}^{20}+\|Z_M\|_{\mathcal{A}}^{20}+|\sigma|\|\mathfrak{Z}_N\|_{\mathcal{A}}^{20}\big)\\
&\geq-C\varepsilon^5M^{10}-C\big(\|Y_N\|_{\mathcal{A}}^{20}+\|Z_M\|_{\mathcal{A}}^{20}+|\sigma|\|\mathfrak{Z}_N\|_{\mathcal{A}}^{20}\big)\\
&=-C_0-C\big(\|Y_N\|_{\mathcal{A}}^{20}+\|Z_M\|_{\mathcal{A}}^{20}+|\sigma|\|\mathfrak{Z}_N\|_{\mathcal{A}}^{20}\big).
\end{aligned}\tag{4.3.18}$$

Therefore, from (4.3.15), (4.3.17), and (4.3.18) together with (4.3.11), Lemma 4.2.2 and (3.2.17), we obtain

$$\mathbb{E}\big[-\delta\min(\mathrm{I},\mathrm{II})\big]\leq C(\delta,\sigma).\tag{4.3.19}$$

Next, we treat the third term under the expectation in (4.3.5). This term gives the main contribution. From (4.3.14) and Young's inequality with Lemma 4.3.2, we have

$$\begin{aligned}
&\sigma\int_{\mathbb{T}^3}\Upsilon_N^3dx-|\sigma|\alpha_M^{\frac32}\int_{\mathbb{T}^3}f_M^3dx\\
&\quad=-\sigma\int_{\mathbb{T}^3}Z_M^3dx+3|\sigma|\int_{\mathbb{T}^3}Z_M^2\sqrt{\alpha_M}f_Mdx-3\sigma\int_{\mathbb{T}^3}Z_M\alpha_Mf_M^2dx\\
&\quad\geq-\eta|\sigma|\alpha_M^{\frac32}\int_{\mathbb{T}^3}f_M^3dx-C_\eta|\sigma|\int_{\mathbb{T}^3}|Z_M|^3dx
\end{aligned}\tag{4.3.20}$$

for any $0 < \eta < 1$. Then, it follows from (4.3.20) with $\eta = \frac{1}{2}$ and Lemmas 4.3.2 and 4.3.3 that

$$\begin{aligned}\mathbb{E}\Big[\sigma \int_{\mathbb{T}^3} \Upsilon_N^3 dx\Big] &\geq (1-\eta)|\sigma|\alpha_M^{\frac{3}{2}} \int_{\mathbb{T}^3} f_M^3 dx - C_\eta |\sigma| \mathbb{E}\Big[\int_{\mathbb{T}^3} |Z_M|^3 dx\Big] \\ &\gtrsim |\sigma| M^3 - |\sigma| M^{\frac{3}{2}} \\ &\gtrsim |\sigma| M^3 \end{aligned} \tag{4.3.21}$$

for $M \gg 1$.

We now treat the fourth and sixth terms under the expectation in (4.3.5). From (4.3.14), we have $\Upsilon_N \in \mathcal{H}_{\leq 1}$. Then, by the Wiener chaos estimate (Lemma 2.3.1) and (4.3.14) with Lemmas 4.3.2 and 4.3.3, we have

$$\mathbb{E}\big[\|\Upsilon_N\|_{L^2}^3\big] \lesssim \mathbb{E}\big[\|\Upsilon_N\|_{L^2}^2\big]^{\frac{3}{2}} \lesssim M^{\frac{3}{2}}. \tag{4.3.22}$$

Recall that both $\widehat{Z}_M$ and $\widehat{f}_M$ are supported on $\{|n| \leq M\}$. Then, from (4.3.13), (4.3.14), and Lemmas 4.3.2 and 4.3.3 as above, we have

$$\mathbb{E}\Big[\int_0^1 \|\dot{\Upsilon}^N(t)\|_{H_x^1}^2 dt\Big] \lesssim M^2 \mathbb{E}\big[\|\Upsilon^N\|_{L^2}^2\big] \lesssim M^3. \tag{4.3.23}$$

We state a lemma which controls the fifth term under the expectation in (4.3.5). We present the proof of this lemma at the end of this section.

Lemma 4.3.4. *Let $\gamma > 0$. Then, we have*

$$\mathbb{E}\Big[\Big|\int_{\mathbb{T}^3} :(Y_N + \Upsilon_N + \sigma \widetilde{\mathfrak{Z}}_N)^2: \, dx\Big|^\gamma\Big] \leq C(\sigma, \gamma) < \infty, \tag{4.3.24}$$

uniformly in $N \geq M \geq 1$.[3]

Therefore, putting (4.3.5), (4.3.19), (4.3.21), (4.3.22), (4.3.23), and Lemma 4.3.4 together, we obtain

$$\begin{aligned} &-\log \mathbb{E}\big[\exp\big(\delta \min\big(\|\rho_\varepsilon * u\|_{\mathcal{A}}^{20}, L\big) - \delta \|u_N\|_{\mathcal{A}}^{20} - R_N^\diamond(u)\big)\big] \\ &\quad \leq -C_1 |\sigma| M^3 + C_2 M^3 + C(\delta, \sigma, \gamma) \end{aligned} \tag{4.3.25}$$

for some $C_1, C_2 > 0$, provided that $L \gg M^5 \gg 1$ and $\varepsilon = \varepsilon(M) > 0$ sufficiently small. By taking the limits in N and L, we conclude from (4.3.25) that

$$\begin{aligned} &\limsup_{L\to\infty} \lim_{N\to\infty} \mathbb{E}_\mu\big[\exp\big(\delta \min\big(\|\rho_\varepsilon * u_N\|_{\mathcal{A}}^{20}, L\big) - \delta \|u_N\|_{\mathcal{A}}^{20} - R_N^\diamond(u)\big)\big] \\ &\quad \geq \exp\big(C_1 |\sigma| M^3 - C_2 M^3 - C_0(\sigma)\big) \to \infty, \end{aligned}$$

[3] Recall from (4.3.14) that the definition of Υ_N depends on M.

as $M \to \infty$, provided that $|\sigma|$ is sufficiently large. This proves (4.3.1) and thus we conclude the proof of Proposition 4.1.2. ∎

We conclude this section by presenting the proof of Lemma 4.3.4.

Proof of Lemma 4.3.4. From (3.2.3) and (3.2.11), we have

$$\begin{aligned}\int_{\mathbb{T}^3} \Upsilon_N \widetilde{\mathfrak{Z}}_N dx &= \int_0^1 \int_{\mathbb{T}^3} \langle\nabla\rangle^{-\frac{3}{4}} \Upsilon_N \cdot \langle\nabla\rangle^{-\frac{5}{4}} \pi_N^2 (:Y_N^2(t):) dx dt \\ &\le \|\Upsilon_N\|_{H^{-\frac{3}{4}}} \int_0^1 \| :Y_N^2(t): \|_{H^{-\frac{5}{4}}} dt. \end{aligned} \tag{4.3.26}$$

As for the first factor, it follows from (4.3.14), (2.1.3), (4.3.12), and Lemma 4.3.2 that

$$\begin{aligned}\|\Upsilon_N\|_{H^{-\frac{3}{4}}} &\lesssim \|Z_M\|_{H^{-\frac{3}{4}}} + \sqrt{\alpha_M}\|f_M\|_{H^{-\frac{3}{4}}} \\ &\lesssim \|Z_M\|_{H^{-\frac{3}{4}}} + \sqrt{\alpha_M}\|f_M\|_{H^{-1}}^{\frac{3}{4}} \|f_M\|_{L^2}^{\frac{1}{4}} \\ &\lesssim \|Z_M\|_{H^{-\frac{3}{4}}} + M^{-\frac{1}{4}}. \end{aligned} \tag{4.3.27}$$

Hence, from (4.3.26), (4.3.27), (4.3.11), and Lemma 3.1.2, we obtain

$$\mathbb{E}\bigg[\bigg|\int_{\mathbb{T}^3} \Upsilon_N \widetilde{\mathfrak{Z}}_N dx\bigg|^2\bigg] \lesssim \mathbb{E}\big[\|\Upsilon_N\|_{H^{-\frac{3}{4}}}^2\big] + \mathbb{E}\big[\| :Y_N^2(t): \|^2_{L^1_t([0,1];H_x^{-\frac{5}{4}})}\big] \lesssim 1. \tag{4.3.28}$$

From (4.3.14), we have

$$\begin{aligned}&\Upsilon_N^2 + 2Y_N\Upsilon^N \\ &\quad = Z_M^2 - 2\,\mathrm{sgn}(\sigma)\sqrt{\alpha_M} Z_M f_M + \alpha_M f_M^2 \\ &\qquad - 2Y_N Z_M + 2\,\mathrm{sgn}(\sigma)\sqrt{\alpha_M} Y_N f_M \\ &\quad = (Z_M^2 - \alpha_M) - 2\,\mathrm{sgn}(\sigma)\sqrt{\alpha_M} Z_M f_M + \alpha_M(-1 + f_M^2) + 2\alpha_M \\ &\qquad - 2(Y_N Z_M - Z_M^2) - 2(Z_M^2 - \alpha_M) - 2\alpha_M + 2\,\mathrm{sgn}(\sigma)\sqrt{\alpha_M} Y_N f_M \\ &\quad = -(Z_M^2 - \alpha_M) - 2\,\mathrm{sgn}(\sigma)\sqrt{\alpha_M} Z_M f_M + \alpha_M(-1 + f_M^2) \\ &\qquad - 2(Y_N Z_M - Z_M^2) + 2\,\mathrm{sgn}(\sigma)\sqrt{\alpha_M} Y_N f_M. \end{aligned} \tag{4.3.29}$$

Note from (3.2.3) and (4.3.14) that $\int_{\mathbb{T}^3} :(Y_N + \Upsilon_N + \sigma\widetilde{\mathfrak{Z}}_N)^2: dx \in \mathcal{H}_{\le 4}$. Then, from the Wiener chaos estimate (Lemma 2.3.1), (4.3.14), (4.3.28), (4.3.29), and

Lemmas 3.1.2 and 4.3.3 with (4.3.7), we have

$$\begin{aligned}
&\mathbb{E}\bigg[\bigg|\int_{\mathbb{T}^3} :(Y_N + \Upsilon_N + \sigma\widetilde{\mathfrak{Z}}_N)^2: \,dx\bigg|^\gamma\bigg] \\
&\le C(\gamma)\bigg\{\mathbb{E}\bigg[\bigg|\int_{\mathbb{T}^3} :(Y_N + \Upsilon_N + \sigma\widetilde{\mathfrak{Z}}_N)^2: \,dx\bigg|^2\bigg]\bigg\}^{\frac{\gamma}{2}} \\
&= C(\gamma)\bigg\{\mathbb{E}\bigg[\bigg|\int_{\mathbb{T}^3} :Y_N^2: \,dx + \int_{\mathbb{T}^3}(\Upsilon_N^2 + 2Y_N\Upsilon_N)dx + \sigma^2\int_{\mathbb{T}^3}\widetilde{\mathfrak{Z}}_N^2 dx \\
&\qquad + 2\sigma\int_{\mathbb{T}^3}\Upsilon_N\widetilde{\mathfrak{Z}}_N dx + 2\sigma\int_{\mathbb{T}^3} Y_N\widetilde{\mathfrak{Z}}_N dx\bigg|^2\bigg]\bigg\}^{\frac{\gamma}{2}} \\
&\le C(\gamma)\bigg\{\mathbb{E}\bigg[\bigg(\int_{\mathbb{T}^3} :Y_N^2: \,dx\bigg)^2\bigg] + \sigma^4\mathbb{E}\bigg[\bigg(\int_{\mathbb{T}^3}\widetilde{\mathfrak{Z}}_N^2 dx\bigg)^2\bigg] \\
&\qquad + \sigma^2\mathbb{E}\bigg[\bigg(\int_{\mathbb{T}^3}\Upsilon_N\widetilde{\mathfrak{Z}}_N dx\bigg)^2\bigg] + \sigma^2\mathbb{E}\bigg[\bigg(\int_{\mathbb{T}^3} Y_N\widetilde{\mathfrak{Z}}_N dx\bigg)^2\bigg] \\
&\qquad + \mathbb{E}\bigg[\bigg(-\int_{\mathbb{T}^3} Y_N Z_M dx + \int_{\mathbb{T}^3} Z_M^2 dx\bigg)^2\bigg] \\
&\qquad + \mathbb{E}\bigg[\bigg(\int_{\mathbb{T}^3} Z_M^2 dx - \alpha_M\bigg)^2\bigg] + \alpha_M^2\bigg(-1 + \int_{\mathbb{T}^3} f_M^2 dx\bigg)^2 \\
&\qquad + \alpha_M\mathbb{E}\bigg[\bigg(\int_{\mathbb{T}^3} Y_N f_M dx\bigg)^2\bigg] + \alpha_M\mathbb{E}\bigg[\bigg(\int_{\mathbb{T}^3} Z_M f_M dx\bigg)^2\bigg]\bigg\}^{\frac{\gamma}{2}} \\
&\le C(\sigma,\gamma),
\end{aligned}$$

which yields the bound (4.3.24). ∎

4.4 Non-convergence of the truncated Φ_3^3-measures

In this section, we present the proof of Proposition 4.1.4 on non-convergence of the truncated Φ_3^3-measures $\{\rho_N\}_{N\in\mathbb{N}}$.

We first define a slightly different tamed version of the truncated Φ_3^3-measure by setting

$$d\nu_\delta^{(N)}(u) = (Z_\delta^{(N)})^{-1}\exp\big(-\delta\|u\|_{\mathcal{A}}^{20} - R_N^\diamond(u)\big)d\mu(u) \tag{4.4.1}$$

for $N \in \mathbb{N}$ and $\delta > 0$, where the $\mathcal{A}$-norm and $R_N^\diamond$ are as in (4.1.3) and (1.2.10), respectively, and

$$Z_\delta^{(N)} = \int \exp\big(-\delta\|u\|_{\mathcal{A}}^{20} - R_N^\diamond(u)\big)d\mu(u).$$

As compared to $\nu_{N,\delta}$ in (4.1.6), there is no frequency cutoff π_N in the taming $-\delta\|u\|_{\mathcal{A}}^{20}$ in (4.4.1). As a corollary to the proof of Proposition 4.1.1, we obtain the following convergence result for $\nu_\delta^{(N)}$.

Lemma 4.4.1. *Let $\delta > 0$, Then, as measures on $\mathcal{C}^{-100}(\mathbb{T}^3)$, the sequence of measures $\{\nu_\delta^{(N)}\}_{N\in\mathbb{N}}$ defined in (4.4.1) converges weakly to the limiting measure ν_δ constructed in Proposition 4.1.1.*

Proof. By the definitions (4.1.6) and (4.4.1) of $\nu_{N,\delta}$ and $\nu_\delta^{(N)}$, it suffices to prove

$$\lim_{N\to\infty}\bigg\{\int F(u)\exp\big(-\delta\|u\|_{\mathcal{A}}^{20} - R_N^\diamond(u)\big)d\mu(u) \\ - \int F(u)\exp\big(-\delta\|u_N\|_{\mathcal{A}}^{20} - R_N^\diamond(u)\big)d\mu(u)\bigg\} = 0$$

for any bounded continuous function $F : \mathcal{C}^{-100}(\mathbb{T}^3) \to \mathbb{R}$. In the following, we prove

$$\lim_{N\to\infty}\int \big|\exp\big(-\delta\|u\|_{\mathcal{A}}^{20} - R_N^\diamond(u)\big) - \exp\big(-\delta\|u_N\|_{\mathcal{A}}^{20} - R_N^\diamond(u)\big)\big|d\mu(u) = 0. \quad (4.4.2)$$

By the uniform boundedness of the frequency projector π_N on $\mathcal{A}$, we have

$$\|u_N\|_{\mathcal{A}} \lesssim \|u\|_{\mathcal{A}}, \quad (4.4.3)$$

uniformly in $N \in \mathbb{N}$. Then, it follows from the mean-value theorem, (4.4.3), and the Schauder estimate (4.1.5) that there exists $c_0 > 0$ such that

$$\begin{aligned}
&\int \big|\exp\big(-\delta\|u\|_{\mathcal{A}}^{20} - R_N^\diamond(u)\big) - \exp\big(-\delta\|u_N\|_{\mathcal{A}}^{20} - R_N^\diamond(u)\big)\big|d\mu(u) \\
&\quad\lesssim \delta\int \exp\big(-\delta\min\big(\|u\|_{\mathcal{A}}^{20}, \|u_N\|_{\mathcal{A}}^{20}\big) - R_N^\diamond(u)\big)\big|\|u\|_{\mathcal{A}}^{20} - \|u_N\|_{\mathcal{A}}^{20}\big|d\mu(u) \\
&\quad\lesssim \delta\int \exp\big(-\delta c_0\|u_N\|_{\mathcal{A}}^{20} - R_N^\diamond(u)\big)\|u - u_N\|_{\mathcal{A}}\|u\|_{\mathcal{A}}^{19}d\mu(u) \\
&\quad\lesssim \delta\int \exp\big(-\delta c_0\|u_N\|_{\mathcal{A}}^{20} - R_N^\diamond(u)\big)N^{-\frac{1}{8}}\|u\|_{W^{-\frac{5}{8},3}}^{20}d\mu(u).
\end{aligned} \quad (4.4.4)$$

In the last step, we used the following bound:

$$\|u - u_N\|_{\mathcal{A}} \lesssim \|\pi_N^\perp u\|_{W^{-\frac{3}{4},3}} \lesssim N^{-\frac{1}{8}}\|u\|_{W^{-\frac{5}{8},3}},$$

which follows from (4.1.3), (4.1.5), and the fact that $\pi_N^\perp u = u - u_N$ has the frequency support $\{|n| \gtrsim N\}$. Therefore, by (4.1.6), Proposition 4.1.1, and (4.2.16), we obtain

$$\begin{aligned}
&\limsup_{N\to\infty} \int \big|\exp\big(-\delta\|u\|_{\mathcal{A}}^{20} - R_N^\diamond(u)\big) - \exp\big(-\delta\|u_N\|_{\mathcal{A}}^{20} - R_N^\diamond(u)\big)\big| d\mu(u) \\
&\lesssim \delta \lim_{N\to\infty} \int \exp\big(-\delta c_0 \|u_N\|_{\mathcal{A}}^{20} - R_N^\diamond(u)\big) N^{-\frac{1}{8}} \|u\|_{W^{-\frac{5}{8},3}}^{20} d\mu(u) \\
&= \delta \lim_{N\to\infty} N^{-\frac{1}{8}} Z_{N,c_0\delta} \int \|u\|_{W^{-\frac{5}{8},3}}^{20} d\nu_{N,c_0\delta} \\
&= 0.
\end{aligned}$$

This proves (4.4.2). ∎

Remark 4.4.2. In the penultimate step of (4.4.4), we used the boundedness of the cube frequency projector $\pi_N = \pi_N^{\text{cube}}$ on $L^3(\mathbb{T}^3)$ and hence this argument does not work for the ball frequency projector π_N^{ball} defined in (1.4.1).

We conclude this chapter by presenting the proof of Proposition 4.1.4.

Proof of Proposition 4.1.4. Suppose by contradiction that, as probability measures on $\mathcal{A}$, $\{\rho_{N_k}\}_{k\in\mathbb{N}}$ has a weak limit ν_0. Then, given any $\delta > 0$, from Lemma 4.4.1 with (4.4.1) and (1.2.11), we have

$$\begin{aligned}
d\nu_\delta &= \lim_{k\to\infty} \frac{\exp(-\delta\|u\|_{\mathcal{A}}^{20} - R_{N_k}^\diamond(u))}{\int \exp(-\delta\|v\|_{\mathcal{A}}^{20} - R_{N_k}^\diamond(v)) d\mu(v)} d\mu(u) \\
&= \lim_{k\to\infty} \frac{\exp(-\delta\|u\|_{\mathcal{A}}^{20})}{\int \exp(-\delta\|v\|_{\mathcal{A}}^{20}) d\rho_{N_k}(v)} d\rho_{N_k}(u) \\
&= \frac{\exp(-\delta\|u\|_{\mathcal{A}}^{20})}{\int \exp(-\delta\|v\|_{\mathcal{A}}^{20}) d\nu_0(v)} d\nu_0(u),
\end{aligned} \tag{4.4.5}$$

where the limits are interpreted as weak limits of measures on $\mathcal{C}^{-100}(\mathbb{T}^3)$. Note that, in the last step, we used the weak convergence in $\mathcal{A}$ of the truncated Φ^3_3-measures ρ_{N_k}, since $\exp(-\delta\|u\|_{\mathcal{A}}^{20})$ is continuous on $\mathcal{A}$, but not on $\mathcal{C}^{-100}(\mathbb{T}^3)$. Therefore, from (4.4.5) and (4.1.9), we obtain

$$d\nu_0(u) = \Big(\int \exp\big(-\delta\|v\|_{\mathcal{A}}^{20}\big) d\nu_0(v)\Big) d\overline{\rho}_\delta(u). \tag{4.4.6}$$

By assumption, ν_0 is a probability measure on $\mathcal{A}$ and thus $\|u\|_{\mathcal{A}} < \infty$, ν_0-almost surely. By the fact that ν_0 is a probability measure, (4.4.6), and Proposition 4.1.2, we obtain

$$\begin{aligned} 1 &= \int 1 d\nu_0 \\ &= \int \exp\big(-\delta \|u\|_{\mathcal{A}}^{20}\big) d\nu_0(u) \int 1 d\,\overline{\rho}_\delta(u) \\ &= \infty, \end{aligned}$$

which yields a contradiction. Therefore, no subsequence of the truncated Φ_3^3-measures ρ_N has a weak limit as probability measures on $\mathcal{A}$. ■