Chapter 5

Local well-posedness

5.1 Overview of the chapter

In this chapter, we present the proof of Theorem 1.3.1 on local well-posedness of the (renormalized) hyperbolic Φ^3_3-model (1.3.1):

$$\partial_t^2 u + \partial_t u + (1-\Delta)u - \sigma :u^2: + M(:u^2:)u = \sqrt{2}\xi, \tag{5.1.1}$$

where M is defined as in (1.3.2). For the local theory, the size of $\sigma \neq 0$ does not play any role and hence we set $\sigma = 1$ in the remaining part of this chapter. As mentioned in Chapter 1, local well-posedness of (5.1.1) follows from a slight modification of the argument in [36, 54]. We, however, point out that the argument in [36] on the quadratic SNLW alone is not sufficient due to the additional term $M(:u^2:)u$, coming from the taming in constructing the Φ^3_3-measure.

5.2 Paracontrolled approach

In this section, we go over a paracontrolled approach to rewrite the equation (5.1.1) into a system of three unknowns. While our presentation closely follows those in [36, 54], we present some details for readers' convenience. Proceeding in the spirit of [18, 36, 47, 54], we transform the quadratic SdNLW (5.1.1) to a system of PDEs. In order to treat the additional term $M(:u^2:)u$ in (5.1.1), which contains an ill-defined product in $:u^2:$, we follow the approach in our previous work [54] on the focusing Hartree Φ^4_3-model, which leads to the system of three equations; see (5.2.27) below. Compare this with [18, 36, 47], where the resulting systems consist of two equations. At the end of this section, we state a local well-posedness result of the resulting system.

The main difficulty in studying the hyperbolic Φ^3_3-model (5.1.1) comes from the roughness of the space-time white noise. This is already manifested at the level of the linear equation. Let Ψ denote the stochastic convolution, satisfying the following linear stochastic damped wave equation:

$$\begin{cases} \partial_t^2 \Psi + \partial_t \Psi + (1-\Delta)\Psi = \sqrt{2}\xi \\ (\Psi, \partial_t \Psi)|_{t=0} = (\phi_0, \phi_1), \end{cases}$$

where $(\phi_0, \phi_1) = (\phi_0^\omega, \phi_1^\omega)$ is a pair of the Gaussian random distributions with $\operatorname{Law}(\phi_0^\omega, \phi_1^\omega) = \vec{\mu} = \mu \otimes \mu_0$ in (1.2.2). Define the linear damped wave propagator $\mathcal{D}(t)$ by

$$\mathcal{D}(t) = e^{-\frac{t}{2}} \frac{\sin\left(t\sqrt{\frac{3}{4} - \Delta}\right)}{\sqrt{\frac{3}{4} - \Delta}}$$

viewed as a Fourier multiplier operator. By setting

$$[\![n]\!] = \sqrt{\frac{3}{4} + |n|^2}, \tag{5.2.1}$$

we have

$$\mathcal{D}(t) f = e^{-\frac{t}{2}} \sum_{n \in \mathbb{Z}^3} \frac{\sin(t [\![n]\!])}{[\![n]\!]} \hat{f}(n) e_n. \tag{5.2.2}$$

Then, the stochastic convolution Ψ can be expressed as

$$\Psi(t) = S(t)(\phi_0, \phi_1) + \sqrt{2} \int_0^t \mathcal{D}(t - t') dW(t'), \tag{5.2.3}$$

where $S(t)$ is defined by

$$S(t)(f, g) = \partial_t \mathcal{D}(t) f + \mathcal{D}(t)(f + g) \tag{5.2.4}$$

and W denotes a cylindrical Wiener process on $L^2(\mathbb{T}^3)$ defined in (3.1.1). It is easy to see that Ψ almost surely lies in $C(\mathbb{R}_+; W^{-\frac{1}{2}-\varepsilon,\infty}(\mathbb{T}^3))$ for any $\varepsilon > 0$; see Lemma 5.4.1 below. In the following, we use $\varepsilon > 0$ to denote a small positive constant, which can be arbitrarily small.

In the following, we adopt Hairer's convention to denote the stochastic terms by trees; the vertex "$\cdot$" corresponds to the space-time white noise ξ, while the edge denotes the Duhamel integral operator $\mathcal{I}$ given by

$$\begin{aligned} \mathcal{I}(F)(t) &= \int_0^t \mathcal{D}(t - t') F(t') dt' \\ &= \int_0^t e^{-\frac{t-t'}{2}} \frac{\sin\big((t - t')\sqrt{\frac{3}{4} - \Delta}\big)}{\sqrt{\frac{3}{4} - \Delta}} F(t') dt'. \end{aligned} \tag{5.2.5}$$

With a slight abuse of notation, we set

$$\uparrow \, := \Psi, \tag{5.2.6}$$

where Ψ is as in (5.2.3), with the understanding that $\uparrow$ in (5.2.6) includes the random linear solution $S(t)(\phi_0, \phi_1)$. As mentioned above, $\uparrow$ has (spatial) regularity[1] $-\frac{1}{2}-$.

Given $N \in \mathbb{N}$, we define the truncated stochastic terms $\uparrow_N$ and $\curlyvee_N$ by

$$\uparrow_N := \pi_N \uparrow \quad \text{and} \quad \curlyvee_N := \mathcal{I}(\vee_N) = \int_0^t \mathcal{D}(t - t') \vee_N(t') dt', \tag{5.2.7}$$

[1]We only discuss spatial regularities of various stochastic objects in this part. Hereafter, we use $a-$ to denote $a - \varepsilon$ for arbitrarily small $\varepsilon > 0$.

where π_N is the frequency projector defined in (1.2.5) and $\vee_N$ is the Wick power defined by

$$\vee_N := \uparrow_N^2 - \sigma_N \tag{5.2.8}$$

with

$$\sigma_N = \mathbb{E}\big[\uparrow_N^2(x,t)\big] = \sum_{n\in\mathbb{Z}^3} \frac{\chi_N^2(n)}{\langle n\rangle^2} \sim N \to \infty, \tag{5.2.9}$$

as $N \to \infty$. Note that σ_N in (5.2.9) is independent[2] of $(x,t) \in \mathbb{T}^3 \times \mathbb{R}_+$ and agrees with σ_N defined in (1.2.8). Note that we have

$$\vee = \lim_{N\to\infty} \vee_N \quad \text{in } C([0,T]; W^{-1-,\infty}(\mathbb{T}^3))$$

almost surely. See Lemma 5.4.1.

Next, we define the second order stochastic term $\curlyvee$:

$$\curlyvee := \mathcal{I}(\vee) = \int_0^t \mathcal{D}(t-t')\vee(t')dt',$$

as a limit of $\curlyvee_N$ defined in (5.2.7). With a naive regularity counting, with one degree of smoothing from the damped wave Duhamel integral operator $\mathcal{I}$ in (5.2.5), one may expect that $\curlyvee$ has regularity $0- = 2(-\frac{1}{2}-) + 1$. However, by exploiting the multilinear dispersive smoothing effect, Gubinelli, Koch, and the first author showed that there is an extra $\frac{1}{2}$-smoothing for $\curlyvee$ and that $\curlyvee$ has regularity $\frac{1}{2}-$. See Lemma 5.4.3 below. See also [14, 52, 65] for analogous multilinear dispersive smoothing for the random wave equations. In particular, see [14, 65], where multilinear smoothing has been studied extensively for higher order stochastic objects in the cubic case.

If we proceed with the second order expansion as in [36]:

$$u = \uparrow + \curlyvee + v,$$

the residual term v satisfies the equation of the form:

$$(\partial_t^2 + \partial_t + 1 - \Delta)v = 2v\uparrow + 2\uparrow\curlyvee + \text{other terms}.$$

Inheriting the worse regularity $-\frac{1}{2}-$ of $\uparrow$, the second term $\uparrow\curlyvee$ has regularity $-\frac{1}{2}-$. Hence, we expect v to have regularity at most $\frac{1}{2}- = (-\frac{1}{2}-) + 1$. In particular, the product $v\uparrow$ is not well defined since $(\frac{1}{2}-) + (-\frac{1}{2}-) < 0$.

In order to overcome this problem, we now introduce a paracontrolled ansatz as in [36, 47]:

$$u = \uparrow + \curlyvee + X + Y, \tag{5.2.10}$$

[2] This comes from the space-time translation invariance of the truncated stochastic convolution $\uparrow_N$.

where X and Y satisfy

$$(\partial_t^2 + \partial_t + 1 - \Delta)X = 2(X + Y + \Upsilon)\textcircled{$<$}\, \upharpoonleft - M(:u^2:)\upharpoonleft, \tag{5.2.11}$$

$$(\partial_t^2 + \partial_t + 1 - \Delta)Y = (X + Y + \Upsilon)^2 + 2(X + Y + \Upsilon)\textcircled{$\geqslant$}\, \upharpoonleft - M(:u^2:)(X + Y + \Upsilon) \tag{5.2.12}$$

with the understanding that

$$:u^2: = (X + Y + \Upsilon)^2 + 2(X + Y)\upharpoonleft + 2\upharpoonleft\Upsilon + \vee. \tag{5.2.13}$$

Here, $\textcircled{$\geqslant$} = \textcircled{$>$} + \textcircled{$=$}$. Note that, in the X-equation (5.2.11), we collected the worst terms from the v-equation, while all the terms in the Y-equation (5.2.12) are expected to behave better (that is, if the resonant product in (5.2.12) can be given a meaning). We point out that the problematic term $M(:u^2:)$ appears in *both* equations, unlike the situation in [36].

There are two resonant products in the system (5.2.11)–(5.2.12), which do not a priori make sense: $\Upsilon\textcircled{$=$}\,\upharpoonleft$ and $X\textcircled{$=$}\,\upharpoonleft$. We can use stochastic analysis and multilinear harmonic analysis to give a meaning to the first resonant product:

$$\Upsilon_{\textcircled{$=$}} := \Upsilon\textcircled{$=$}\,\upharpoonleft$$

as a distribution of regularity $0- = (\frac{1}{2}-) + (-\frac{1}{2}-)$ (without renormalization). See Lemma 5.4.4 below. This in particular says that Y has expected regularity $1-$.

In view of Lemma 2.1.2, the right-hand side of (5.2.11) has regularity $-\frac{1}{2}-$ (if we pretend that $M(:u^2:)$ makes sense), and thus we expect that X has regularity $\frac{1}{2}-$. In particular, the resonant product $X\textcircled{$=$}\,\upharpoonleft$ in the Y-equation is not well defined since the sum of the regularities is negative. In [36], this issue was overcome by substituting the Duhamel formulation of the X-equation into the resonant product $X\textcircled{$=$}\,\upharpoonleft$ and then introducing certain paracontrolled operators (see (5.2.19), (5.2.20), and (5.2.22) below). This was possible in [36] since there was no additional term $M(:u^2:)$ in the system, in particular in the X-equation. In our current problem, the problematic resonant product $X\textcircled{$=$}\,\upharpoonleft$ also appears in $M(:u^2:)$, in particular, in the X-equation. Thus, a strategy in [36,47] of substituting the Duhamel formulation of the X-equation into $X\textcircled{$=$}\,\upharpoonleft$ would lead to an infinite iteration of such substitutions. We point out that such an infinite iteration of the Duhamel formulation works in certain situations but we choose an alternative approach which is simpler.

The main idea is to follow the strategy in our previous work [54] and introduce a new unknown, representing the problematic resonant product:

$$\text{“}\mathfrak{R} = X\textcircled{$=$}\,\upharpoonleft\text{”} \tag{5.2.14}$$

which leads to a system of three unknowns $(X, Y, \mathfrak{R})$.

We now turn our attention to $:u^2:$ in (5.2.13). Let $Q_{X,Y}$ to denote a good part of $:u^2:$ defined by

$$Q_{X,Y} = (X+Y)^2 + 2(X+Y)\Upsilon^{\bullet} + 2X \olessthan \upharpoonleft + 2X \ogreaterthan \upharpoonleft + 2Y \upharpoonleft. \tag{5.2.15}$$

In view of $X \olessthan \upharpoonleft$ and $Y \upharpoonleft$, $Q_{X,Y}$ has (expected) regularity $-\frac{1}{2}-$ From (5.2.10), (5.2.14), and (5.2.15), we can write $:u^2:$ as

$$:u^2: = Q_{X,Y} + 2\mathfrak{R} + \Upsilon^2 + 2\Upsilon\!\!\!\cdot + \vee, \tag{5.2.16}$$

where $\Upsilon\!\!\!\cdot$ denotes the product of Υ and $\upharpoonleft$ given by

$$\Upsilon\!\!\!\cdot = \Upsilon \olessthan \upharpoonleft + \Upsilon\!\!\!\cdot_{\circleddash} + \Upsilon \ogreaterthan \upharpoonleft.$$

By substituting the Duhamel formulation of the X-equation (5.2.11) and (5.2.16) into (5.2.14), we obtain

$$\begin{aligned} \mathfrak{R} &= 2\mathcal{I}\big((X+Y+\Upsilon) \olessthan \upharpoonleft\big) \circleddash \upharpoonleft \\ &\quad - \mathcal{I}\big(M\big(Q_{X,Y} + 2\mathfrak{R} + \Upsilon^2 + 2\Upsilon\!\!\!\cdot + \vee\big)\upharpoonleft\big) \circleddash \upharpoonleft. \end{aligned} \tag{5.2.17}$$

As we see below, both resonant products on the right-hand side are not well defined at this point.

Let us consider the first term on the right-hand side of (5.2.17):

$$\mathcal{I}\big((X+Y+\Upsilon) \olessthan \upharpoonleft\big) \circleddash \upharpoonleft. \tag{5.2.18}$$

Due to the paraproduct structure (with the high frequency part given by $\upharpoonleft$) under the Duhamel integral operator $\mathcal{I}$, we see that the resonant product in (5.2.18) is not well defined at this point since a term $\mathcal{I}(w \olessthan \upharpoonleft)$ has (at best) regularity $\frac{1}{2}-$. In order to give a precise meaning to the right-hand side of (5.2.17), we now recall the paracontrolled operators introduced in [36].[3] We point out that in the parabolic setting, it is at this step where one would introduce commutators and exploit their smoothing properties. For our dispersive problem, however, one of the commutators does not provide any smoothing and thus such an argument does not seem to work. See [36, Remark 1.17].

Given a function w on $\mathbb{T}^3 \times \mathbb{R}_+$, define

$$\begin{aligned} \mathfrak{I}_{\olessthan}(w)(t) &:= \mathcal{I}(w \olessthan \upharpoonleft)(t) \\ &= \sum_{n\in\mathbb{Z}^3} e_n \sum_{\substack{n=n_1+n_2\\ |n_1|\ll|n_2|}} \int_0^t e^{-\frac{t-t'}{2}} \frac{\sin((t-t')[\![n]\!])}{[\![n]\!]} \widehat{w}(n_1,t')\widehat{\upharpoonleft}(n_2,t')dt', \end{aligned} \tag{5.2.19}$$

[3]Strictly speaking, the paracontrolled operators introduced in [36] are for the undamped wave equation. Since the local-in-time mapping property remains unchanged, we ignore this minor point.

where $[\![n]\!]$ is as in (5.2.1). Here, $|n_1| \ll |n_2|$ signifies the paraproduct $\otimes$ in the definition of $\mathfrak{I}_{\otimes}$.[4] As mentioned above, the regularity of $\mathfrak{I}_{\otimes}(w)$ is (at best) $\frac{1}{2}-$ and thus the resonant product $\mathfrak{I}_{\otimes}(w) \ominus \Upsilon$ does not make sense in terms of deterministic analysis. Proceeding as in [36], we divide the paracontrolled operator $\mathfrak{I}_{\otimes}$ into two parts. Fix small $\theta > 0$. Denoting by n_1 and n_2 the spatial frequencies of w and Υ as in (5.2.19), we define $\mathfrak{I}^{(1)}_{\otimes}$ and $\mathfrak{I}^{(2)}_{\otimes}$ as the restrictions of $\mathfrak{I}_{\otimes}$ onto $\{|n_1| \gtrsim |n_2|^\theta\}$ and $\{|n_1| \ll |n_2|^\theta\}$. More concretely, we set

$$\begin{aligned} &\mathfrak{I}^{(1)}_{\otimes}(w)(t) \\ &:= \sum_{n\in\mathbb{Z}^3} e_n \sum_{\substack{n=n_1+n_2 \\ |n_2|^\theta \lesssim |n_1| \ll |n_2|}} \int_0^t e^{-\frac{t-t'}{2}} \frac{\sin((t-t')[\![n]\!])}{[\![n]\!]} \widehat{w}(n_1,t')\widehat{\Upsilon}(n_2,t')dt' \end{aligned} \tag{5.2.20}$$

and

$$\mathfrak{I}^{(2)}_{\otimes}(w) := \mathfrak{I}_{\otimes}(w) - \mathfrak{I}^{(1)}_{\otimes}(w). \tag{5.2.21}$$

As for the first paracontrolled operator $\mathfrak{I}^{(1)}_{\otimes}$, the lower bound $|n_1| \gtrsim |n_2|^\theta$ and the positive regularity of w allow us to prove a smoothing property such that the resonant product $\mathfrak{I}^{(1)}_{\otimes}(w) \ominus \Upsilon$ is well defined. See Lemma 5.4.5 below.

As noted in [36], the second paracontrolled operator $\mathfrak{I}^{(2)}_{\otimes}$ does not seem to possess a (deterministic) smoothing property. One of the main novelties in [36] was then to directly study the random operator $\mathfrak{I}_{\otimes,\ominus}$ defined by

$$\begin{aligned} \mathfrak{I}_{\otimes,\ominus}(w)(t) &:= \mathfrak{I}^{(2)}_{\otimes}(w) \ominus \Upsilon(t) \\ &= \sum_{n\in\mathbb{Z}^3} e_n \int_0^t \sum_{n_1\in\mathbb{Z}^3} \widehat{w}(n_1,t')\mathcal{A}_{n,n_1}(t,t')dt', \end{aligned} \tag{5.2.22}$$

where $\mathcal{A}_{n,n_1}(t,t')$ is given by

$$\begin{aligned} &\mathcal{A}_{n,n_1}(t,t') \\ &= \mathbf{1}_{[0,t]}(t') \sum_{\substack{n-n_1=n_2+n_3 \\ |n_1| \ll |n_2|^\theta \\ |n_1+n_2| \sim |n_3|}} e^{-\frac{t-t'}{2}} \frac{\sin((t-t')[\![n_1+n_2]\!])}{[\![n_1+n_2]\!]} \widehat{\Upsilon}(n_2,t')\widehat{\Upsilon}(n_3,t). \end{aligned} \tag{5.2.23}$$

Here, the condition $|n_1+n_2| \sim |n_3|$ is used to denote the spectral multiplier corresponding to the resonant product $\ominus$ in (5.2.22). See (5.4.6) and (5.4.7) for the precise

[4] For simplicity of the presentation, we use the less precise definitions of paracontrolled operators. For example, see (5.4.4) for the precise definition of the paracontrolled operator $\mathfrak{I}^{(1)}_{\otimes}$.

definitions. The almost sure bounded property of the random operator $\mathfrak{I}_{\circledless,\circleddash}$ was studied in [36, 54]. See Lemma 5.4.6 below.

Next, we consider the second term on the right-hand side of (5.2.17):

$$\mathcal{I}\big(M\big(Q_{X,Y} + 2\mathfrak{R} + Y^2 + 2\Psi + \vee\big)\uparrow\big) \circleddash \uparrow. \tag{5.2.24}$$

Once again, the resonant product is not well defined since the sum of regularities is negative. The term (5.2.24) appeared in our previous work [54] on the focusing Hartree Φ^4_3-model, where we introduced the following stochastic term:

$$\mathbb{A}(x,t,t') = \sum_{n\in\mathbb{Z}^3} e_n(x) \sum_{\substack{n=n_1+n_2\\ |n_1|\sim|n_2|}} e^{-\frac{t-t'}{2}} \frac{\sin((t-t')[\![n_1]\!])}{[\![n_1]\!]} \widehat{\uparrow}(n_1,t')\widehat{\uparrow}(n_2,t) \tag{5.2.25}$$

for $t \ge t' \ge 0$, where $|n_1| \sim |n_2|$ signifies the resonant product. Then, we have

$$\big(\mathcal{I}\big(M(w)\uparrow\big)\circleddash \uparrow\big)(t) = \int_0^t M(w)(t')\mathbb{A}(t,t')dt'. \tag{5.2.26}$$

We point out that the Fourier transform $\widehat{\mathbb{A}}(n,t,t')$ corresponds to $\mathcal{A}_{n,0}(t,t')$ defined in (5.2.23) and thus the analysis for $\mathbb{A}$ is closely related to that for the paracontrolled operator $\mathfrak{I}_{\circledless,\circleddash}$ in (5.2.22). See Lemma 5.4.7 below for the almost sure regularity of $\mathbb{A}$.

Finally, we are ready to present the full system for the three unknowns $(X, Y, \mathfrak{R})$. Putting together (5.2.11), (5.2.12), (5.2.15), (5.2.17), (5.2.20), (5.2.22), and (5.2.26), we arrive at the following system:

$$\begin{aligned}
(\partial_t^2+\partial_t+1-\Delta)X &= 2(X + Y + Y)\circledless \uparrow \\
&\quad - M\big(Q_{X,Y} + 2\mathfrak{R} + Y^2 + 2\Psi + \vee\big)\uparrow,\\
(\partial_t^2+\partial_t+1-\Delta)Y &= (X + Y + Y)^2 + 2(\mathfrak{R} + Y\circleddash \uparrow + \Psi_{\circleddash})\\
&\quad + 2(X + Y + Y)\circledgtr \uparrow \\
&\quad - M\big(Q_{X,Y} + 2\mathfrak{R} + Y^2 + 2\Psi + \vee\big)(X + Y + Y),\\
\mathfrak{R} &= 2\mathfrak{I}^{(1)}_{\circledless}(X + Y + Y)\circleddash \uparrow + 2\mathfrak{I}_{\circledless,\circleddash}(X + Y + Y)\\
&\quad - \int_0^t M\big(Q_{X,Y} + 2\mathfrak{R} + Y^2 + 2\Psi + \vee\big)\mathbb{A}(t,t')dt',\\
(X, \partial_t X, Y, \partial_t Y)|_{t=0} &= (X_0, X_1, Y_0, Y_1).
\end{aligned} \tag{5.2.27}$$

By viewing the following random distributions and operator in the system above:

$$\uparrow, \quad \vee, \quad Y, \quad \Psi_{\circleddash}, \quad \mathbb{A}, \quad \text{and} \quad \mathfrak{I}_{\circledless,\circleddash}, \tag{5.2.28}$$

as predefined deterministic data with certain regularity/mapping properties, we prove the following local well-posedness of the system (5.2.27).

Theorem 5.2.1. *Let $\frac{1}{4} < s_1 < \frac{1}{2} < s_2 \leq s_1 + \frac{1}{4}$ and $s_2 - 1 \leq s_3 < 0$. Then, there exist $\theta = \theta(s_3) > 0$ and $\varepsilon = \varepsilon(s_1, s_2, s_3) > 0$ such that if*

- $\<1>$ *is a distribution-valued function belonging to* $C([0,1]; W^{-\frac{1}{2}-\varepsilon,\infty}(\mathbb{T}^3)) \cap C^1([0,1]; W^{-\frac{3}{2}-\varepsilon,\infty}(\mathbb{T}^3))$,
- $\<2>$ *is a distribution-valued function belonging to* $C([0,1]; W^{-1-\varepsilon,\infty}(\mathbb{T}^3))$,
- $\<21>$ *is a distribution-valued function belonging to* $C([0,1]; W^{\frac{1}{2}-\varepsilon,\infty}(\mathbb{T}^3)) \cap C^1([0,1]; W^{-1-\varepsilon,\infty}(\mathbb{T}^3))$,
- $\<320>$ *is a distribution-valued function belonging to* $C([0,1]; H^{-\varepsilon}(\mathbb{T}^3))$,
- $\mathbb{A}(t,t')$ *is a distribution-valued function belonging to* $L^\infty_{t'} L^3_t(\Delta_2(1); H^{-\varepsilon}(\mathbb{T}^3))$, *where* $\Delta_2(T) \subset [0,T]^2$ *is defined by*

$$\Delta_2(T) = \{(t,t') \in \mathbb{R}^2_+ : 0 \leq t' \leq t \leq T\}, \tag{5.2.29}$$

- *the operator* $\mathfrak{I}_{\circledless, \circleddash}$ *belongs to the class* $\mathcal{L}_2(\frac{3}{2}, 1)$, *where* $\mathcal{L}_2(q,T)$ *is defined by*

$$\mathcal{L}_2(q,T) := \mathcal{L}(L^q([0,T]; L^2(\mathbb{T}^3)); L^\infty([0,T]; H^{s_3}(\mathbb{T}^3))), \tag{5.2.30}$$

then the system (5.2.27) *is locally well-posed in* $\mathcal{H}^{s_1}(\mathbb{T}^3) \times \mathcal{H}^{s_2}(\mathbb{T}^3)$. *More precisely, given any* $(X_0, X_1, Y_0, Y_1) \in \mathcal{H}^{s_1}(\mathbb{T}^3) \times \mathcal{H}^{s_2}(\mathbb{T}^3)$, *there exist* $T > 0$ *and a unique solution* $(X, Y, \mathfrak{R})$ *to the hyperbolic* Φ^3_3*-system* (5.2.27) *on* $[0,T]$ *in the class*:

$$Z^{s_1,s_2,s_3}(T) = X^{s_1}(T) \times Y^{s_2}(T) \times L^3([0,T]; H^{s_3}(\mathbb{T}^3)). \tag{5.2.31}$$

Here, $X^{s_1}(T)$ *and* $Y^{s_2}(T)$ *are the energy spaces at the regularities* s_1 *and* s_2 *intersected with appropriate Strichartz spaces defined in* (5.5.1) *below. Furthermore, the solution* $(X, Y, \mathfrak{R})$ *depends Lipschitz-continuously on the enhanced data set*:

$$\big(X_0, X_1, Y_0, Y_1, \<1>, \<2>, \<21>, \<320>, \mathbb{A}, \mathfrak{I}_{\circledless, \circleddash}\big) \tag{5.2.32}$$

in the class:

$$\begin{aligned}
\mathcal{X}^{s_1,s_2,\varepsilon}_T = \mathcal{H}^{s_1}(\mathbb{T}^3) \times \mathcal{H}^{s_2}(\mathbb{T}^3) \\
&\times \big(C([0,T]; W^{-\frac{1}{2}-\varepsilon,\infty}(\mathbb{T}^3)) \cap C^1([0,T]; W^{-\frac{3}{2}-\varepsilon,\infty}(\mathbb{T}^3))\big) \\
&\times C([0,T]; W^{-1-\varepsilon,\infty}(\mathbb{T}^3)) \\
&\times \big(C([0,T]; W^{\frac{1}{2}-\varepsilon,\infty}(\mathbb{T}^3)) \cap C^1([0,T]; W^{-1-\varepsilon,\infty}(\mathbb{T}^3))\big) \\
&\times C([0,T]; H^{-\varepsilon}(\mathbb{T}^3)) \times L^\infty_{t'} L^3_t(\Delta_2(T); H^{-\varepsilon}(\mathbb{T}^3)) \times \mathcal{L}_2\bigg(\frac{3}{2}, T\bigg).
\end{aligned}$$

Given the a priori regularities of the enhanced data, Theorem 5.2.1 follows from the standard energy and Strichartz estimates for the wave equation. While the proof is a slight modification of those in [36, 54], we present the proof of Theorem 5.2.1 in

Section 5.5 for readers' convenience. The local well-posedness of the hyperbolic Φ^3_3-model (Theorem 1.3.1) follows from Theorem 5.2.1 and the almost sure convergence of the truncated stochastic objects:

$$\uparrow_N, \quad \vee_N, \quad \curlyvee_N, \quad \curlyvee\!\!\!\circ_N, \quad \mathbb{A}_N, \quad \text{and} \quad \mathfrak{I}^N_{\circledless, \circleddash} \tag{5.2.33}$$

to the elements in the enhanced data set in (5.2.28); see Lemmas 5.4.1, 5.4.3, 5.4.4, 5.4.5, 5.4.6, and 5.4.7 in Section 5.4. See Remark 5.2.2 below.

Remark 5.2.2. (i) For the sake of the well-posedness of the system (5.2.27), we considered general initial data $(X_0, X_1, Y_0, Y_1) \in \mathcal{H}^{s_1}(\mathbb{T}^3) \times \mathcal{H}^{s_2}(\mathbb{T}^3)$ in Theorem 5.2.1. However, in order to go back from the system (5.2.27) to the hyperbolic Φ^3_3-model (5.1.1) with the identification (5.2.14) (in the limiting sense) we need to set $(X_0, X_1) = (0, 0)$ since the resonant product of the linear solution $S(t)(X_0, X_1)$ and $\uparrow$ is not well defined in general. As we see in Chapter 6, we simply use the zero initial data for the system (5.2.27) in constructing global-in-time invariant Gibbs dynamics for the hyperbolic Φ^3_3-model (5.1.1).

(ii) Our choice of the norms for $\curlyvee\!\!\!\circ$ is crucial in the globalization argument. See Proposition 6.2.4 and Remark 6.2.5.

(iii) In proving the local well-posedness result of the system (5.2.27) stated in Theorem 5.2.1, we do not need to use the C^1_T-norms for $\uparrow$ and $\curlyvee$. However, we will need these C^1_T-norms for $\uparrow$ and $\curlyvee$ in the globalization argument presented in Chapter 6 and thus have included them in the hypothesis and the definition of Theorem 5.2.1 of the space $\mathcal{X}^{s_1,s_2,\varepsilon}_T$. See also (5.5.3) and Remark 5.5.1.

Furthermore, with this definition of the space $\mathcal{X}^{s_1,s_2,\varepsilon}_T$, the map from an enhanced data set in (5.2.32) (with $(X_0, X_1, Y_0, Y_1) = (0, 0, u_0, u_1)$) to $(u, \partial_t u)$, where $u = \uparrow + \curlyvee + X + Y$ as in (5.2.10) becomes a continuous map from $\mathcal{X}^{s_1,s_2,\varepsilon}_T$ to $C([0, T]; \mathcal{H}^{-\frac{1}{2}-\varepsilon}(\mathbb{T}^3))$.

5.3 Strichartz estimates

Given $0 \le s \le 1$, we say that a pair (q, r) is s-admissible (a pair $(\tilde{q}, \tilde{r})$ is dual s-admissible,[5] respectively) if $1 \le \tilde{q} < 2 < q \le \infty$, $1 < \tilde{r} \le 2 \le r < \infty$,

$$\frac{1}{q} + \frac{3}{r} = \frac{3}{2} - s = \frac{1}{\tilde{q}} + \frac{3}{\tilde{r}} - 2, \quad \frac{1}{q} + \frac{1}{r} \le \frac{1}{2}, \quad \text{and} \quad \frac{1}{\tilde{q}} + \frac{1}{\tilde{r}} \ge \frac{3}{2}.$$

We say that u is a solution to the following nonhomogeneous linear damped wave equation:

$$\begin{cases} (\partial_t^2 + \partial_t + 1 - \Delta)u = F \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \end{cases} \tag{5.3.1}$$

[5]Here, we define the notion of dual s-admissibility for the convenience of the presentation. Note that $(\tilde{q}, \tilde{r})$ is dual s-admissible if and only if $(\tilde{q}', \tilde{r}')$ is $(1 - s)$-admissible.

on a time interval containing $t = 0$, if u satisfies the following Duhamel formulation:

$$u = S(t)(u_0, u_1) + \int_0^t \mathcal{D}(t - t')F(t')dt',$$

where $S(t)$ and $\mathcal{D}(t)$ are as in (5.2.4) and (5.2.2), respectively. We now recall the Strichartz estimates for solutions to the nonhomogeneous linear damped wave equation (5.3.1).

Lemma 5.3.1. *Given* $0 \le s \le 1$, *let* (q, r) *and* $(\tilde{q}, \tilde{r})$ *be* s*-admissible and dual* s*-admissible pairs, respectively. Then, a solution* u *to the nonhomogeneous linear damped wave equation* (5.3.1) *satisfies*

$$\|(u, \partial_t u)\|_{L^\infty_T \mathcal{H}^s_x} + \|u\|_{L^q_T L^r_x} \lesssim \|(u_0, u_1)\|_{\mathcal{H}^s} + \|F\|_{L^{\tilde{q}}_T L^{\tilde{r}}_x} \tag{5.3.2}$$

for all $0 < T \le 1$. *The following estimate also holds:*

$$\|(u, \partial_t u)\|_{L^\infty_T \mathcal{H}^s_x} + \|u\|_{L^q_T L^r_x} \lesssim \|(u_0, u_1)\|_{\mathcal{H}^s} + \|F\|_{L^1_T H^{s-1}_x} \tag{5.3.3}$$

for all $0 < T \le 1$. *The same estimates also hold for for any finite* $T > 1$ *but with the implicit constants depending on* T.

The Strichartz estimates on $\mathbb{R}^d$ are well known; see [30, 41, 46] in the context of the undamped wave equation (with the linear part $\partial_t^2 - \Delta$). For the undamped Klein–Gordon equation (with the linear part $\partial_t^2 + 1 - \Delta$), see [42]. Thanks to the finite speed of propagation, these estimates on $\mathbb{T}^3$ follow from the corresponding estimates on $\mathbb{R}^3$.

As for the current damped case, by setting $v(t) = e^{\frac{t}{2}}u(t)$, the damped wave equation (5.3.1) becomes

$$\begin{cases} (\partial_t^2 + \frac{3}{4} - \Delta)v = e^{\frac{t}{2}} F \\ (v, \partial_t v)|_{t=0} = (u_0, u_1), \end{cases}$$

to which the Strichartz estimates for the Klein–Gordon equation apply. By undoing the transformation, we then obtain the Strichartz estimates for the damped equation (5.3.1) on finite time intervals $[0, T]$, where the implicit constants depend on T.

In proving Theorem 5.2.1, we use the fact that $(8, \frac{8}{3})$ and $(4, 4)$ are $\frac{1}{4}$-admissible and $\frac{1}{2}$-admissible, respectively. We also use a dual $\frac{1}{2}$-admissible pair $(\frac{4}{3}, \frac{4}{3})$.

5.4 Stochastic terms and paracontrolled operators

In this section, we collect regularity properties of stochastic terms and the paracontrolled operators. See [36, 54] for the proofs. Note that the stochastic objects are constructed from the stochastic convolution $\uparrow = \Psi$ in (5.2.3). In particular, in the following, probabilities of various events are measured with respect to the Gaussian initial data and the space-time white noise.[6]

First, we state the regularity properties of $\upharpoonleft$ and $\vee$. See [36, Lemma 3.1] and [54, Lemma 4.1].

Lemma 5.4.1. *Let $T > 0$.*

(i) *For any $\varepsilon > 0$, $\upharpoonleft_N$ in (5.2.7) converges to $\upharpoonleft$ in $C([0,T]; W^{-\frac{1}{2}-\varepsilon,\infty}(\mathbb{T}^3)) \cap C^1([0,T]; W^{-\frac{3}{2}-\varepsilon,\infty}(\mathbb{T}^3))$ almost surely. In particular, we have*

$$\upharpoonleft \in C([0,T]; W^{-\frac{1}{2}-\varepsilon,\infty}(\mathbb{T}^3)) \cap C^1([0,T]; W^{-\frac{3}{2}-\varepsilon,\infty}(\mathbb{T}^3))$$

almost surely. Moreover, we have the following tail estimate:

$$\mathbb{P}\Big(\|\upharpoonleft_N\|_{C_T W_x^{-\frac{1}{2}-\varepsilon,\infty} \cap C_T^1 W_x^{-\frac{3}{2}-\varepsilon,\infty}} > \lambda \Big) \le C(1+T)\exp\big(-c\lambda^2\big) \tag{5.4.1}$$

for any $T > 0$ and $\lambda > 0$, uniformly in $N \in \mathbb{N} \cup \{\infty\}$ with the understanding that $\upharpoonleft_\infty = \upharpoonleft$.

(ii) *For any $\varepsilon > 0$, $\vee_N$ in (5.2.8) converges to $\vee$ in $C([0,T]; W^{-1-\varepsilon,\infty}(\mathbb{T}^3))$ almost surely. In particular, we have*

$$\vee \in C([0,T]; W^{-1-\varepsilon,\infty}(\mathbb{T}^3))$$

almost surely. Moreover, we have the following tail estimate:

$$\mathbb{P}\Big(\|\vee_N\|_{C_T W_x^{-1-\varepsilon,\infty}} > \lambda \Big) \le C(1+T)\exp\big(-c\lambda\big)$$

for any $T > 0$ and $\lambda > 0$, uniformly in $N \in \mathbb{N} \cup \{\infty\}$ with the understanding that $\vee_\infty = \vee$.

Remark 5.4.2. A slight modification of the proof of the exponential tail estimate (5.4.1) shows that there exists small $\delta > 0$ such that

$$\mathbb{P}\Big(N_2^{\delta} \|\upharpoonleft_{N_1} - \upharpoonleft_{N_2}\|_{C_T W_x^{-\frac{1}{2}-\varepsilon,\infty} \cap C_T^1 W_x^{-\frac{3}{2}-\varepsilon,\infty}} > \lambda \Big) \le C(1+T)\exp\big(-c\lambda^2\big)$$

for any $T > 0$ and $\lambda > 0$, uniformly in $N_1 \ge N_2 \ge 1$. A similar comment applies to the other elements $\vee_N$, Y_N, $Y^{\circledcirc}_N$, $\mathbb{A}_N$, and $\mathfrak{I}^N_{\circledless, \circleddash}$ in the truncated enhanced data set in (5.2.33).

The next two lemmas treat Y and the resonant product $Y^{\circledcirc}$, exhibiting an extra $\frac{1}{2}$-smoothing. See [36, Propositions 1.6 and 1.8]. While the exponential tail estimates (5.4.2) and (5.4.3) were not proven in [36], they follow from the second moment

[6] With the notation in Chapter 6 (see (6.1.4)), this is equivalent to saying that we measure various events with respect to $\vec{\mu} \otimes \mathbb{P}_2$.

bounds on the Fourier coefficients of Υ_N and $\Upsilon^{\bullet}_{\ominus N}$ obtained in [36] and arguing as in the proof of [37, Lemma 2.3], using a version of the Garsia–Rodemich–Rumsey inequality (see [37, Lemma 2.2]) with the fact that $\Upsilon_N \in \mathcal{H}_2$ and $\Upsilon^{\bullet}_{\ominus N} \in \mathcal{H}_{\leq 3}$. Since the required argument is verbatim from [37], we omit details.

Lemma 5.4.3. *Let $T > 0$. Then, Υ_N converges to Υ in $C([0,T]; W^{\frac{1}{2}-\varepsilon,\infty}(\mathbb{T}^3)) \cap C^1([0,T]; W^{-1-\varepsilon,\infty}(\mathbb{T}^3))$ almost surely for any $\varepsilon > 0$. In particular, we have*

$$\Upsilon \in C([0,T]; W^{\frac{1}{2}-\varepsilon,\infty}(\mathbb{T}^3)) \cap C^1([0,T]; W^{-1-\varepsilon,\infty}(\mathbb{T}^3))$$

almost surely for any $\varepsilon > 0$. Moreover, we have the following tail estimate:

$$\mathbb{P}\Big(\|\Upsilon_N\|_{C_T W_x^{\frac{1}{2}-\varepsilon,\infty} \cap C_T^1 W_x^{-1-\varepsilon,\infty}} > \lambda\Big) \leq C(1+T)\exp\big(-c\lambda\big) \tag{5.4.2}$$

for any $T > 0$ and $\lambda > 0$, uniformly in $N \in \mathbb{N} \cup \{\infty\}$ with the understanding that $\Upsilon_\infty = \Upsilon$.

Lemma 5.4.4. *Let $T > 0$. Then, $\Upsilon^{\bullet}_{\ominus N} := \Upsilon_N \circleddash \upharpoonleft_N$ converges to $\Upsilon^{\bullet}_{\ominus}$ in $C([0,T]; W^{-\varepsilon,\infty}(\mathbb{T}^3))$ almost surely for any $\varepsilon > 0$. In particular, we have*

$$\Upsilon^{\bullet}_{\ominus} \in C([0,T]; W^{-\varepsilon,\infty}(\mathbb{T}^3))$$

almost surely for any $\varepsilon > 0$. Moreover, we have the following tail estimate:

$$\mathbb{P}\Big(\|\Upsilon^{\bullet}_{\ominus N}\|_{C_T W_x^{-\varepsilon,\infty}} > \lambda\Big) \leq C(1+T)\exp\big(-c\lambda^{\frac{2}{3}}\big) \tag{5.4.3}$$

for any $T > 0$ and $\lambda > 0$, uniformly in $N \in \mathbb{N} \cup \{\infty\}$ with the understanding that $\Upsilon^{\bullet}_{\ominus \infty} = \Upsilon^{\bullet}_{\ominus}$.

Next, we state the almost sure mapping properties of the paracontrolled operators. We first consider the paracontrolled operator $\mathfrak{I}^{(1)}_{\otimes}$ defined in (5.2.20). By writing out the frequency relation $|n_2|^\theta \lesssim |n_1| \ll |n_2|$ in a more precise manner, we have

$$\begin{aligned} \mathfrak{I}^{(1)}_{\otimes}(w)(t) = \sum_{n \in \mathbb{Z}^3} e_n \sum_{n = n_1 + n_2} \sum_{\theta k + c_0 \leq j < k-2} \varphi_j(n_1)\varphi_k(n_2) \\ \times \int_0^t e^{-\frac{t-t'}{2}} \frac{\sin((t-t')[\![n]\!])}{[\![n]\!]} \widehat{w}(n_1,t')\widehat{\upharpoonleft}(n_2,t')dt', \end{aligned} \tag{5.4.4}$$

where φ_j is as in (2.1.1) and $c_0 \in \mathbb{R}$ is some fixed constant. Given a pathwise regularity of $\upharpoonleft$, the mapping property of $\mathfrak{I}^{(1)}_{\otimes}$ can be established in a deterministic manner. See [54, Lemma 7.1]. See also [36, Corollary 5.2].

Lemma 5.4.5. *Let $s > 0$ and $T > 0$. Then, given small $\theta > 0$, there exists small $\varepsilon = \varepsilon(s,\theta) > 0$ such that the following deterministic estimate holds the paracontrolled* *operator* $\mathfrak{I}^{(1)}_{\olessthan}$ *defined in* (5.2.20):

$$\|\mathfrak{I}^{(1)}_{\olessthan}(w)\|_{L^\infty_T H^{\frac{1}{2}+3\varepsilon}_x} \lesssim \|w\|_{L^2_T H^s_x} \|\upharpoonleft\|_{L^2_T W^{-\frac{1}{2}-\varepsilon,\infty}_x}. \tag{5.4.5}$$

In particular, $\mathfrak{I}^{(1)}_{\olessthan}$ *belongs almost surely to the class*

$$\mathcal{L}_1(T) = \mathcal{L}(L^2([0,T]; H^s(\mathbb{T}^3)); C([0,T]; H^{\frac{1}{2}+3\varepsilon}(\mathbb{T}^3))).$$

Moreover, by letting $\mathfrak{I}^{(1),N}_{\olessthan}$, $N \in \mathbb{N}$, *denote the paracontrolled operator in* (5.2.20) *with* $\upharpoonleft$ *replaced by the truncated stochastic convolution* $\upharpoonleft_N$ *in* (5.2.7)*, the truncated paracontrolled operator* $\mathfrak{I}^{(1),N}_{\olessthan}$ *converges almost surely to* $\mathfrak{I}^{(1)}_{\olessthan}$ *in* $\mathcal{L}_1(T)$.

Next, we consider the random operator $\mathfrak{I}_{\olessthan,\circledequal}$ defined in (5.2.22). By writing out the frequency relations more carefully as in (5.4.4), we have

$$\mathfrak{I}_{\olessthan,\circledequal}(w)(t) = \sum_{n\in\mathbb{Z}^3} e_n \int_0^t \sum_{j=0}^\infty \sum_{n_1\in\mathbb{Z}^3} \varphi_j(n_1)\hat{w}(n_1,t')\mathcal{A}_{n,n_1}(t,t')dt', \tag{5.4.6}$$

where $\mathcal{A}_{n,n_1}(t,t')$ is given by

$$\begin{aligned}\mathcal{A}_{n,n_1}(t,t') = \mathbf{1}_{[0,t]}(t') \sum_{\substack{k=0\\0\le j<\theta k+c_0}}^\infty \sum_{\substack{\ell,m=0\\|\ell-m|\le 2}}^\infty \sum_{n-n_1=n_2+n_3} \varphi_k(n_2)\varphi_\ell(n_1+n_2)\varphi_m(n_3)\\ \times e^{-\frac{t-t'}{2}} \frac{\sin((t-t')[\![n_1+n_2]\!])}{[\![n_1+n_2]\!]} \widehat{\upharpoonleft}(n_2,t')\widehat{\upharpoonleft}(n_3,t).\end{aligned} \tag{5.4.7}$$

Then, we have the following almost sure mapping property of the random operator $\mathfrak{I}_{\olessthan,\circledequal}$. See [54, Proposition 2.5]. See also [36, Proposition 1.11].

Lemma 5.4.6. *Let* $s_3 < 0$ *and* $T > 0$. *Then, there exists small* $\theta = \theta(s_3) > 0$ *such that, for any finite* $q > 1$, *the paracontrolled operator* $\mathfrak{I}_{\olessthan,\circledequal}$ *defined by* (5.2.22) *and* (5.2.23) *belongs to* $\mathcal{L}_2(q,T)$ *defined in* (5.2.30)*, almost surely. Furthermore, the following tail estimate holds for some* $C, c > 0$*:*

$$\mathbb{P}\big(\|\mathfrak{I}_{\olessthan,\circledequal}\|_{\mathcal{L}_2(q,T)} > \lambda\big) \le C(1+T)\exp(-\lambda) \tag{5.4.8}$$

for any $\lambda \gg 1$.

If we define the truncated paracontrolled operator $\mathfrak{I}^N_{\olessthan,\circledequal}$, $N \in \mathbb{N}$, *by replacing* $\upharpoonleft$ *in* (5.2.22) *and* (5.2.23) *with the truncated stochastic convolution* $\upharpoonleft_N$ *in* (5.2.7)*, then the truncated paracontrolled operators* $\mathfrak{I}^N_{\olessthan,\circledequal}$ *converge almost surely to* $\mathfrak{I}_{\olessthan,\circledequal}$ *in* $\mathcal{L}_2(q,T)$. *Furthermore, the tail estimate* (5.4.8) *holds for the truncated paracontrolled operators* $\mathfrak{I}^N_{\olessthan,\circledequal}$, *uniformly in* $N \in \mathbb{N}$.

Finally, we state the regularity property of $\mathbb{A}$ defined in (5.2.25). See [54, Lemma 7.2]. Given $N \in \mathbb{N}$, we define the truncated version $\mathbb{A}_N$:

$$\mathbb{A}_N(x,t,t') = \sum_{n\in\mathbb{Z}^3} e_n(x) \sum_{\substack{n=n_1+n_2\\|n_1|\sim|n_2|}} e^{-\frac{t-t'}{2}} \frac{\sin((t-t')[\![n_1]\!])}{[\![n_1]\!]} \widehat{\upharpoonleft}_N(n_1,t')\widehat{\upharpoonleft}_N(n_2,t) \tag{5.4.9}$$

by replacing $\upharpoonleft$ by $\upharpoonleft_N$ in (5.2.25).

Lemma 5.4.7. *Fix finite $q \ge 2$. Then, given any $T, \varepsilon > 0$ and finite $p \ge 1$, $\{\mathbb{A}_N\}_{N\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\Omega; L^\infty_{t'} L^q_t(\Delta_2(T); H^{-\varepsilon}(\mathbb{T}^3)))$, converging to some limit $\mathbb{A}$ (formally defined by (5.2.25)) in $L^p(\Omega; L^\infty_{t'} L^q_t(\Delta_2(T); H^{-\varepsilon}(\mathbb{T}^3)))$, where $\Delta_2(T)$ is as in equation (5.2.29). Moreover, $\mathbb{A}_N$ converges almost surely to the same limit in $L^\infty_{t'} L^q_t(\Delta_2(T); H^{-\varepsilon}(\mathbb{T}^3))$. Furthermore, we have the following uniform tail estimate:*

$$\mathbb{P}\big(\|\mathbb{A}_N\|_{L^\infty_{t'} L^q_t(\Delta_2(T); H^{-\varepsilon}_x)} > \lambda\big) \le C(1+T)\exp(-\lambda)$$

for any $\lambda \gg 1$, and $N \in \mathbb{N} \cup \{\infty\}$, where $\mathbb{A}_\infty = \mathbb{A}$.

5.5 Proof of local well-posedness

In this section, we present the proof of Theorem 5.2.1. In the following, we assume that $s_3 < 0 < s_1 < s_2 < 1$. Recall that $(8, \frac{8}{3})$ and $(4,4)$ are $\frac{1}{4}$-admissible and $\frac{1}{2}$-admissible, respectively. Given $0 < T \le 1$, we define $X^{s_1}(T)$ (and $Y^{s_2}(T)$) as the intersection of the energy spaces of regularity s_1 (and s_2, respectively) and the Strichartz space:

$$\begin{aligned} X^{s_1}(T) &= C([0,T]; H^{s_1}(\mathbb{T}^3)) \cap C^1([0,T]; H^{s_1-1}(\mathbb{T}^3)) \\ &\qquad \cap L^8([0,T]; W^{s_1-\frac{1}{4},\frac{8}{3}}(\mathbb{T}^3)), \\ Y^{s_2}(T) &= C([0,T]; H^{s_2}(\mathbb{T}^3)) \cap C^1([0,T]; H^{s_2-1}(\mathbb{T}^3)) \\ &\qquad \cap L^4([0,T]; W^{s_2-\frac{1}{2},4}(\mathbb{T}^3)), \end{aligned} \tag{5.5.1}$$

and set

$$Z^{s_1,s_2,s_3}(T) = X^{s_1}(T) \times Y^{s_2}(T) \times L^3([0,T]; H^{s_3}(\mathbb{T}^3)).$$

By writing (5.2.27) in the Duhamel formulation, we have

$$\begin{aligned} X &= \Phi_1(X, Y, \mathfrak{R}) \\ &:= S(t)(X_0, X_1) + 2\mathcal{I}\big((X + Y + \Upsilon) \circledless \upharpoonleft\big) \\ &\qquad - \mathcal{I}\big(M\big(Q_{X,Y} + 2\mathfrak{R} + \Upsilon^2 + 2\Upsilon\!\!\cdot + \vee\big)\upharpoonleft\big), \end{aligned}$$

$$
\begin{aligned}
Y &= \Phi_2(X, Y, \mathfrak{R}) \\
&:= S(t)(Y_0, Y_1) + \mathcal{I}\big((X + Y + \curlyvee)^2\big) \\
&\quad + 2\mathcal{I}\big(\mathfrak{R} + Y \circledequal \upharpoonleft + \curlyvee_{\circledequal}\big) + 2\mathcal{I}\big((X + Y + \curlyvee) \circledgtr \upharpoonleft\big) \\
&\quad - \mathcal{I}\big(M\big(\mathcal{Q}_{X,Y} + 2\mathfrak{R} + \curlyvee^2 + 2\curlyvee_{\bullet} + \vee\big)(X + Y + \curlyvee)\big), \\
\mathfrak{R} &= \Phi_3(X, Y, \mathfrak{R}) \\
&:= 2\mathfrak{I}^{(1)}_{\circledless}(X + Y + \curlyvee) \circledequal \upharpoonleft + 2\mathfrak{I}_{\circledless, \circledequal}(X + Y + \curlyvee) \\
&\quad - \int_0^t M\big(\mathcal{Q}_{X,Y} + 2\mathfrak{R} + \curlyvee^2 + 2\curlyvee_{\bullet} + \vee\big)\mathbb{A}(t, t')dt'.
\end{aligned}
\tag{5.5.2}
$$

In the following, we use $\varepsilon = \varepsilon(s_1, s_2, s_3) > 0$ to denote a small positive number. Given an enhanced data set as in (5.2.32), we set

$$
\Xi = \big(\upharpoonleft, \vee, \curlyvee, \curlyvee_{\circledequal}, \mathbb{A}, \mathfrak{I}_{\circledless, \circledequal}\big)
$$

and

$$
\begin{aligned}
\|\Xi\|_{\mathcal{X}^\varepsilon_T} &= \|\upharpoonleft\|_{C_T W_x^{-\frac{1}{2}-\varepsilon,\infty} \cap C_T^1 W_x^{-\frac{3}{2}-\varepsilon,\infty}} + \|\vee\|_{C_T W_x^{-1-\varepsilon,\infty}} \\
&\quad + \|\curlyvee\|_{C_T W_x^{\frac{1}{2}-\varepsilon,\infty} \cap C_T^1 W_x^{-1-\varepsilon,\infty}} + \|\curlyvee_{\circledequal}\|_{C_T H_x^{-\varepsilon}} \\
&\quad + \|\mathbb{A}\|_{L^\infty_{t'} L^3_t(\Delta_2; H_x^{-\varepsilon})} + \|\mathfrak{I}_{\circledless, \circledequal}\|_{\mathcal{L}_2(\frac{3}{2}, T)}
\end{aligned}
\tag{5.5.3}
$$

for some small $\varepsilon = \varepsilon(s_1, s_2, s_3) > 0$. Moreover, we assume that

$$
\|(X_0, X_1)\|_{\mathcal{H}^{s_1}} + \|(Y_0, Y_1)\|_{\mathcal{H}^{s_2}} + \|\Xi\|_{\mathcal{X}^\varepsilon_1} \le K \tag{5.5.4}
$$

for some $K \ge 1$. Here, we assume the bound on Ξ for the time interval $[0, 1]$.

Remark 5.5.1. As for proving local well-posedness stated in Theorem 5.2.1, we do not need to use the $C_T^1 W_x^{-\frac{3}{2}-\varepsilon,\infty}$-norm for $\upharpoonleft$ and the $C_T^1 W_x^{-1-\varepsilon,\infty}$-norm for $\curlyvee$. However, in constructing global-in-time dynamics, we need to make use of these norms and thus we have included them in the definition of the $\mathcal{X}^\varepsilon_T$-norm in (5.5.3).

We first establish preliminary estimates. By Sobolev's inequality, we have

$$
\|f^2\|_{H^{-a}} \lesssim \|f^2\|_{L^{\frac{6}{3+2a}}} = \|f\|^2_{L^{\frac{12}{3+2a}}} \lesssim \|f\|^2_{H^{\frac{3-2a}{4}}} \tag{5.5.5}
$$

for any $0 \le a < \frac{3}{2}$. By (5.2.15), (5.5.5), Lemma 2.1.2, Lemma 2.1.3 (ii), and Hölder's inequality with (5.5.4), we have

$$
\begin{aligned}
&\|\mathcal{Q}_{X,Y}\|_{L^\infty_T H_x^{-100}} \\
&\quad \lesssim \|(X + Y)^2\|_{L^\infty_T H_x^{-100}} + \|X\curlyvee\|_{L^\infty_T H_x^{-100}} + \|Y\curlyvee\|_{L^\infty_T H_x^{-100}} \\
&\qquad + \|X \circledless \upharpoonleft\|_{L^\infty_T H_x^{-100}} + \|X \circledgtr \upharpoonleft\|_{L^\infty_T H_x^{-100}} + \|Y \upharpoonleft\|_{L^\infty_T H_x^{-100}}
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \|X\|_{L^\infty_T H^\varepsilon_x}^2 + \|Y\|_{L^\infty_T H^\varepsilon_x}^2 \\
&\quad + \big(\|X\|_{L^\infty_T L^2_x} + \|Y\|_{L^\infty_T L^2_x}\big)\|\curlyvee\|_{L^\infty_T L^\infty_x} \\
&\quad + \|X\|_{L^\infty_T L^2_x}\|\uparrow\|_{L^\infty_T W_x^{-\frac12-\varepsilon,\infty}} + \|Y\|_{L^\infty_T H_x^{\frac12+\varepsilon}}\|\uparrow\|_{L^\infty_T W_x^{-\frac12-\varepsilon,\infty}} \\
&\lesssim \|(X,Y,\mathfrak{R})\|^2_{Z^{s_1,s_2,s_3}(T)} + K^2,
\end{aligned}
\tag{5.5.6}
$$

provided that $s_1 \geq \varepsilon$ and $s_2 \geq \frac12 + \varepsilon$.

We now estimate $\Phi_1(X,Y,\mathfrak{R})$ in (5.5.2). By (5.5.1), Lemmas 5.3.1 and 2.1.2, (1.3.2), and (5.5.6) with (5.5.4), we have

$$
\begin{aligned}
&\|\Phi_1(X,Y,\mathfrak{R})\|_{X^{s_1}(T)} \\
&\quad \lesssim \|(X_0,X_1)\|_{\mathcal{H}^{s_1}} + \big\|(X+Y+\curlyvee)\olessthan \uparrow\big\|_{L^1_T H^{s_1-1}_x} \\
&\qquad + \big\|M\big(Q_{X,Y}+2\mathfrak{R}+\curlyvee^2+2\curlyvee\!\!\cdot+\vee\big)\uparrow\big\|_{L^1_T H^{s_1-1}_x} \\
&\quad \lesssim \|(X_0,X_1)\|_{\mathcal{H}^{s_1}} + T\|X+Y+\curlyvee\|_{L^\infty_T L^2_x}\|\uparrow\|_{L^\infty_T W_x^{-\frac12-\varepsilon,\infty}} \\
&\qquad + T^{\frac13}\|Q_{X,Y}+2\mathfrak{R}+\curlyvee^2+2\curlyvee\!\!\cdot+\vee\|^2_{L^{\frac32}_T H^{-100}_x}\|\uparrow\|_{L^\infty_T H^{s_1-1}_x} \\
&\quad \lesssim \|(X_0,X_1)\|_{\mathcal{H}^{s_1}} + T^{\frac13}K\big(\|(X,Y,\mathfrak{R})\|^4_{Z^{s_1,s_2,s_3}(T)} + K^4\big),
\end{aligned}
\tag{5.5.7}
$$

provided that $\varepsilon \leq s_1 < \frac12 - \varepsilon$, $s_2 \geq \frac12 + \varepsilon$, and $s_3 \geq -100$.

Next, we estimate $\Phi_2(X,Y,\mathfrak{R})$ in (5.5.2). By (5.5.1) and Lemma 5.3.1 with the fractional Leibniz rule (Lemma 2.1.3 (i)), we have

$$
\begin{aligned}
&\big\|\mathcal{I}\big((X+Y+\curlyvee)^2\big)\big\|_{Y^{s_2}(T)} \\
&\quad \lesssim \big\|\langle\nabla\rangle^{s_2-\frac12}(X+Y+\curlyvee)^2\big\|_{L^{\frac43}_{T,x}} \\
&\quad \lesssim T^{\frac14}\Big(\big\|\langle\nabla\rangle^{s_2-\frac12}X\big\|^2_{L^8_T L^{\frac83}_x} + \big\|\langle\nabla\rangle^{s_2-\frac12}Y\big\|^2_{L^4_{T,x}} + \big\|\langle\nabla\rangle^{s_2-\frac12}\curlyvee\big\|^2_{L^\infty_{T,x}}\Big) \\
&\quad \lesssim T^{\frac14}\big(\|(X,Y,\mathfrak{R})\|^2_{Z^{s_1,s_2,s_3}(T)} + K^2\big),
\end{aligned}
\tag{5.5.8}
$$

provided that $\frac12 \leq s_2 \leq \min(1-\varepsilon, s_1+\frac14)$. By Lemmas 5.3.1 and 2.1.2, (5.5.8), and (5.5.6) with (5.5.4), we have

$$
\begin{aligned}
&\|\Phi_2(X,Y,\mathfrak{R})\|_{Y^{s_2}(T)} \\
&\quad \lesssim \|(Y_0,Y_1)\|_{\mathcal{H}^{s_2}} + \big\|\mathcal{I}\big((X+Y+\curlyvee)^2\big)\big\|_{Y^{s_2}(T)} + \|\mathfrak{R}\|_{L^1_T H^{s_2-1}_x} \\
&\qquad + \|Y \circeq \uparrow\|_{L^1_T H^{s_2-1}_x} + \|\curlyvee\!\!\circ\|_{L^1_T H^{s_2-1}_x} + \|(X+Y+\curlyvee)\ogreaterthan\uparrow\|_{L^1_T H^{s_2-1}_x} \\
&\qquad + \big\|M\big(Q_{X,Y}+2\mathfrak{R}+\curlyvee^2+2\curlyvee\!\!\cdot+\vee\big)(X+Y+\curlyvee)\big\|_{L^1_T H^{s_2-1}_x}
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \|(Y_0,Y_1)\|_{\mathcal{H}^{s_2}} + T^{\frac14}\big(\|(X,Y,\mathfrak{R})\|^2_{Z^{s_1,s_2,s_3}(T)} + K^2\big) + T^{\frac23}\|\mathfrak{R}\|_{L^{3}_T H^{s_3}_x}\\
&\quad + T\|\Upsilon_{\circleddash}\|_{L^\infty_T H^{-\varepsilon}_x} + T\big(\|X\|_{L^\infty_T H^{s_1}_x} + \|Y\|_{L^\infty_T H^{s_2}_x} + \|\Upsilon\|_{L^\infty_T W^{\frac12-\varepsilon}_x}\big)\|\,\!\uparrow\!\|_{L^\infty_T W^{-\frac12-\varepsilon}_x}\\
&\quad + T^{\frac13}\|Q_{X,Y} + 2\mathfrak{R} + \Upsilon^2 + 2\Upsilon\!\!\cdot + \mathrm{V}\|^2_{L^3_T H^{-100}_x}\\
&\quad \times \big(\|X\|_{L^\infty_T H^{s_1}_x} + \|Y\|_{L^\infty_T H^{s_2}_x} + \|\Upsilon\|_{L^\infty_T W^{\frac12-\varepsilon,\infty}_x}\big)\\
&\lesssim \|(Y_0,Y_1)\|_{\mathcal{H}^{s_2}} + T^{\frac14}\big(\|(X,Y,\mathfrak{R})\|^5_{Z^{s_1,s_2,s_3}(T)} + K^5\big),
\end{aligned}
\tag{5.5.9}
$$

provided that $s_1 \geq \varepsilon$, $\frac12 + \varepsilon < s_2 \leq \min(1-3\varepsilon, s_1 + \frac14, s_3 + 1)$, and $s_3 \geq -100$.

Finally, we estimate $\Phi_3(X,Y,\mathfrak{R})$ in (5.5.2). By Lemmas 2.1.2, 5.4.5 (in particular (5.4.5)), and (5.5.6) with (5.5.4), we have

$$
\begin{aligned}
&\|\Phi_3(X,Y,\mathfrak{R})\|_{L^3_T H^{s_3}_x}\\
&\quad\lesssim \big\|\mathfrak{I}^{(1)}_{\circledless}(X+Y+\Upsilon)\circleddash \uparrow\big\|_{L^3_T H^{s_3}_x} + \big\|\mathfrak{I}_{\circledless,\circleddash}\big(X+Y+\Upsilon\big)\big\|_{L^3_T H^{s_3}_x}\\
&\qquad + \bigg\|\int_0^t M\big(Q_{X,Y} + 2\mathfrak{R} + \Upsilon^2 + 2\Upsilon\!\!\cdot + \mathrm{V}\big)\mathbb{A}(t,t')dt'\bigg\|_{L^3_T H^{s_3}_x}\\
&\quad\lesssim T^{\frac13}\big\|\mathfrak{I}^{(1)}_{\circledless}\big(X+Y+\Upsilon\big)\big\|_{L^\infty_T H^{\frac12+3\varepsilon}_x}\|\uparrow\|_{L^\infty_T W^{-\frac12-\varepsilon,\infty}_x} + T^{\frac13}K\|X+Y+\Upsilon\|_{L^\infty_T L^2_x}\\
&\qquad + \int_0^T \big|M(Q_{X,Y} + 2\mathfrak{R} + \Upsilon^2 + 2\Upsilon\!\!\cdot + \mathrm{V})(t')\big|\cdot\|\mathbb{A}(t,t')\|_{L^3_t([t',T];H^{s_3}_x)}dt'\\
&\quad\lesssim T^{\frac13}K^2\big(\|X\|_{L^\infty_T H^{s_1}_x} + \|Y\|_{L^\infty_T H^{s_2}_x} + \|\Upsilon\|_{L^\infty_T W^{\frac12-\varepsilon,\infty}_x}\big)\\
&\qquad + T^{\frac13}K\|Q_{X,Y} + 2\mathfrak{R} + \Upsilon^2 + 2\Upsilon\!\!\cdot + \mathrm{V}\|^2_{L^3_T H^{-100}_x}\\
&\quad\lesssim T^{\frac13}K\big(\|(X,Y,\mathfrak{R})\|^4_{Z^{s_1,s_2,s_3}(T)} + K^4\big)
\end{aligned}
\tag{5.5.10}
$$

provided that $s_1 > 0$ with sufficiently small $\varepsilon = \varepsilon(s_1) > 0$ (in view of Lemma 5.4.5), $s_2 \geq \frac12 + \varepsilon$, and $-100 \leq s_3 \leq -\varepsilon$.

Note that $|x|x$ is differentiable with a locally bounded derivative. In view of (1.3.2), this allows us to estimate the difference $M(w_1) - M(w_2)$. By repeating a similar computation, we also obtain the difference estimate:

$$
\begin{aligned}
&\|\vec{\Phi}(X,Y,\mathfrak{R}) - \vec{\Phi}(\widetilde{X},\widetilde{Y},\widetilde{\mathfrak{R}})\|_{Z^{s_1,s_2,s_3}(T)}\\
&\qquad\lesssim T^{\frac14}\big(\|(X,Y,\mathfrak{R})\|^4_{Z^{s_1,s_2,s_3}(T)} + K^4\big)\|(X,Y,\mathfrak{R}) - (\widetilde{X},\widetilde{Y},\widetilde{\mathfrak{R}})\|_{Z^{s_1,s_2,s_3}(T)},
\end{aligned}
\tag{5.5.11}
$$

where

$$\vec{\Phi} := (\Phi_1,\Phi_2,\Phi_3).$$

Therefore, by choosing $T = T(K) > 0$ sufficiently small, we conclude from (5.5.7), (5.5.9), (5.5.10), and (5.5.11) that $\vec{\Phi} = (\Phi_1,\Phi_2,\Phi_3)$ is a contraction on the closed ball

$B_R \subset Z^{s_1,s_2,s_3}(T)$ of radius $R \sim 1 + \|(X_0, X_1)\|_{\mathcal{H}^{s_1}} + \|(Y_0, Y_1)\|_{\mathcal{H}^{s_2}}$ centered at the origin. A similar computation yields Lipschitz continuous dependence of the solution $(X, Y, \mathfrak{R})$ on the enhanced data set $(X_0, X_1, Y_0, Y_1, \Xi)$ measured in the $\mathcal{X}_T^{s_1,s_2,\varepsilon}$-norm by possibly making $T > 0$ smaller. This concludes the proof of Theorem 5.2.1.