Chapter 6

Invariant Gibbs dynamics

6.1 Overview of the chapter

In this chapter, we present the proof of Theorem 1.3.2. In the remaining part of this chapter, we work in the weakly nonlinear regime. Namely, we fix $\sigma \neq 0$ such that $|\sigma| \le \sigma_0$, where σ_0 is as in Theorem 1.2.1 (i). We also fix sufficiently large $A \gg 1$ as in Theorem 1.2.1 (i) such that the Φ^3_3-measure ρ is constructed as the limit of the truncated Φ^3_3-measures ρ_N in (1.2.11). With these parameters, consider the truncated Gibbs measure $\vec{\rho}_N$:

$$\vec{\rho}_N = \rho_N \otimes \mu_0 \tag{6.1.1}$$

for $N \in \mathbb{N}$, where μ_0 is the white noise measure; see (1.2.1) with $s = 0$. A standard argument [37, 54, 59] shows that the truncated Gibbs measure $\vec{\rho}_N$ is invariant under the truncated hyperbolic Φ^3_3-model (1.3.6):

$$\begin{aligned} &\partial_t^2 u_N + \partial_t u_N + (1-\Delta) u_N \\ &\qquad - \sigma \pi_N\big(:(\pi_N u_N)^2:\big) + M(:(\pi_N u_N)^2:)\pi_N u_N = \sqrt{2}\xi, \end{aligned} \tag{6.1.2}$$

where $:(\pi_N u_N)^2: := (\pi_N u_N)^2 - \sigma_N$ and π_N and σ_N are as in (1.2.5) and (1.2.8), respectively. See Lemma 6.2.3 below. Moreover, as a corollary to Theorem 1.2.1 (i), the truncated Gibbs measure $\vec{\rho}_N$ in (6.1.1) converges weakly to the Gibbs measure $\vec{\rho} = \rho \otimes \mu_0$ in (1.2.18).

Our main goal is to construct global-in-time dynamics for the limiting hyperbolic Φ^3_3-model (1.3.1) almost surely with respect to the Gibbs measure $\vec{\rho}$, and prove invariance of the Gibbs measure $\vec{\rho}$ under the limiting hyperbolic Φ^3_3-dynamics. A naive approach would be to apply Bourgain's invariant measure argument [9, 10], by exploiting the invariance of the truncated Gibbs measure $\vec{\rho}_N$ under the truncated hyperbolic Φ^3_3-dynamics, and to try to construct global-in-time limiting dynamics for the limiting process $u = \lim_{N\to\infty} u_N$. There are, however, two issues in the current situation: (i) the truncated Gibbs measure $\vec{\rho}_N$ converges to the limiting Gibbs measure $\vec{\rho}$ *only weakly* and (ii) the Gibbs measure $\vec{\rho}$ and the base Gaussian measure $\vec{\mu} = \mu \otimes \mu_0$ in (1.2.2) are *mutually singular*. Moreover, our local theory relies on the paracontrolled approach, which gives additional difficulty. As a result, Bourgain's invariant measure argument [9, 10] is not directly applicable to our problem. In [14], Bringmann encountered a similar problem in the context of the defocusing Hartree NLW on $\mathbb{T}^3$, where he overcame this issue by introducing a new globalization argument, by using the fact that the (truncated) Gibbs measure is absolutely continuous with respect to a shifted measure (as in Appendix A below) [13, 54] in a uniform manner and establishing a (rather involved) large time stability theory, where sets of large probabilities are characterized via the shifted measures.

In the following, we introduce a new alternative globalization argument. This new argument has the advantage of being conceptually simple and straightforward. Our approach consists of several steps:

Step 1. In the first step, we establish a uniform (in N) exponential integrability of the truncated enhanced data set Ξ_N (see (6.1.10) below) with respect to the truncated measure $\vec{\rho}_N \otimes \mathbb{P}_2$ (Proposition 6.2.4). Here, $\mathbb{P}_2$ is the measure for the stochastic forcing defined in (6.1.4) below. By combining the variational approach with space-time estimates, we prove this uniform exponential integrability *without* any reference to (the truncated version of) the shifted measure $\text{Law}(Y(1) + \sigma \mathfrak{Z}(1) + \mathcal{W}(1))$ constructed in Appendix A. As a corollary, we construct the limiting enhanced data set Ξ associated with the Gibbs measure $\vec{\rho}$ (see (6.1.11) below) by establishing convergence of the truncated enhanced data set Ξ_N almost surely with respect to the limiting measure $\vec{\rho} \otimes \mathbb{P}_2$.

Step 2. In the second step, we establish a stability result (Proposition 6.3.1). We prove this stability result by a simple contraction argument, where we use a norm with an exponentially decaying weight in time. As a result, the proof follows from a small modification of that of the local well-posedness (Theorem 5.2.1). As compared to [14], our stability argument is very simple (both in terms of the statements and the proofs).

Step 3. In the third step, we establish a uniform (in N) control on the solution $(X_N, Y_N, \mathfrak{R}_N)$ to the truncated system (see (6.3.2) below) with respect to the truncated measure $\vec{\rho}_N \otimes \mathbb{P}_2$ (Proposition 6.3.2). The proof is based on the invariance of the truncated Gibbs measure $\vec{\rho}_N$ and a discrete Gronwall argument.

Step 4. In the fourth step, we study the pushforward measures $(\Xi_N)_\#(\vec{\rho}_N \otimes \mathbb{P}_2)$ and $(\Xi)_\#(\vec{\rho} \otimes \mathbb{P}_2)$. In particular, by using ideas from theory of optimal transport (the Kantorovich duality) and the Boué–Dupuis variational formula, we prove that the pushforward measure $(\Xi_N)_\#(\vec{\rho}_N \otimes \mathbb{P}_2)$ converges to $(\Xi)_\#(\vec{\rho} \otimes \mathbb{P}_2)$ in the Wasserstein-1 distance, as $N \to \infty$; see Proposition 6.3.3 below.

Once we establish Steps 1–4, the proof of Theorem 1.3.2 follows in a straightforward manner. In Section 6.2, we first study the truncated dynamics (6.1.2) and briefly go over almost sure global well-posedness of (6.1.2) and invariance of the truncated Gibbs measure $\vec{\rho}_N$ (Lemma 6.2.3). We then discuss the details of Step 1 above. In Section 6.3, we first go over the details of Steps 2, 3, and 4 and then present the proof of Theorem 1.3.2.

Notations. By assumption, the Gaussian field $\vec{\mu} = \mu \otimes \mu_0$ in (1.2.2) and hence the (truncated) Gibbs measure are independent of (the distribution of) the space-time white noise ξ in (1.3.1) and (6.1.2). Hence, we can write the probability space Ω as

$$\Omega = \Omega_1 \times \Omega_2 \tag{6.1.3}$$

such that the random Fourier series in (1.2.4) depend only on $\omega_1 \in \Omega_1$, while the cylindrical Wiener process W in (3.1.1) depends only on $\omega_2 \in \Omega_2$. In view of (6.1.3), we also write the underlying probability measure $\mathbb{P}$ on Ω as

$$\mathbb{P} = \mathbb{P}_1 \otimes \mathbb{P}_2, \tag{6.1.4}$$

where $\mathbb{P}_j$ is the marginal probability measure on Ω_j, $j = 1, 2$.

With the decomposition (6.1.3) in mind, we set

$$\upharpoonleft(t; \vec{u}_0, \omega_2) = S(t)\vec{u}_0 + \sqrt{2}\int_0^t \mathcal{D}(t-t')dW(t', \omega_2) \tag{6.1.5}$$

for $\vec{u}_0 = (u_0, u_1) \in \mathcal{H}^{-\frac{1}{2}-\varepsilon}(\mathbb{T}^3)$ and $\omega_2 \in \Omega_2$, where $S(t)$ and $\mathcal{D}(t)$ are as in (5.2.4) and (5.2.2), respectively. When it is clear from the context, we may suppress the dependence on $\vec{u}_0$ and/or ω_2. Given $N \in \mathbb{N}$, we set

$$\upharpoonleft_N(\vec{u}_0, \omega_2) = \pi_N \upharpoonleft(\vec{u}_0, \omega_2), \tag{6.1.6}$$

where π_N is as in (1.2.5). We also set

$$\begin{aligned} \vee_N(\vec{u}_0, \omega_2) &= \upharpoonleft_N^2(\vec{u}_0, \omega_2) - \sigma_N, \\ \Upsilon_N(\vec{u}_0, \omega_2) &= \pi_N \mathcal{I}(\vee_N(\vec{u}_0, \omega_2)), \\ \Upsilon^{\circ}_N(\vec{u}_0, \omega_2) &= \Upsilon_N(\vec{u}_0, \omega_2) \ominus \upharpoonleft_N(\vec{u}_0, \omega_2), \end{aligned} \tag{6.1.7}$$

and define $\mathbb{A}_N(\vec{u}_0, \omega_2)$ as in (5.4.9) by replacing $\upharpoonleft_N$ with $\upharpoonleft_N(\vec{u}_0, \omega_2)$. We define the paracontrolled operator $\widetilde{\mathfrak{I}}^N_{\olessthan, \ominus} = \widetilde{\mathfrak{I}}^N_{\olessthan, \ominus}(\vec{u}_0, \omega_2)$ in a manner analogous to $\mathfrak{I}^N_{\olessthan, \ominus}$ in Lemma 5.4.6, but with an extra frequency cutoff π_N. Namely, instead of (5.2.19), we first define $\widetilde{\mathfrak{I}}^N_{\olessthan}$ by

$$\widetilde{\mathfrak{I}}^N_{\olessthan}(w)(t) = \mathcal{I}(\pi_N(w \olessthan \upharpoonleft_N))(t), \tag{6.1.8}$$

where $\upharpoonleft_N = \upharpoonleft_N(\vec{u}_0, \omega_2)$ is as in (6.1.6). We then define $\widetilde{\mathfrak{I}}^{(1),N}_{\olessthan}$ and $\widetilde{\mathfrak{I}}^{(2),N}_{\olessthan}$ as in (5.2.20) and (5.2.21) with an extra frequency cutoff $\chi_N(n)$, depending on $|n_1| \gtrsim |n_2|^\theta$ or $|n_1| \ll |n_2|^\theta$. Note that the conclusion of Lemma 5.4.5 (in particular the estimate (5.4.5)) holds for $\widetilde{\mathfrak{I}}^{(1),N}_{\olessthan}$, uniformly in $N \in \mathbb{N}$. Finally, we define $\widetilde{\mathfrak{I}}^N_{\olessthan, \ominus}$ by

$$\widetilde{\mathfrak{I}}^N_{\olessthan, \ominus}(w)(t) = \widetilde{\mathfrak{I}}^{(2),N}_{\olessthan}(w) \ominus \upharpoonleft_N(t), \tag{6.1.9}$$

namely, by inserting a frequency cutoff $\chi_N(n_1 + n_2)$ and replacing $\upharpoonleft$ by $\upharpoonleft_N = \upharpoonleft_N(\vec{u}_0, \omega_2)$ in (5.2.23). We then define the truncated enhanced data set $\Xi_N(\vec{u}_0, \omega_2)$ by

$$\Xi_N(\vec{u}_0, \omega_2) = \big(\upharpoonleft_N, \vee_N, \Upsilon_N, \Upsilon^{\circ}_N, \mathbb{A}_N, \widetilde{\mathfrak{I}}^N_{\olessthan, \ominus}\big), \tag{6.1.10}$$

where, on the right-hand side, we suppressed the dependence on $(\vec{u}_0, \omega_2)$ for notational simplicity. Note that, given $\vec{u}_0 \in \mathcal{H}^{-\frac{1}{2}-\varepsilon}(\mathbb{T}^3)$, the enhanced data set $\Xi_N(\vec{u}_0, \omega_2)$ does not converge in general. Nonetheless, for the notational purpose, let us *formally* define the (untruncated) enhanced data set $\Xi(\vec{u}_0, \omega_2)$ by setting

$$\Xi(\vec{u}_0, \omega_2) = \big(\uparrow, \vee, \mathrm{Y}, \mathring{\mathrm{Y}}, \mathbb{A}, \mathfrak{I}_{\otimes,\ominus}\big), \tag{6.1.11}$$

where each term on the right-hand side is a limit of the corresponding term in (6.1.10) (if it exists). In Corollary 6.2.6, we will construct the enhanced data set $\Xi(\vec{u}_0, \omega_2)$ in (6.1.11) as a limit of the truncated enhanced data set $\Xi_N(\vec{u}_0, \omega_2)$ in (6.1.10) almost surely with respect to $\vec{\rho} \otimes \mathbb{P}_2$.

In the remaining part of this chapter, we fix $s_1, s_2, s_3 \in \mathbb{R}$ satisfying

$$\frac{1}{4} < s_1 < \frac{1}{2} < s_2 < s_1 + \frac{1}{4} \quad \text{and} \quad s_2 - 1 < s_3 < 0. \tag{6.1.12}$$

Furthermore, we take both s_1 and s_2 to be sufficiently close to $\frac{1}{2}$ (such that the conditions in (6.3.26) are satisfied, say with $r_1 = r_2 = 3$).

Remark 6.1.1. (i) In view of (6.1.6) with (1.2.5) we have $\uparrow_N(\vec{u}_0, \omega_2) = \uparrow_N(\pi_N \vec{u}_0, \omega_2)$ and thus

$$\Xi_N(\vec{u}_0, \omega_2) = \Xi_N(\pi_N \vec{u}_0, \omega_2).$$

Namely, the truncated enhanced data set $\Xi_N(\vec{u}_0, \omega_2)$ in (6.1.10) depends only on the low frequency part $\pi_N \vec{u}_0$ of the initial data.

(ii) Note that the terms Y_N, $\mathring{\mathrm{Y}}_N$, and $\widetilde{\mathfrak{I}}^N_{\otimes,\ominus}$ in (6.1.10) come with an extra frequency cutoff as compared to the corresponding terms studied in Chapter 5. When $\mathrm{Law}(\vec{u}_0) = \vec{\mu}$, the results in Lemmas 5.4.3, 5.4.4, and 5.4.6, and Remark 5.4.2 from Section 5.4 also apply to $\mathrm{Y}_N(\vec{u}_0, \omega_2)$, $\mathring{\mathrm{Y}}_N(\vec{u}_0, \omega_2)$, and $\widetilde{\mathfrak{I}}^N_{\otimes,\ominus}(\vec{u}_0, \omega_2)$.

(iii) Note that the $\mathcal{X}^\varepsilon_T$-norm for enhanced data sets defined in (5.5.3) also measures the time derivatives of $\uparrow_N$ and Y_N in appropriate space-time norms. In view of (6.1.7) and (5.2.5), the time derivative of $\mathrm{Y}_N(\vec{u}_0, \omega_2)$ is given by

$$\partial_t \mathrm{Y}_N(t; \vec{u}_0, \omega_2) = \pi_N \int_0^t \partial_t \mathcal{D}(t - t') \vee_N(t'; \vec{u}_0, \omega_2) dt'.$$

As for the stochastic convolution, recall that, unlike the heat or Schrödinger case, the stochastic convolution for the damped wave equation is differentiable in time and the time derivative of $\uparrow_N(\vec{u}_0, \omega_2)$ is given by

$$\partial_t \uparrow_N(t; \vec{u}_0, \omega_2) = \pi_N \partial_t S(t) \vec{u}_0 + \sqrt{2}\pi_N \int_0^t \partial_t \mathcal{D}(t - t') dW(t', \omega_2). \tag{6.1.13}$$

The formula (6.1.13) easily follows from viewing the stochastic integral in (6.1.5) (with an extra frequency cutoff π_N) as a Paley–Wiener–Zygmund integral and taking a time derivative.

6.2 On the truncated dynamics

In this section, we study the truncated hyperbolic Φ^3_3-model (6.1.2). We first go over local well-posedness of the truncated equation (6.1.2) and then almost sure global well-posedness and invariance of the truncated Gibbs measure $\vec{\rho}_N$; see Lemmas 6.2.1 and 6.2.3. Then, by combining the Boué–Dupuis variational formula (Lemma 3.1.1) and space-time estimates, we prove uniform (in N) exponential integrability of the truncated enhanced data set $\Xi_N(\vec{u}_0, \omega_2)$ with respect to $\vec{\rho}_N \otimes \mathbb{P}_2$ on $(\vec{u}_0, \omega_2)$; see Proposition 6.2.4. As a corollary, we prove that the truncated enhanced data set $\Xi_N(\vec{u}_0, \omega_2)$ in (6.1.10) converges to the limiting enhanced data set $\Xi(\vec{u}_0, \omega_2)$ in (6.1.11) almost surely with respect to the limiting measure $\vec{\rho} \otimes \mathbb{P}_2$ (Corollary 6.2.6).

Given $N \in \mathbb{N}$, let $\vec{u}_0 = (u_0, u_1)$ be a pair of random distributions such that $\mathrm{Law}((u_0, u_1)) = \vec{\rho}_N = \rho_N \otimes \mu_0$. Let u_N be a solution to the truncated equation (6.1.2) with $(u_N, \partial_t u_N)|_{t=0} = \vec{u}_0$. With $:(\pi_N u_N)^2: \,:= (\pi_N u_N)^2 - \sigma_N$, we write (6.1.2) as

$$\begin{cases} \partial_t^2 u_N + \partial_t u_N + (1-\Delta) u_N \\ \quad -\sigma \pi_N\big((\pi_N u_N)^2 - \sigma_N\big) + M\big((\pi_N u_N)^2 - \sigma_N\big)\pi_N u_N = \sqrt{2}\xi \\ (u_N, \partial_t u_N)|_{t=0} = \vec{u}_0, \end{cases} \tag{6.2.1}$$

where M is as in (1.3.2). Note that, due to the presence of the frequency projector π_N, the dynamics (6.2.1) on high frequencies $\{|n| \gtrsim N\}$ and low frequencies $\{|n| \lesssim N\}$ are decoupled. The high frequency part of the dynamics (6.2.1) is given by

$$\begin{cases} \partial_t^2 \pi_N^\perp u_N + \partial_t \pi_N^\perp u_N + (1-\Delta)\pi_N^\perp u_N = \sqrt{2}\pi_N^\perp \xi \\ (\pi_N^\perp u_N, \partial_t \pi_N^\perp u_N)|_{t=0} = \pi_N^\perp \vec{u}_0. \end{cases} \tag{6.2.2}$$

The solution $\pi_N^\perp u_N$ to (6.2.2) is given by

$$\pi_N^\perp u_N = \pi_N^\perp \upharpoonleft(\vec{u}_0), \tag{6.2.3}$$

where $\upharpoonleft(\vec{u}_0)$ is as in (6.1.5) with the ω_2-dependence suppressed. With $v_N = \pi_N u_N$, the low frequency part of the dynamics (6.2.1) is given by

$$\begin{cases} \partial_t^2 v_N + \partial_t v_N + (1-\Delta) v_N \\ \quad -\sigma \pi_N\big((\pi_N v_N)^2 - \sigma_N\big) + M\big((\pi_N v_N)^2 - \sigma_N\big)\pi_N v_N = \sqrt{2}\pi_N \xi \\ (v_N, \partial_t v_N)|_{t=0} = \pi_N \vec{u}_0, \end{cases} \tag{6.2.4}$$

where we kept π_N in several places to emphasize that (6.2.4) depends only on finite many frequencies $\{n \in NQ\}$ with Q as in (1.2.7). By writing (6.2.4) in the Duhamel formulation, we have

$$v_N(t) = \pi_N S(t)\vec{u}_0 + \int_0^t \mathcal{D}(t-t')\mathcal{N}_N(v_N)(t')dt' + \upharpoonleft_N(t;0), \tag{6.2.5}$$

where the truncated nonlinearity $\mathcal{N}_N(v_N)$ is given by

$$\mathcal{N}_N(v_N) = \sigma\pi_N\big((\pi_N v_N)^2 - \sigma_N\big) - M\big((\pi_N v_N)^2 - \sigma_N\big)\pi_N v_N, \tag{6.2.6}$$

and $\upharpoonright_N(t;0)$ is as in (6.1.6) with $\vec{u}_0 = 0$:

$$\upharpoonright_N(t;0,\omega_2) = \sqrt{2}\int_0^t \mathcal{D}(t-t')\pi_N dW(t',\omega_2).$$

For each fixed $N \in \mathbb{N}$, we have $\upharpoonright_N(t;0) = \pi_N \upharpoonright(t;0) \in C^1(\mathbb{R}_+; C^\infty(\mathbb{T}^3))$; see Remark 6.1.1. By viewing $\upharpoonright_N(t;0)$ in (6.2.5) as a perturbation, it suffices to study the following damped NLW with a deterministic perturbation:

$$v_N(t) = \pi_N S(t)(v_0, v_1) + \int_0^t \mathcal{D}(t-t')\mathcal{N}_N(v_N)(t')dt' + F, \tag{6.2.7}$$

where $(v_0, v_1) \in \mathcal{H}^1(\mathbb{T}^3)$, σ_N is as in (1.2.8), and $F \in C^1(\mathbb{R}_+; C^\infty(\mathbb{T}^3))$ is a given deterministic function.

A standard contraction argument with the one degree of smoothing from the Duhamel integral operator $\mathcal{I}$ in (5.2.5) and Sobolev's inequality yields the following local well-posedness of (6.2.7). Since the argument is standard, we omit details. See, for example, the proof of [54, Lemma 9.1].

Lemma 6.2.1. *Let $N \in \mathbb{N}$. Given any $(v_0, v_1) \in \mathcal{H}^1(\mathbb{T}^3)$ and $F \in C^1([0,1]; H^1(\mathbb{T}^3))$ with*

$$\|(v_0, v_1)\|_{\mathcal{H}^1} \le R \quad \textit{and} \quad \|F\|_{C^1([0,1];H^1)} \le K$$

for some $R, K \ge 1$, there exist $\tau = \tau(R,K,N) > 0$ and a unique solution v_N to (6.2.7) on $[0,\tau]$, satisfying the bound:

$$\|v_N\|_{\widetilde{X}^1(\tau)} \lesssim R + K,$$

where

$$\widetilde{X}^1(\tau) = C([0,\tau]; H^1(\mathbb{T}^3)) \cap C^1([0,\tau]; L^2(\mathbb{T}^3)).$$

Moreover, the solution v_N is unique in $\widetilde{X}^1(\tau)$.

Remark 6.2.2. (i) A standard contraction argument gives $\tau = \tau(R,K,N) \sim (R + K + N)^{-\theta}$ for some $\theta > 0$, in particular the local existence depends on $N \in \mathbb{N}$.

(ii) We also point out that the uniqueness statement for v_N in Lemma 6.2.1 is unconditional, namely, the uniqueness of the solution v_N holds in the entire class $\widetilde{X}^1(\tau)$. Then, from (6.2.3) and the unconditional uniqueness of the solution $v_N = v_N(\pi_N\vec{u}_0)$ to (6.2.4), we obtain the *unique* representation of u_N:

$$u_N = \pi_N^\perp \upharpoonright(\vec{u}_0) + v_N(\pi_N\vec{u}_0).$$

See for example (6.3.73) below, where we use a different representation of u_N.

Before proceeding further, let us introduce some notations. Given the cylindrical Wiener process W in (3.1.1), by possibly enlarging the probability space Ω_2, there exists a family of translations $\tau_{t_0} : \Omega_2 \to \Omega_2$ such that

$$W(t, \tau_{t_0}(\omega_2)) = W(t + t_0, \omega_2) - W(t_0, \omega_2)$$

for $t, t_0 \geq 0$ and $\omega_2 \in \Omega_2$. Denote by $\Phi^N(t)$ the stochastic flow map to the truncated hyperbolic Φ^3_3-model (6.1.2) constructed in Lemma 6.2.1 (which is not necessarily global at this point). Namely,

$$\begin{aligned} \vec{u}_N(t) = (u_N(t), \partial_t u_N(t)) &= \Phi^N(t)(\vec{u}_0, \omega_2) \\ &= \big(\Phi^N_1(t)(\vec{u}_0, \omega_2), \Phi^N_2(t)(\vec{u}_0, \omega_2)\big) \end{aligned} \tag{6.2.8}$$

is the solution to (6.1.2) with $\vec{u}_N|_{t=0} = \vec{u}_0$, satisfying $\mathrm{Law}(\vec{u}_0) = \vec{\rho}_N$, and the noise $\xi(\omega_2)$. We now extend $\Phi^N(t)$ as

$$\widehat{\Phi}^N(t)(\vec{u}_0, \omega_2) = \big(\Phi^N(t)(\vec{u}_0, \omega_2), \tau_t(\omega_2)\big). \tag{6.2.9}$$

Note that by the uniqueness of the solution to (6.1.2), we have

$$\begin{aligned} \Phi^N(t_1 + t_2)(\vec{u}_0, \omega_2) &= \Phi^N(t_2)\big(\Phi^N(t_1)(\vec{u}_0, \omega_2), \tau_{t_1}(\omega_2)\big) \\ &= \Phi^N(t_2)\big(\widehat{\Phi}^N(t_1)(\vec{u}_0, \omega_2)\big) \end{aligned}$$

for $t_1, t_2 \geq 0$ as long as the flow is well defined.

By writing the truncated dynamics (6.1.2) as a superposition of the deterministic NLW:

$$\partial_t^2 u_N + (1 - \Delta) u_N - \mathcal{N}_N(u_N) = 0, \tag{6.2.10}$$

where $\mathcal{N}_N(u_N)$ is as in (6.2.6), and the Ornstein–Uhlenbeck process (for $\partial_t u_N$):

$$\partial_t(\partial_t u_N) = -\partial_t u_N + \sqrt{2}\xi, \tag{6.2.11}$$

we see that the truncated Gibbs measure $\vec{\rho}_N$ in (6.1.1) is formally[1] invariant under the dynamics of (6.1.2), since $\vec{\rho}_N$ is invariant under the NLW dynamics (6.2.10), while the white noise measure μ_0 on $\partial_t u_N$ (and hence $\vec{\rho}_N = \rho_N \otimes \mu_0$ on $(u_N, \partial_t u_N)$) is invariant under the Ornstein–Uhlenbeck dynamics (6.2.11). Then, by exploiting the formal invariance of the truncated Gibbs measure $\vec{\rho}_N$, Bourgain's invariant measure argument [9] yields the following result on almost sure global well-posedness of the truncated hyperbolic Φ^3_3-model (6.1.2) and invariance of the truncated Gibbs measure $\vec{\rho}_N$. Since the argument is standard (for fixed $N \in \mathbb{N}$), we omit details. See the proof of [54, Lemma 9.3] for details.

[1]Namely, as long as the dynamics is well defined.

Lemma 6.2.3. *Let $N \in \mathbb{N}$. Then, the truncated hyperbolic Φ^3_3-model (6.1.2) is almost surely globally well-posed with respect to the random initial data distributed by the truncated Gibbs measure $\vec{\rho}_N$ in (6.1.1). Furthermore, $\vec{\rho}_N$ is invariant under the resulting dynamics and, as a consequence, the measure $\vec{\rho}_N \otimes \mathbb{P}_2$ is invariant under the extended stochastic flow map $\widehat{\Phi}^N(t)$ defined in (6.2.9). More precisely, there exists $\Sigma_N \subset \Omega = \Omega_1 \times \Omega_2$ with $\vec{\rho}_N \otimes \mathbb{P}_2(\Sigma_N) = 1$ such that the solution $u_N = u_N(\vec{u}_0, \omega_2)$ to (6.1.2) exists globally in time and* $\mathrm{Law}(u_N(t), \partial_t u_N(t)) = \vec{\rho}_N$ *for any $t \in \mathbb{R}_+$.*

Next, we establish uniform exponential integrability of the truncated enhanced data set $\Xi_N(\vec{u}_0, \omega_2)$ in (6.1.10) with respect to the truncated measure $\vec{\rho}_N \otimes \mathbb{P}_2$. We also establish uniform exponential integrability for the difference of the truncated enhanced data sets.

Proposition 6.2.4. *Let $T > 0$. Then, we have*

$$\int \mathbb{E}_{\mathbb{P}_2}\big[\exp\big(\|\Xi_N(\vec{u}_0,\omega_2)\|^{\alpha}_{\mathcal{X}^{\varepsilon}_T}\big)\big] d\vec{\rho}_N(\vec{u}_0) \le C(T,\varepsilon,\alpha) < \infty \tag{6.2.12}$$

for $0 < \alpha < \frac{1}{3}$, uniformly in $N \in \mathbb{N}$, where the $\mathcal{X}^{\varepsilon}_T$-norm and the truncated enhanced data set $\Xi_N(\vec{u}_0, \omega_2)$ are as in (5.5.3) and (6.1.10), respectively. Here, $\mathbb{E}_{\mathbb{P}_2}$ denotes an expectation with respect to the probability measure $\mathbb{P}_2$ on $\omega_2 \in \Omega_2$ defined in (6.1.4).

Moreover, there exists small $\beta > 0$ such that

$$\begin{aligned} \int \mathbb{E}_{\mathbb{P}_2}\big[\exp\big(N_2^{\beta}\|\Xi_{N_1}(\vec{u}_0,\omega_2) - \Xi_{N_2}(\vec{u}_0,\omega_2)\|^{\alpha}_{\mathcal{X}^{\varepsilon}_T}\big)\big] d\vec{\rho}_N(\vec{u}_0) \\ \le C(T,\varepsilon,\alpha) < \infty \end{aligned} \tag{6.2.13}$$

for $0 < \alpha < \frac{1}{3}$, uniformly in $N, N_1, N_2 \in \mathbb{N}$ with $N \ge N_1 \ge N_2$.

Proof. For simplicity, we only prove (6.2.12) and (6.2.13) for the random operator $\widetilde{\mathfrak{I}}^N_{⧀,⊜}$ defined in (6.1.9). The other terms in $\Xi_N(\vec{u}_0, \omega_2)$ can be estimated in an analogous manner. See Remark 6.2.5.

We break the proof into two parts.

Part 1. We first prove the following uniform exponential integrability:

$$\int \mathbb{E}_{\mathbb{P}_2}\Big[\exp\Big(\|\widetilde{\mathfrak{I}}^N_{⧀,⊜}\|^{\alpha}_{\mathcal{L}_2(q,T)}\Big)\Big] d\vec{\rho}_N(\vec{u}_0) \le C(T,\varepsilon,\alpha) < \infty \tag{6.2.14}$$

for any $T > 0$, any finite $q > 1$, and $0 < \alpha < \frac{1}{2}$, uniformly in $N \in \mathbb{N}$. Note that the range $0 < \alpha < \frac{1}{2}$ of the exponent in (6.2.14) comes from the presence of $\|\mathfrak{Z}_N\|^2_{W^{1-\varepsilon,\infty}}$ in (6.2.28) and (6.2.32), since $\mathfrak{Z}_N$ defined in one line below (3.2.3) belongs to $\mathcal{H}_{\le 2}$. Similarly, the overall restriction $0 < \alpha < \frac{1}{3}$ in this proposition comes from the terms involving ψ_1 in (6.2.38), where ψ_1 is defined in (6.2.23) with (6.2.21). Namely, the worst contribution in (6.2.38) behaves like $\|\mathfrak{Z}_N\|_{W^{1-\varepsilon,\infty}}^{3\alpha}$ which is exponentially integrable only for $\alpha < \frac{1}{3}$; see (6.2.39).

From (6.1.8) and (6.1.9), we see that $\widetilde{\mathfrak{I}}^N_{\circledless,\circleddash}$ depends on two entries of $\uparrow_N = \pi_N \uparrow(\vec{u}_0, \omega_2)$. We now generalize the definition of $\widetilde{\mathfrak{I}}^N_{\circledless,\circleddash}$ to allow general entries. Given $\psi_j \in C(\mathbb{R}_+; \mathcal{D}'(\mathbb{T}^3))$, $j = 1, 2$, we first define $\widetilde{\mathfrak{I}}^N_{\circledless}[\psi_1]$ by

$$\widetilde{\mathfrak{I}}^N_{\circledless}[\psi_1](w) = \mathcal{I}\big(\pi_N(w \circledless (\pi_N \psi_1))\big). \tag{6.2.15}$$

As in (5.2.20) and (5.2.21), define $\widetilde{\mathfrak{I}}^{(2),N}_{\circledless}[\psi_1]$ to be the restriction of $\widetilde{\mathfrak{I}}^N_{\circledless}[\psi_1]$ onto $\{|n_1| \ll |n_2|^\theta\}$:

$$\widetilde{\mathfrak{I}}^{(2),N}_{\circledless}[\psi_1](w) = \mathcal{I}\big(\pi_N(\mathcal{K}^\theta(w, \pi_N \psi_1))\big), \tag{6.2.16}$$

where $\mathcal{K}^\theta$ is the bilinear Fourier multiplier operator with the multiplier $\mathbf{1}_{\{|n_1| \ll |n_2|^\theta\}}$. More precisely, we have

$$\begin{aligned} &\widetilde{\mathfrak{I}}^{(2),N}_{\circledless}[\psi_1](w)(t) \\ &\quad = \sum_{n \in \mathbb{Z}^3} \chi_N(n) e_n \sum_{n = n_1 + n_2} \sum_{0 \le j < \theta k + c_0} \varphi_j(n_1) \varphi_k(n_2) \chi_N(n_2) \\ &\qquad \times \int_0^t e^{-\frac{t-t'}{2}} \frac{\sin((t-t')[\![n]\!])}{[\![n]\!]} \hat{w}(n_1, t') \hat{\psi}_1(n_2, t') dt', \end{aligned} \tag{6.2.17}$$

where χ_N is as in (1.2.6) and $c_0 \in \mathbb{R}$ is as in (5.4.4). Then, we define $\widetilde{\mathfrak{I}}^N_{\circledless,\circleddash}[\psi_1, \psi_2]$ by

$$\widetilde{\mathfrak{I}}^N_{\circledless,\circleddash}[\psi_1, \psi_2](w) = \widetilde{\mathfrak{I}}^{(2),N}_{\circledless}[\psi_1](w) \circleddash (\pi_N \psi_2). \tag{6.2.18}$$

Note that $\widetilde{\mathfrak{I}}^N_{\circledless,\circleddash}[\psi_1, \psi_2]$ is bilinear in ψ_1 and ψ_2. We also set

$$\widetilde{\mathfrak{I}}^N_{\circledless,\circleddash}[\psi] = \widetilde{\mathfrak{I}}^N_{\circledless,\circleddash}[\psi, \psi] \tag{6.2.19}$$

for simplicity. With this notation, we can write $\widetilde{\mathfrak{I}}^N_{\circledless,\circleddash}$ in (6.2.14) as $\widetilde{\mathfrak{I}}^N_{\circledless,\circleddash}[\uparrow(\vec{u}_0, \omega_2)]$, where $\vec{u}_0 = (u_0, u_1)$. Note that we have

$$\widetilde{\mathfrak{I}}^N_{\circledless,\circleddash}[\pi_N \psi] = \widetilde{\mathfrak{I}}^N_{\circledless,\circleddash}[\psi].$$

Before proceeding further, we record the following boundedness of $\mathcal{K}^\theta$ defined in (6.2.16) and (6.2.17); a slight modification of the proof of (2.1.7) in Lemma 2.1.2 yields

$$\|\mathcal{K}^\theta(f, g)\|_{B^{s_2}_{p,q}} \lesssim \|f\|_{L^{p_1}} \|g\|_{B^{s_2}_{p_2,q}} \tag{6.2.20}$$

for any $s_2 \in \mathbb{R}$ and $1 \le p, p_1, p_2, q \le \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.

By the Boué–Dupuis variational formula (Lemma 3.1.1) with the change of variables (3.2.4), we have

$$
\begin{aligned}
&-\log \int \exp\Big(\big\|\widetilde{\mathfrak{I}}^N_{\circledless,\circleddash}[\upharpoonleft(\vec{u}_0,\omega_2)]\big\|^{\alpha}_{\mathcal{L}_2(q,T)}\Big)d\rho_N(u_0)\\
&= \inf_{\dot{\Upsilon}^N\in\mathbb{H}^1_a} \mathbb{E}\Big[-\big\|\widetilde{\mathfrak{I}}^N_{\circledless,\circleddash}[\upharpoonleft(Y+\Theta,u_1,\omega_2)]\big\|^{\alpha}_{\mathcal{L}_2(q,T)}\\
&\qquad + \widehat{R}^{\diamond}_N(Y+\Upsilon^N+\sigma\mathfrak{Z}_N) + \frac{1}{2}\int_0^1 \|\dot{\Upsilon}^N(t)\|^2_{H^1_x}dt\Big] + \log Z_N,
\end{aligned}
$$

where $\widehat{R}^{\diamond}_N$ is as in (3.2.25) and

$$
\Theta = \Upsilon^N + \sigma\mathfrak{Z}_N. \tag{6.2.21}
$$

Recall the notation $Y_N = \pi_N Y$ and $\Upsilon_N = \pi_N \Upsilon^N$. Then, from Lemmas 3.2.2 and 3.2.3 with Lemma 3.1.2 and (3.2.17), there exists $\varepsilon_0, C_0 > 0$ such that

$$
\begin{aligned}
&-\log \int \exp\Big(\big\|\widetilde{\mathfrak{I}}^N_{\circledless,\circleddash}[\upharpoonleft(\vec{u}_0,\omega_2)]\big\|^{\alpha}_{\mathcal{L}_2(q,T)}\Big)d\rho_N(u_0)\\
&\geq \inf_{\dot{\Upsilon}^N\in\mathbb{H}^1_a} \mathbb{E}\Big[-\big\|\widetilde{\mathfrak{I}}^N_{\circledless,\circleddash}[\upharpoonleft(Y+\Theta,u_1,\omega_2)]\big\|^{\alpha}_{\mathcal{L}_2(q,T)}\\
&\qquad + \varepsilon_0\big(\|\Upsilon^N\|^2_{H^1} + \|\Upsilon_N\|^6_{L^2}\big)\Big] - C_0,
\end{aligned} \tag{6.2.22}
$$

uniformly in u_1 and ω_2.

In view of (6.1.5), we write $\upharpoonleft(Y+\Theta,u_1,\omega_2)$ as

$$
\upharpoonleft(Y+\Theta,u_1,\omega_2) = \upharpoonleft(Y,u_1,\omega_2) + S(t)(\Theta,0) =: \psi_0 + \psi_1, \tag{6.2.23}
$$

where $S(t)$ is as in (5.2.4). By (6.2.19), we have

$$
\begin{aligned}
&\big\|\widetilde{\mathfrak{I}}^N_{\circledless,\circleddash}[\upharpoonleft(Y+\Theta,u_1,\omega_2)]\big\|_{\mathcal{L}_2(q,T)}\\
&\quad\leq \big\|\widetilde{\mathfrak{I}}^N_{\circledless,\circleddash}[\psi_0,\psi_0]\big\|_{\mathcal{L}_2(q,T)} + \big\|\widetilde{\mathfrak{I}}^N_{\circledless,\circleddash}[\psi_0,\psi_1]\big\|_{\mathcal{L}_2(q,T)}\\
&\qquad + \big\|\widetilde{\mathfrak{I}}^N_{\circledless,\circleddash}[\psi_1,\psi_0]\big\|_{\mathcal{L}_2(q,T)} + \big\|\widetilde{\mathfrak{I}}^N_{\circledless,\circleddash}[\psi_1,\psi_1]\big\|_{\mathcal{L}_2(q,T)}.
\end{aligned} \tag{6.2.24}
$$

Under the truncated Gibbs measure $\vec{\rho}_N$, we have $\mathrm{Law}(u_1) = \mu_0$ and thus we have $\mathrm{Law}(Y,u_1) = \vec{\mu} = \mu\otimes\mu_0$. Then, from the uniform exponential tail estimates in Lemmas 5.4.1 and 5.4.6 (see also Remark 6.1.1) with (3.2.3), there exists $K(Y,u_1,\omega_2)$ such that

$$
\begin{aligned}
&\big\|\widetilde{\mathfrak{I}}^N_{\circledless,\circleddash}[\psi_0]\big\|_{\mathcal{L}_2(q,T)} + \|\psi_0\|^2_{L^\infty_T W_x^{-\frac{1}{2}-\varepsilon,\infty}} + \|\mathfrak{Z}_N\|_{W^{1-\varepsilon,\infty}}\\
&\quad\leq K(Y,u_1,\omega_2)
\end{aligned} \tag{6.2.25}
$$

and

$$
\mathbb{E}_{\vec{\mu}\otimes\mathbb{P}_2}\big[\exp(\delta K(Y,u_1,\omega_2))\big] < \infty \tag{6.2.26}
$$

for sufficiently small $\delta > 0$.

We now estimate the last three terms on the right-hand side of (6.2.24). Let $s_3 < 0$. By Sobolev's inequality, (6.2.18), Hölder's inequality,[2] (6.2.16), Sobolev's inequality, Lemma 5.3.1, and (6.2.20) with (6.2.23), we have

$$\begin{aligned}
&\big\| \widetilde{\mathfrak{J}}^N_{\textcircled{<},\ominus}[\psi_0,\psi_1](w)\big\|_{L^\infty_T H^{s_3}_x} \\
&\lesssim \big\| \widetilde{\mathfrak{J}}^{(2),N}_{\textcircled{<}}[\psi_0](w) \ominus (\pi_N \psi_1)\big\|_{L^\infty_T L_x^{\frac{6}{3-2s_3}}} \\
&\lesssim \big\| \widetilde{\mathfrak{J}}^{(2),N}_{\textcircled{<}}[\psi_0](w)\big\|_{L^\infty_T L_x^{\frac{3}{1-s_3-\varepsilon}}} \|\pi_N \psi_1\|_{L^\infty_T L_x^{\frac{6}{1+2\varepsilon}}} \\
&\lesssim \|\mathcal{I}(\mathcal{K}^\theta(w,\pi_N\psi_0))\|_{L^\infty_T H_x^{s_3+\frac{1}{2}+\varepsilon}} \|\psi_1\|_{L^\infty_T H^{1-\varepsilon}_x} \\
&\lesssim \|\mathcal{K}^\theta(w,\pi_N\psi_0)\|_{L^1_T H_x^{s_3-\frac{1}{2}+\varepsilon}} \|\Theta\|_{H^{1-\varepsilon}} \\
&\lesssim \|w\|_{L^1_T L^2_x} \|\psi_0\|_{L^\infty_T W_x^{-\frac{1}{2}-2\varepsilon,\infty}} \|\Theta\|_{H^{1-\varepsilon}},
\end{aligned} \tag{6.2.27}$$

for $\varepsilon > 0$ sufficiently small such that $4\varepsilon \le -s_3$. Hence, by the definition (5.2.30) of the $\mathcal{L}(q,T)$-norm, Cauchy's inequality, and (6.2.21), we obtain

$$\begin{aligned}
&\big\| \widetilde{\mathfrak{J}}^N_{\textcircled{<},\ominus}[\psi_0,\psi_1]\big\|_{\mathcal{L}_2(q,T)} \\
&\lesssim T^{\frac{q-1}{q}} \|\psi_0\|_{L^\infty_T W_x^{-\frac{1}{2}-2\varepsilon,\infty}} \|\Theta\|_{H^{1-\varepsilon}} \\
&\lesssim T^{\frac{q-1}{q}} \big(\|\psi_0\|^2_{L^\infty_T W_x^{-\frac{1}{2}-\varepsilon,\infty}} + \|\Upsilon^N\|^2_{H^1} + \|\mathfrak{Z}_N\|^2_{W^{1-\varepsilon,\infty}} \big).
\end{aligned} \tag{6.2.28}$$

Proceeding as in (6.2.27) and applying Sobolev's embedding theorem with (6.2.21) and (6.2.23), we have

$$\begin{aligned}
&\big\| \widetilde{\mathfrak{J}}^N_{\textcircled{<},\ominus}[\psi_1,\psi_1]\big\|_{\mathcal{L}_2(q,T)} \\
&\lesssim T^{\frac{q-1}{q}} \|\psi_1\|_{L^\infty_T W_x^{-\frac{1}{2}-2\varepsilon,\infty}} \|\Theta\|_{H^{1-\varepsilon}} \lesssim T^{\frac{q-1}{q}} \|\Theta\|^2_{H^{1-\varepsilon}} \\
&\lesssim T^{\frac{q-1}{q}} \big(\|\Upsilon^N\|^2_{H^1} + \|\mathfrak{Z}_N\|^2_{W^{1-\varepsilon,\infty}} \big).
\end{aligned} \tag{6.2.29}$$

[2] To be more precise, this is the Coifman–Meyer theorem on $\mathbb{T}^3$ to estimate a resonant product. The Coifman–Meyer theorem on $\mathbb{T}^3$ follows from the Coifman–Meyer theorem for functions on $\mathbb{R}^d$ [31, Theorem 7.5.3] and the transference principle [26, Theorem 3]. We may equally proceed with (2.1.9) in Lemma 2.1.2 with a slight loss of derivative which does not affect the estimate.

Finally, from Lemmas 2.1.2, 5.3.1, Sobolev's inequality, and (6.2.20), we have

$$\begin{aligned}
&\big\|\widetilde{\mathfrak{I}}^N_{\otimes,\ominus}[\psi_1,\psi_0](w)\big\|_{L^\infty_T H^{s_3}_x}\\
&\quad\le \big\|\widetilde{\mathfrak{I}}^{(2),N}_{\otimes}[\psi_1](w)\ominus(\pi_N\psi_0)\big\|_{L^\infty_T L^2_x}\\
&\quad\lesssim \|\mathcal{I}(\mathcal{K}^\theta(w,\pi_N\psi_1))\|_{L^\infty_T H^{\frac12+2\varepsilon}_x}\|\psi_0\|_{L^\infty_T W^{-\frac12-\varepsilon,\infty}_x}\\
&\quad\lesssim \|\mathcal{K}^\theta(w,\pi_N\psi_1)\|_{L^1_T H^{-\frac12+2\varepsilon}_x}\|\psi_0\|_{L^\infty_1 W^{-\frac12-\varepsilon,\infty}_x}\\
&\quad\lesssim \|\mathcal{K}^\theta(w,\pi_N\psi_1)\|_{L^1_T L^{\frac{3}{2-2\varepsilon}}_x}\|\psi_0\|_{L^\infty_T W^{-\frac12-\varepsilon,\infty}_x}\\
&\quad\lesssim \|w\|_{L^q_T L^2_x}\|\psi_1\|_{L^{q'}_T B^0_{\frac{6}{1-4\varepsilon},2}}\|\psi_0\|_{L^\infty_T W^{-\frac12-\varepsilon,\infty}_x}.
\end{aligned} \tag{6.2.30}$$

Note that $(\frac{1}{3\varepsilon},\frac{6}{1-4\varepsilon})$ is $(1-\varepsilon)$-admissible. Since $q>1$, we can choose $\varepsilon>0$ sufficiently small such that $q'\le\frac{1}{3\varepsilon}$. Then, by Minkowski's integral inequality, (6.2.23), and Lemma 5.3.1, we have

$$\|\psi_1\|_{L^{q'}_T B^0_{\frac{6}{1-4\varepsilon},2}}\le\bigg(\sum_{j=0}^\infty\|S(t)(\mathbf{P}_j\Theta,0)\|^2_{L^{q'}_T L^{\frac{6}{1-4\varepsilon}}_x}\bigg)^{\frac12}\lesssim\|\Theta\|_{H^{1-\varepsilon}}, \tag{6.2.31}$$

where $\mathbf{P}_j$ is the Littlewood–Paley projector onto the frequencies $\{|n|\sim 2^j\}$. Hence, from (5.2.30), (6.2.30), (6.2.31), and Cauchy's inequality with (6.2.21), we obtain

$$\begin{aligned}
&\big\|\widetilde{\mathfrak{I}}^N_{\otimes,\ominus}[\psi_1,\psi_0]\big\|_{\mathcal{L}_2(q,T)}\\
&\quad\le C(T)\|\psi_0\|_{L^\infty_T W^{-\frac12-\varepsilon,\infty}_x}\|\Theta\|_{H^{1-\varepsilon}}\\
&\quad\le C(T)\big(\|\psi_0\|^2_{L^\infty_T W^{-\frac12-\varepsilon,\infty}_x}+\|\Upsilon^N\|^2_{H^1}+\|\mathfrak{Z}_N\|^2_{W^{1-\varepsilon,\infty}}\big).
\end{aligned} \tag{6.2.32}$$

By (6.2.24), (6.2.25), (6.2.28), (6.2.29), (6.2.32), and Young's inequality (with $\alpha<1$) we have

$$\begin{aligned}
&\inf_{\dot\Upsilon^N\in\mathbb{H}^1_a}\mathbb{E}\Big[-\big\|\widetilde{\mathfrak{I}}^N_{\otimes,\ominus}[\upharpoonleft(Y+\Theta,u_1,\omega_2)]\big\|^\alpha_{\mathcal{L}_2(q,T)}+\varepsilon_0\big(\|\Upsilon^N\|^2_{H^1}+\|\Upsilon_N\|^6_{L^2}\big)\Big]\\
&\ge -c\mathbb{E}\big[K(Y,u_1,\omega_2)^{2\alpha}\big]+\inf_{\dot\Upsilon^N\in\mathbb{H}^1_a}\big(-c\|\Upsilon^N\|^{2\alpha}_{H^1}+\varepsilon_0\|\Upsilon^N\|^2_{H^1}\big)-C_1\\
&\gtrsim -\mathbb{E}\big[K(Y,u_1,\omega_2)^{2\alpha}\big]-C_2.
\end{aligned} \tag{6.2.33}$$

Therefore, from (6.2.22), (6.2.33), Young's inequality, and Jensen's inequality, we obtain

$$\begin{aligned}
&\int \exp\Big(\big\|\mathfrak{I}^{N}_{\circledcirc,\ominus}[\upharpoonleft(\vec{u}_0,\omega_2)]\big\|^{\alpha}_{\mathcal{L}_2(q,1)}\Big)d\rho_N(u_0)\\
&\lesssim \exp\Big(C\mathbb{E}\big[K(Y,u_1,\omega_2)^{2\alpha}\big]\Big)\\
&\le \exp\Big(\delta\mathbb{E}\big[K(Y,u_1,\omega_2)\big]\Big)\\
&\le \int \exp\big(\delta K(Y,u_1,\omega_2)\big)d\mu(Y)
\end{aligned}$$

for $0<\alpha<\frac{1}{2}$. Finally, by integrating in (u_1,ω_2) with respect to $\mu_2\otimes\mathbb{P}_2$, we obtain the desired bound (6.2.14) from (6.2.26).

Part 2. Next, we briefly discuss how to prove (6.2.13) for the random operator $\widetilde{\mathfrak{I}}^{N}_{\circledcirc,\ominus}$. For $N\ge N_1\ge N_2\ge 1$, proceeding as in Part 1, we arrive at

$$\begin{aligned}
&-\log\int \exp\Big(N_2^{\beta}\big\|\widetilde{\mathfrak{I}}^{N_1}_{\circledcirc,\ominus}[\upharpoonleft(\vec{u}_0,\omega_2)]-\widetilde{\mathfrak{I}}^{N_2}_{\circledcirc,\ominus}[\upharpoonleft(\vec{u}_0,\omega_2)]\big\|^{\alpha}_{\mathcal{L}_2(q,T)}\Big)d\rho_N(u_0)\\
&\ge \inf_{\dot{\Upsilon}^N\in\mathbb{H}^1_a}\mathbb{E}\Big[-N_2^{\beta}\big\|\widetilde{\mathfrak{I}}^{N_1}_{\circledcirc,\ominus}[\upharpoonleft(Y+\Theta,u_1,\omega_2)]\\
&\qquad\qquad -\widetilde{\mathfrak{I}}^{N_2}_{\circledcirc,\ominus}[\upharpoonleft(Y+\Theta,u_1,\omega_2)]\big\|^{\alpha}_{\mathcal{L}_2(q,T)}\\
&\qquad\qquad +\varepsilon_0\big(\|\Upsilon^N\|^2_{H^1}+\|\Upsilon_N\|^6_{L^2}\big)\Big]-C_0,
\end{aligned}$$

uniformly in u_1 and ω_2. See (6.2.22). With ψ_0 and ψ_1 as in (6.2.23), we write

$$\begin{aligned}
&N_2^{\frac{\beta}{\alpha}}\big\|\widetilde{\mathfrak{I}}^{N_1}_{\circledcirc,\ominus}[\upharpoonleft(Y+\Theta,u_1,\omega_2)]-\widetilde{\mathfrak{I}}^{N_2}_{\circledcirc,\ominus}[\upharpoonleft(Y+\Theta,u_1,\omega_2)]\big\|_{\mathcal{L}_2(q,T)}\\
&\quad\le N_2^{\frac{\beta}{\alpha}}\big\|\widetilde{\mathfrak{I}}^{N_1}_{\circledcirc,\ominus}[\psi_0,\psi_0]-\widetilde{\mathfrak{I}}^{N_2}_{\circledcirc,\ominus}[\psi_0,\psi_0]\big\|_{\mathcal{L}_2(q,T)}\\
&\qquad+N_2^{\frac{\beta}{\alpha}}\big\|\widetilde{\mathfrak{I}}^{N_1}_{\circledcirc,\ominus}[\psi_0,\psi_1]-\widetilde{\mathfrak{I}}^{N_2}_{\circledcirc,\ominus}[\psi_0,\psi_1]\big\|_{\mathcal{L}_2(q,T)}\\
&\qquad+N_2^{\frac{\beta}{\alpha}}\big\|\widetilde{\mathfrak{I}}^{N_1}_{\circledcirc,\ominus}[\psi_1,\psi_0]-\widetilde{\mathfrak{I}}^{N_2}_{\circledcirc,\ominus}[\psi_1,\psi_0]\big\|_{\mathcal{L}_2(q,T)}\\
&\qquad+N_2^{\frac{\beta}{\alpha}}\big\|\widetilde{\mathfrak{I}}^{N_1}_{\circledcirc,\ominus}[\psi_0,\psi_1]-\widetilde{\mathfrak{I}}^{N_2}_{\circledcirc,\ominus}[\psi_1,\psi_1]\big\|_{\mathcal{L}_2(q,T)}.
\end{aligned}\tag{6.2.34}$$

In view of Remark 6.1.1 (see also Lemma 5.4.6 and Remark 5.4.2), we see that there exists $K(Y,u_1,\omega_2)$ such that

$$\begin{aligned}
&N_2^{\frac{\beta}{\alpha}}\big\|\widetilde{\mathfrak{I}}^{N_1}_{\circledcirc,\ominus}[\psi_0,\psi_0]-\widetilde{\mathfrak{I}}^{N_2}_{\circledcirc,\ominus}[\psi_0,\psi_0]\big\|_{\mathcal{L}_2(q,T)}\\
&\qquad+\|\psi_0\|^2_{L^\infty_T W_x^{-\frac{1}{2}-\varepsilon,\infty}}+\|\mathfrak{Z}_N\|_{W^{1-\varepsilon,\infty}}\le \widetilde{K}(Y,u_1,\omega_2)
\end{aligned}\tag{6.2.35}$$

and

$$\mathbb{E}_{\vec{\mu}\otimes\mathbb{P}_2}\big[\exp(\delta\widetilde{K}(Y,u_1,\omega_2))\big]<\infty \tag{6.2.36}$$

for sufficiently small $\delta > 0$, provided that $\beta > 0$ is sufficiently small. The last three terms on the right-hand side of (6.2.34) can be handled as in (6.2.28), (6.2.29), and (6.2.32). By noting that one of the factors comes with $\pi_{N_1} - \pi_{N_2}$, we gain a small negative power of N_2 by losing small regularity in (6.2.28), (6.2.29), and (6.2.32), while keeping the resulting regularities on the right-hand sides unchanged. This allows us to hide $N_2^{\frac{\beta}{\alpha}}$ in (6.2.34). The rest of the argument follows precisely as in Part 1. ∎

Remark 6.2.5. In the proof of Proposition 6.2.4, we only treated $\widetilde{\mathfrak{Z}}^N_{\circledcirc, \ominus}$ from the truncated enhanced data set $\Xi_N(\vec{u}_0, \omega_2)$ in (6.1.10). Let us briefly discuss how to treat the other terms in $\Xi_N(\vec{u}_0, \omega_2)$ to get the exponential integrability bound (6.2.12). The second bound (6.2.13) follows in a similar manner. The terms $\uparrow_N$, $\vee_N$, $\curlyvee_N$, and $\mathbb{A}_N$ can be estimated in a similar manner since they are (at most) quadratic in $\uparrow(Y + \Theta, u_1, \omega_2)$ and the product $\psi_0\psi_1$ is well defined, where ψ_j, $j = 0, 1$, is as in (6.2.23).

As for $\curlyvee_N^{\ominus}$, with the notation above and (6.2.23), we have

$$
\begin{aligned}
&\curlyvee^{\ominus}_N[\uparrow(Y + \Theta, u_1, \omega_2)] \\
&\quad = \curlyvee^{\ominus}_N[\psi_0 + \psi_1] \\
&\quad = \curlyvee_N[\psi_0 + \psi_1] \ominus (\pi_N\psi_0) + \curlyvee_N[\psi_0 + \psi_1] \ominus (\pi_N\psi_1).
\end{aligned}
\tag{6.2.37}
$$

Let $0 < \alpha < \frac{1}{3}$. Then, by Lemma 2.1.2 and Young's inequality, we can estimate the second term on the right-hand side as

$$
\begin{aligned}
\|\curlyvee_N[\psi_0 + \psi_1] \ominus (\pi_N\psi_1)\|^{\alpha}_{C_T H_x^{-\varepsilon}}
&\lesssim \|\curlyvee_N[\psi_0 + \psi_1]\|^{\alpha}_{C_T W_x^{\frac{1}{2}-\varepsilon,\infty}} \|\psi_1\|^{\alpha}_{C_T H_x^{1-\varepsilon}} \\
&\lesssim \|\curlyvee_N[\psi_0 + \psi_1]\|^{\frac{3}{2}\alpha}_{C_T W_x^{\frac{1}{2}-\varepsilon,\infty}} + \|\psi_1\|^{3\alpha}_{C_T H_x^{1-\varepsilon}}.
\end{aligned}
\tag{6.2.38}
$$

Noting that $\frac{3}{2}\alpha < \frac{1}{2}$ and $3\alpha < 1$, we can control the first term on the right-hand side of (6.2.38) by the exponential integrability bound for $\curlyvee_N$ under $\vec{\rho}_N \otimes \mathbb{P}_2$, while by Young's inequality with (6.2.23) and (6.2.21), we can bound the second term by

$$
\delta\big(\|\Upsilon^N\|_{H^1} + \|\mathfrak{Z}_N\|_{W^{1-\varepsilon,\infty}}\big) + C_\delta,
\tag{6.2.39}
$$

for any small $\delta > 0$.

Let us consider the first term on the right-hand side of (6.2.37). In view of (6.1.7), by writing

$$
\begin{aligned}
\curlyvee_N[\psi_0 + \psi_1] \ominus (\pi_N\psi_0) = \curlyvee_N[\psi_0] \ominus (\pi_N\psi_0)
&+ 2\big(\pi_N \mathcal{I}\big((\pi_N\psi_0)(\pi_N\psi_1)\big)\big) \ominus (\pi_N\psi_0) \\
&+ \big(\pi_N \mathcal{I}\big((\pi_N\psi_1)^2\big)\big) \ominus (\pi_N\psi_0).
\end{aligned}
\tag{6.2.40}
$$

Note that we have $\curlyvee_N[\psi_0] \circleddash (\pi_N \psi_0) = \curlyvee_N((Y, u_1), \omega_2)$, where the latter term is as in (6.1.7). While there is an extra frequency cutoff as compared to $\curlyvee_N$ in Lemma 5.4.4, the conclusion of Lemma 5.4.4 also holds for $\curlyvee_N[\psi_0] \circleddash (\pi_N \psi_0) = \curlyvee_N((Y, u_1), \omega_2)$. Hence, we can control the first term on the right-hand side of (6.2.40) by the exponential tail estimate in Lemma 5.4.4 with $0 < \alpha < \frac{1}{3}$. The third term on the right-hand side of (6.2.40) causes no issue since the resonant product of $\pi_N \mathcal{I}((\pi_N \psi_1)^2)$ and $\pi_N \psi_0$ is well defined.

Lastly, let us consider the second term on the right-hand side of (6.2.40). In view of (6.2.15), (6.2.16), and (6.2.18), we have

$$\begin{aligned}
&\big(\pi_N \mathcal{I}\big((\pi_N \psi_0)(\pi_N \psi_1)\big)\big) \circleddash (\pi_N \psi_0) \\
&\quad = \big(\pi_N \mathcal{I}\big((\pi_N \psi_1) \circledgtr (\pi_N \psi_0)\big)\big) \circleddash (\pi_N \psi_0) \\
&\qquad + \widetilde{\mathfrak{I}}^{(1),N}_{\circledless}[\psi_0](\pi_N \psi_1) \circleddash (\pi_N \psi_0) + \widetilde{\mathfrak{I}}^{N}_{\circledless,\circleddash}[\psi_0](\pi_N \psi_1),
\end{aligned} \tag{6.2.41}$$

where $\widetilde{\mathfrak{I}}^{(1),N}_{\circledless}[\psi_0]$ is defined by

$$\widetilde{\mathfrak{I}}^{(1),N}_{\circledless}[\psi_0] := \widetilde{\mathfrak{I}}^{N}_{\circledless}[\psi_0] - \widetilde{\mathfrak{I}}^{(2),N}_{\circledless}[\psi_0]. \tag{6.2.42}$$

From Lemma 2.1.2 and the one degree of smoothing from the Duhamel integral operator $\mathcal{I}$, we see that $\mathcal{I}((\pi_N \psi_1) \circledgtr (\pi_N \psi_0)) \in C([0, T]; H^{\frac{3}{2}-3\varepsilon}(\mathbb{T}^3))$, which allows us to handle the first term on the right-hand side of (6.2.41).

Next, we estimate the second term on the right-hand side of (6.2.41). Recall from (6.2.23) that $\psi_0 = \uparrow(Y, u_1, \omega_2)$ with $\mathrm{Law}(Y, u_1) = \vec{\mu}$. Namely, $\widetilde{\mathfrak{I}}^{(1),N}_{\circledless}[\psi_0]$ defined in (6.2.42) is nothing but $\mathfrak{I}^{(1),N}_{\circledless}$ in Lemma 5.4.5 with an extra frequency cutoff $\chi_N(n)$. Hence, the conclusion of Lemma 5.4.5 (in particular (5.4.5)) holds true for $\widetilde{\mathfrak{I}}^{(1),N}_{\circledless}[\psi_0]$. Then, from Lemmas 2.1.2 and 5.4.5, we have

$$\begin{aligned}
&\big\| \widetilde{\mathfrak{I}}^{(1),N}_{\circledless}[\psi_0](\pi_N \psi_1) \circleddash (\pi_N \psi_0) \big\|^{\alpha}_{C_T H_x^{-\varepsilon}} \\
&\quad \lesssim \big\| \widetilde{\mathfrak{I}}^{(1),N}_{\circledless}[\psi_0](\pi_N \psi_1) \big\|^{\alpha}_{C_T H_x^{\frac{1}{2}+3\varepsilon}} \|\psi_0\|^{\alpha}_{C_T W_x^{-\frac{1}{2}-\varepsilon,\infty}} \\
&\quad \le C(T) \|\psi_1\|^{\alpha}_{C_T H_x^{1-\varepsilon}} \|\psi_0\|^{2\alpha}_{C_T W_x^{-\frac{1}{2}-\varepsilon,\infty}}.
\end{aligned}$$

Then, Young's inequality allows us to handle this term.

Finally, we treat the third term on the right-hand side of (6.2.41). From (5.2.30) and Young's inequality, we have

$$\begin{aligned}
\big\| \widetilde{\mathfrak{I}}^{N}_{\circledless,\circleddash}[\psi_0](\pi_N \psi_1) \big\|^{\alpha}_{C_T H_x^{-\varepsilon}} &\le \big\| \widetilde{\mathfrak{I}}^{N}_{\circledless,\circleddash}[\psi_0] \big\|^{\alpha}_{\mathfrak{L}(\frac{3}{2},T)} \|\psi_1\|^{\alpha}_{L_T^{\frac{3}{2}} L_x^2} \\
&\lesssim C(T)\Big(\big\| \widetilde{\mathfrak{I}}^{N}_{\circledless,\circleddash}[\psi_0] \big\|^{\frac{3}{2}\alpha}_{\mathfrak{L}(\frac{3}{2},T)} + \|\psi_1\|^{3\alpha}_{L_T^{\frac{3}{2}} L_x^2} \Big),
\end{aligned}$$

which can be controlled by (6.2.14) and (6.2.39).

Therefore, Proposition 6.2.4 holds for all the elements in the truncated enhanced data set $\Xi_N(\vec{u}_0, \omega_2)$ in (6.1.10).

We conclude this section by constructing the full enhanced data set $\Xi(\vec{u}_0, \omega_2)$ in (6.1.11) under $\vec{\rho} \otimes \mathbb{P}_2$ as a limit of the truncated enhanced data set $\Xi_N(\vec{u}_0, \omega_2)$ in (6.1.10).

Corollary 6.2.6. *Let $T > 0$. Then, the truncated enhanced data set $\Xi_N(\vec{u}_0, \omega_2)$ in (6.1.10) converges to the enhanced data set $\Xi(\vec{u}_0, \omega_2)$ in (6.1.11), with respect to the $\mathcal{X}_T^\varepsilon$-norm defined in (5.5.3), almost surely and in measure with respect to the limiting measure $\vec{\rho} \otimes \mathbb{P}_2$.*

Proof. Let $0 < \alpha < \frac{1}{3}$ and $\beta > 0$ be as in Proposition 6.2.4. Then, by Fatou's lemma, the weak convergence of $\vec{\rho}_N \otimes \mathbb{P}_2$ to $\vec{\rho} \otimes \mathbb{P}_2$, and Proposition 6.2.4, we have

$$\begin{aligned}
&\int \exp\big(N_2^\beta \|\Xi_{N_1}(\vec{u}_0, \omega_2) - \Xi_{N_2}(\vec{u}_0, \omega_2)\|_{\mathcal{X}_T^\varepsilon}^\alpha\big) d(\vec{\rho} \otimes \mathbb{P}_2)(\vec{u}_0, \omega_2)\\
&\le \liminf_{L\to\infty} \int \exp\Big(\min\big(N_2^\beta \|\Xi_{N_1}(\vec{u}_0, \omega_2) - \Xi_{N_2}(\vec{u}_0, \omega_2)\|_{\mathcal{X}_T^\varepsilon}^\alpha, L\big)\Big) d(\vec{\rho} \otimes \mathbb{P}_2)(\vec{u}_0, \omega_2)\\
&= \liminf_{L\to\infty} \lim_{N\to\infty} \int \exp\Big(\min\big(N_2^\beta \|\Xi_{N_1}(\vec{u}_0, \omega_2)\\
&\qquad\qquad - \Xi_{N_2}(\vec{u}_0, \omega_2)\|_{\mathcal{X}_T^\varepsilon}^\alpha, L\big)\Big) d(\vec{\rho}_N \otimes \mathbb{P}_2)(\vec{u}_0, \omega_2)\\
&\le \lim_{N\to\infty} \int \exp\big(N_2^\beta \|\Xi_{N_1}(\vec{u}_0, \omega_2) - \Xi_{N_2}(\vec{u}_0, \omega_2)\|_{\mathcal{X}_T^\varepsilon}^\alpha\big) d(\vec{\rho}_N \otimes \mathbb{P}_2)(\vec{u}_0, \omega_2)\\
&\lesssim 1,
\end{aligned} \tag{6.2.43}$$

uniformly in $N_1 \ge N_2 \ge 1$. Then, by Chebyshev's inequality, we have

$$\vec{\rho} \otimes \mathbb{P}_2\big(\|\Xi_{N_1}(\vec{u}_0, \omega_2) - \Xi_{N_2}(\vec{u}_0, \omega_2)\|_{\mathcal{X}_T^\varepsilon}^\alpha > \lambda\big) \le Ce^{-cN_2^\beta \lambda^\alpha}$$

for any $\lambda > 0$ and $N_1 \ge N_2 \ge 1$. This shows that $\{\Xi_N(\vec{u}_0, \omega_2)\}_{N\in\mathbb{N}}$ is Cauchy in measure with respect to $\vec{\rho} \otimes \mathbb{P}_2$ and thus converges in measure to the full enhanced data set $\Xi(\vec{u}_0, \omega_2)$ in (6.1.11). By Fatou's lemma and (6.2.43), we also have

$$\int \exp\big(N_2^\beta \|\Xi(\vec{u}_0, \omega_2) - \Xi_{N_2}(\vec{u}_0, \omega_2)\|_{\mathcal{X}_T^\varepsilon}^\alpha\big) d(\vec{\rho} \otimes \mathbb{P}_2)(\vec{u}_0, \omega_2) \lesssim 1,$$

uniformly in $N_1 \ge N_2 \ge 1$, which in turn implies

$$\vec{\rho} \otimes \mathbb{P}_2\big(\|\Xi(\vec{u}_0, \omega_2) - \Xi_{N_2}(\vec{u}_0, \omega_2)\|_{\mathcal{X}_T^\varepsilon}^\alpha > \lambda\big) \le Ce^{-cN_2^\beta \lambda^\alpha}$$

for any $\lambda > 0$ and $N_2 \in \mathbb{N}$. By summing in $N_2 \in \mathbb{N}$ and invoking the Borel–Cantelli lemma, we also conclude almost sure convergence $\Xi_N(\vec{u}_0, \omega_2)$ to $\Xi(\vec{u}_0, \omega_2)$ with respect to $\vec{\rho} \otimes \mathbb{P}_2$. ∎

6.3 Proof of Theorem 1.3.2

In this section, we present the proof of Theorem 1.3.2. The main task is to prove convergence of the solution $(u_N, \partial_t u_N)$ to the truncated hyperbolic Φ^3_3-model (6.1.2). We first carry out Steps 2, 3, and 4 described at the beginning of this chapter. Namely, we first establish a stability result (Proposition 6.3.1) as a slight modification of the local well-posedness argument (Theorem 5.2.1). Next, we establish a uniform (in N) control on the solution $(X_N, Y_N, \mathfrak{R}_N)$ to the truncated system (see (6.3.1) below) with respect to the truncated measure $\rho_N \times \mathbb{P}_2$ (Proposition 6.3.2). Then, by using ideas from theory of optimal transport, we study the convergence property of the pushforward measure $(\Xi_N)_\#(\vec{\rho}_N \otimes \mathbb{P}_2)$ to $(\Xi)_\#(\vec{\rho} \otimes \mathbb{P}_2)$ with respect to the Wasserstein-1 distance (Proposition 6.3.3).

Let $\Phi_1^N(t)(\vec{u}_0, \omega_2)$ be the first component of $\Phi^N(t)(\vec{u}_0, \omega_2)$ in (6.2.8). Then, by decomposing $\Phi_1^N(t)(\vec{u}_0, \omega_2)$ as in (5.2.10):

$$\Phi_1^N(t)(\vec{u}_0, \omega_2) = \uparrow(t; \vec{u}_0, \omega_2) + \sigma \Upsilon_N(t; \vec{u}_0, \omega_2) + X_N(t) + Y_N(t), \tag{6.3.1}$$

we see that X_N, Y_N, and $\mathfrak{R}_N := X_N \mathbin{\textcircled{=}} \uparrow_N(\vec{u}_0, \omega_2)$ satisfy the following system:

$$\begin{aligned}
&(\partial_t^2 + \partial_t + 1 - \Delta) X_N \\
&\quad = 2\sigma \pi_N\big((X_N + Y_N + \sigma \Upsilon_N) \mathbin{\textcircled{<}} \uparrow_N\big) \\
&\qquad - M(Q_{X_N, Y_N} + 2\mathfrak{R}_N + \sigma^2 \Upsilon_N^2 + 2\sigma \Psi_N + \vee_N) \uparrow_N, \\
&(\partial_t^2 + \partial_t + 1 - \Delta) Y_N \\
&\quad = \sigma \pi_N\Big((X_N + Y_N + \sigma \Upsilon_N)^2 + 2\big(\mathfrak{R}_N + Y_N \mathbin{\textcircled{=}} \uparrow_N + \sigma \Psi^{\textcircled{=}}_N\big) \\
&\qquad + 2\big(X_N + Y_N + \sigma \Upsilon_N\big) \mathbin{\textcircled{>}} \uparrow_N\Big) \qquad\qquad (6.3.2)\\
&\qquad - M\big(Q_{X_N, Y_N} + 2\mathfrak{R}_N + \sigma^2 \Upsilon_N^2 + 2\sigma \Psi_N + \vee_N\big)(X_N + Y_N + \sigma \Upsilon_N), \\
&\mathfrak{R}_N = 2\sigma \widetilde{\mathfrak{I}}^{(1), N}_{\textcircled{<}}(X_N + Y_N + \sigma \Upsilon_N) \mathbin{\textcircled{=}} \uparrow_N \\
&\qquad + 2\sigma \widetilde{\mathfrak{I}}^{N}_{\textcircled{<}, \textcircled{=}}(X_N + Y_N + \sigma \Upsilon_N) \\
&\qquad - \int_0^t M\big(Q_{X_N, Y_N} + 2\mathfrak{R}_N + \sigma^2 \Upsilon_N^2 + 2\sigma \Psi_N + \vee_N\big)(t') \mathbb{A}_N(t, t') dt', \\
&(X_N, \partial_t X_N, Y_N, \partial_t Y_N)|_{t=0} = (0, 0, 0, 0),
\end{aligned}$$

where M is as in (1.3.2), Q_{X_N, Y_N} is as in (5.2.15) with $\uparrow$ replaced by $\uparrow_N = \uparrow_N(\vec{u}_0, \omega_2)$ as in (6.1.6), and the enhanced data set is given by $\Xi_N(\vec{u}_0, \omega_2)$ in (6.1.10).

We first establish the following stability result. The main idea is that by introducing a norm with an exponential decaying weight in time (see (6.3.7)), the proof essentially follows from a straightforward modification of the local well-posedness argument (Theorem 5.2.1). A simple, but key observation is (6.3.9) below.

Proposition 6.3.1. *Let $T \gg 1$, $K \gg 1$, and $C_0 \gg 1$. Then, there exist $N_0(T, K, C_0) \in \mathbb{N}$ and small $\kappa_0 = \kappa_0(T, K, C_0) > 0$ such that the following statements hold. Suppose that for some $N \geq N_0$, we have*

$$\|\Xi_N(\vec{u}'_0, \omega'_2)\|_{\mathcal{X}^\varepsilon_T} \leq K \tag{6.3.3}$$

and

$$\|(X_N, Y_N, \mathfrak{R}_N)\|_{Z^{s_1,s_2,s_3}(T)} \leq C_0 \tag{6.3.4}$$

for the solution to $(X_N, Y_N, \mathfrak{R}_N)$ to the truncated system (6.3.2) on $[0, T]$ with the truncated enhanced data set $\Xi_N(\vec{u}'_0, \omega'_2)$. Furthermore, suppose that we have

$$\|\Xi(\vec{u}_0, \omega_2) - \Xi_N(\vec{u}'_0, \omega'_2)\|_{\mathcal{X}^\varepsilon_T} \leq \kappa \tag{6.3.5}$$

for some $0 < \kappa \leq \kappa_0$ and some $(\vec{u}_0, \omega_2)$, where $\Xi(\vec{u}_0, \omega_2)$ denotes the enhanced data set in (6.1.11). Then, there exists a solution $(X, Y, \mathfrak{R})$ to the full system (5.2.27) on $[0, T]$ with the zero initial data and the enhanced data set $\Xi(\vec{u}_0, \omega_2)$, satisfying the bound

$$\|(X, Y, \mathfrak{R})\|_{Z^{s_1,s_2,s_3}(T)} \leq C_0 + 1.$$

Conversely, suppose that

$$\|\Xi(\vec{u}_0, \omega_2)\|_{\mathcal{X}^\varepsilon_T} \leq K$$

and that the full system (5.2.27) with the zero initial data and the enhanced data set $\Xi(\vec{u}_0, \omega_2)$ has a solution $(X, Y, \mathfrak{R})$ on $[0, T]$, satisfying

$$\|(X, Y, \mathfrak{R})\|_{Z^{s_1,s_2,s_3}(T)} \leq C_0.$$

Then, if (6.3.5) holds for some $N \geq N_0$, $0 < \kappa \leq \kappa_0$, and $(\vec{u}'_0, \omega'_2)$, then there exists a solution $(X_N, Y_N, \mathfrak{R}_N)$ to the truncated system (6.3.2) on $[0, T]$ with the enhanced data set $\Xi_N(\vec{u}'_0, \omega'_2)$, satisfying

$$\|(X_N, Y_N, \mathfrak{R}_N) - (X, Y, \mathfrak{R})\|_{Z^{s_1,s_2,s_3}(T)} \leq A(T, K, C_0)(\kappa + N^{-\delta}) \tag{6.3.6}$$

for some $A(T, K, C_0) > 0$ and some small $\delta > 0$.

Proof. Fix $T \gg 1$. Given $\lambda \geq 1$ (to be determined later), we define $\mathcal{Z}^{s_1,s_2,s_3}_\lambda(T)$ by

$$\|(X, Y, \mathfrak{R})\|_{\mathcal{Z}^{s_1,s_2,s_3}_\lambda(T)} = \|(e^{-\lambda t} X, e^{-\lambda t} Y, e^{-\lambda t} \mathfrak{R})\|_{Z^{s_1,s_2,s_3}(T)}. \tag{6.3.7}$$

For notational simplicity, we set $Z = (X, Y, Z)$, $Z_N = (X_N, Y_N, \mathfrak{R}_N)$, $\Xi = \Xi(\vec{u}_0, \omega_2)$, and $\Xi_N = \Xi_N(\vec{u}'_0, \omega'_2)$.

In the following, given $N \in \mathbb{N}$, we assume that (6.3.3), (6.3.4), and (6.3.5) hold. Without loss of generality, assume that $\kappa \leq 1$. Then, from (6.3.3) and (6.3.5), we have

$$\|\Xi(\vec{u}_0, \omega_2)\|_{\mathcal{X}^\varepsilon_T} \leq K + \kappa \leq K + 1 =: K_0. \tag{6.3.8}$$

In the following, we study the difference of the Duhamel formulation[3] (5.5.2) of the system (5.2.27) with the zero initial data (i.e. $(X_0, X_1, Y_0, Y_1) = (0,0,0,0)$) and the Duhamel formulation of the truncated system (6.3.2) with respect to the $\mathcal{Z}_\lambda^{s_1,s_2,s_3}(T)$-norm by choosing appropriate $\lambda = \lambda(T, K_0, R) \gg 1$. See (6.3.16) below.

The main observation is the following bound:

$$e^{-\lambda t}\|e^{\lambda t'}\|_{L^q_{t'}([0,t])} \lesssim \lambda^{-\frac{1}{q}}. \tag{6.3.9}$$

Let $\mathcal{I}$ be the Duhamel integral operator defined in (5.2.5). Then, using (6.3.9), we have

$$\begin{aligned}\|e^{-\lambda t}\mathcal{I}(F)\|_{C_T H^s_x} &\le \bigg\| e^{-\lambda t}\int_0^t e^{\lambda t'}\|e^{-\lambda t'}F(t')\|_{H^{s-1}_x}dt'\bigg\|_{L^\infty_T}\\ &\lesssim \lambda^{-\frac{1}{q}}\|e^{-\lambda t'}F(t')\|_{L^{q'}_T H^{s-1}_x}\end{aligned} \tag{6.3.10}$$

for any $1 \le q \le \infty$. Let (q_1, r_1) be an s_1-admissible pair with $0 < s_1 < 1$. Then, there exists an s_2-admissible pair (q_2, r_2) with $0 < s_1 < s_2 < 1$ such that

$$\frac{1}{q_1} = \frac{\theta}{\infty} + \frac{1-\theta}{q_2}, \quad \frac{1}{r_1} = \frac{\theta}{2} + \frac{1-\theta}{r_2}, \quad \text{and} \quad s_1 = \theta\cdot 0 + (1-\theta)s_2$$

for some $0 < \theta < 1$. By the homogeneous Strichartz estimate ((5.3.2) with $F = 0$), we have

$$\begin{aligned}\|e^{-\lambda t}\mathcal{I}(F)\|_{L^{q_2}_T L^{r_2}_x} &\le \bigg\|\int_0^t e^{-\lambda(t-t')}\mathcal{D}(t-t')(e^{-\lambda t'}F(t'))dt'\bigg\|_{L^{q_2}_T L^{r_2}_x}\\ &\le \int_0^T \|\mathcal{D}(t-t')(e^{-\lambda t'}F(t'))\|_{L^{q_2}_t([0,T];L^{r_2}_x)}dt'\\ &\lesssim \|e^{-\lambda t'}F(t')\|_{L^1_T H^{s_2-1}_x}.\end{aligned} \tag{6.3.11}$$

Thus, given any $\delta > 0$, it follows from interpolating (6.3.10) with large $q \gg 1$ and (6.3.11) that there exists small $\theta = \theta(\delta) > 0$ such that

$$\|e^{-\lambda t}\mathcal{I}(F)\|_{L^{q_1}_T L^{r_1}_x} \le C(T)\lambda^{-\theta}\|e^{-\lambda t'}F(t')\|_{L^{1+\delta}_T H^{s_1-1}_x}. \tag{6.3.12}$$

Recalling that $(4,4)$ is $\frac{1}{2}$-admissible, it follows from (6.3.10), (6.3.12), and Sobolev's inequality that

$$\begin{aligned}\|e^{-\lambda t}\mathcal{I}(F)\|_{C_T\mathcal{H}^{\frac{1}{2}}_x \cap L^4_T L^4_x} &\le C(T)\lambda^{-\theta}\|e^{-\lambda t'}F(t')\|_{L^{1+\delta}_T H^{-\frac{1}{2}}_x}\\ &\le C(T)\lambda^{-\theta}\|e^{-\lambda t'}F(t')\|_{L^{1+\delta}_T L^{\frac{3}{2}}_x}.\end{aligned} \tag{6.3.13}$$

[3]Recall that we set $\sigma = 1$ in Chapter 5 for simplicity and thus need to insert σ in appropriate locations of (5.5.2).

By writing (6.3.2) in the Duhamel formulation, we have

$$
\begin{aligned}
X_N &= \Phi_{1,N}(X_N, Y_N, \mathfrak{R}_N) \\
&:= 2\sigma\pi_N \mathcal{I}\big((X_N + Y_N + \sigma\Upsilon_N) \circledless \upharpoonleft_N\big) \\
&\quad - \mathcal{I}\big(M\big(Q_{X_N,Y_N} + 2\mathfrak{R}_N + \sigma^2\Upsilon_N^2 + 2\sigma\Upsilon\!\!\cdot_N + \curlyvee_N\big)\upharpoonleft_N\big), \\
Y_N &= \Phi_{2,N}(X_N, Y_N, \mathfrak{R}_N) \\
&:= \sigma\pi_N \mathcal{I}\big((X_N + Y_N + \sigma\Upsilon_N)^2\big) \\
&\quad + 2\sigma\pi_N \mathcal{I}\big(\mathfrak{R}_N + Y_N \circleddash \upharpoonleft_N + \sigma \Upsilon_{\circleddash,N}\big) \\
&\quad + 2\sigma\pi_N \mathcal{I}\big((X_N + Y_N + \sigma\Upsilon_N) \circledgtr \upharpoonleft_N\big) \qquad\qquad (6.3.14)\\
&\quad - \mathcal{I}\big(M\big(Q_{X_N,Y_N} + 2\mathfrak{R}_N + \sigma^2\Upsilon_N^2 \\
&\qquad\quad + 2\sigma\Upsilon\!\!\cdot_N + \curlyvee_N\big)(X_N + Y_N + \sigma\Upsilon_N)\big), \\
\mathfrak{R}_N &= \Phi_{3,N}(X_N, Y_N, \mathfrak{R}_N), \\
&:= 2\sigma\widetilde{\mathfrak{I}}^{(1),N}_{\circledless}(X_N + Y_N + \sigma\Upsilon_N) \circleddash \upharpoonleft_N \\
&\quad + 2\sigma\widetilde{\mathfrak{I}}^{N}_{\circledless,\circleddash}(X_N + Y_N + \sigma\Upsilon_N) \\
&\quad - \int_0^t M\big(Q_{X_N,Y_N} + 2\mathfrak{R}_N + \sigma^2\Upsilon_N^2 + 2\sigma\Upsilon\!\!\cdot_N + \curlyvee_N\big)(t')\mathbb{A}_N(t,t')dt'.
\end{aligned}
$$

Then, $Z - Z_N = (X - X_N, Y - Y_N, \mathfrak{R} - \mathfrak{R}_N)$ satisfies the system

$$
\begin{aligned}
X - X_N &= \Phi_1(X, Y, \mathfrak{R}) - \Phi_{1,N}(X_N, Y_N, \mathfrak{R}_N), \\
Y - Y_N &= \Phi_2(X, Y, \mathfrak{R}) - \Phi_{2,N}(X_N, Y_N, \mathfrak{R}_N), \qquad (6.3.15)\\
\mathfrak{R} - \mathfrak{R}_N &= \Phi_3(X, Y, \mathfrak{R}) - \Phi_{3,N}(X_N, Y_N, \mathfrak{R}_N).
\end{aligned}
$$

By setting

$$
\delta X_N = X - X_N, \quad \delta Y_N = Y - Y_N, \quad \text{and} \quad \delta\mathfrak{R}_N = \mathfrak{R} - \mathfrak{R}_N,
$$

we have

$$
X = \delta X_N + X_N, \quad Y = \delta Y_N + Y_N, \quad \text{and} \quad \mathfrak{R} = \delta\mathfrak{R}_N + \mathfrak{R}_N.
$$

Then, we can view the system (6.3.15) for the system for the unknown

$$
\delta Z_N = (\delta X_N, \delta Y_N, \delta\mathfrak{R}_N)
$$

with given source terms $Z_N = (X_N, Y_N, Z_N)$, Ξ_N, and Ξ. We thus rewrite (6.3.15) as

$$
\begin{aligned}
\delta X_N &= \Psi_1(\delta X_N, \delta Y_N, \delta\mathfrak{R}_N), \\
\delta Y_N &= \Psi_2(\delta X_N, \delta Y_N, \delta\mathfrak{R}_N), \qquad (6.3.16)\\
\delta\mathfrak{R}_N &= \Psi_3(\delta X_N, \delta Y_N, \delta\mathfrak{R}_N),
\end{aligned}
$$

where Ψ_j, $j = 1, 2, 3$, is given by

$$\begin{aligned}&\Psi_j(\delta X_N, \delta Y_N, \delta\mathfrak{R}_N)\\&\quad = \Phi_j(\delta X_N + X_N, \delta Y_N + Y_N, \delta\mathfrak{R}_N + \mathfrak{R}_N) - \Phi_{j,N}(X_N, Y_N, \mathfrak{R}_N).\end{aligned} \tag{6.3.17}$$

We now study the system (6.3.16). We basically repeat the computations in Section 5.5 by first multiplying the Duhamel formulation by $e^{-\lambda t}$ and using (6.3.10), (6.3.12), and (6.3.13) as a replacement of the Strichartz estimates (Lemma 5.3.1). This allows us to place $e^{-\lambda t'}$ on one of the factors of $\delta X_N(t')$, $\delta Y_N(t')$, or $\delta\mathfrak{R}_N(t')$ appearing on the right-hand side of (6.3.16) under some integral operator (with integration in the variable t'). Our main goal is to prove that

$$\vec{\Psi} = (\Psi_1, \Psi_2, \Psi_3) \tag{6.3.18}$$

is a contraction on a small ball in $\mathcal{Z}_\lambda^{s_1,s_2,s_3}(T)$. In the following, however, we first establish bounds on Ψ_j in (6.3.17) for $\delta Z_N \in B_1$, where $B_1 \subset Z^{s_1,s_2,s_3}(T)$ denotes the closed ball of radius 1 (with respect to the $Z^{s_1,s_2,s_3}(T)$-norm) centered at the origin. For $\delta Z_N \in B_1$, it follows from (6.3.4) that

$$\begin{aligned}\|Z\|_{Z^{s_1,s_2,s_3}(T)} &\le \|\delta Z_N\|_{Z^{s_1,s_2,s_3}(T)} + \|Z_N\|_{Z^{s_1,s_2,s_3}(T)}\\&\le 1 + C_0 =: R.\end{aligned} \tag{6.3.19}$$

We first study the first equation in (6.3.16). From (6.3.17) with (5.5.2), (6.3.14), and (6.3.17), we have

$$e^{-\lambda t}\Psi_1(\delta X_N, \delta Y_N, \delta\mathfrak{R}_N)(t) = e^{-\lambda t}\mathrm{I}_1(t) + e^{-\lambda t}\mathrm{I}_2(t) + e^{-\lambda t}\mathrm{I}_3(t), \tag{6.3.20}$$

where (i) I_1 contains the difference of one of the elements in the enhanced data sets Ξ and Ξ_N, (ii) I_2 contains the terms with the high frequency projection $\pi_N^\perp = \mathrm{Id} - \pi_N$ onto the frequencies $\{|n| \gtrsim N\}$, and (iii) I_3 consists of the rest, which contains at least one of the differences δX_N, δY_N, or $\delta\mathfrak{R}_N$ (other than those in $Z = \delta Z_N + Z_N$).

In view of (6.3.5), the contribution from I_1 gives a small number κ, while the contribution from I_2 with $\pi_N^\perp$ gives a small negative power of N by losing a small amount of regularity.[4] Proceeding as in (5.5.7) with (6.3.3), (6.3.4), (6.3.5), (6.3.8), and (6.3.19), we have

$$\begin{aligned}\|e^{-\lambda t}\mathrm{I}_1 + e^{-\lambda t}\mathrm{I}_2\|_{X^{s_1}(T)} &\le C(T)(\kappa + N^{-\delta}K_0)(R^4 + K_0^4)\\&\le C(T)(\kappa + N^{-\delta})K_0(R^4 + K_0^4)\end{aligned} \tag{6.3.21}$$

[4]We have sharp inequalities in (6.1.12) as compared to the regularity condition in Theorem 5.2.1. This allows us to gain a small negative power of N, by losing a small amount of regularity and using $\pi_N^\perp$.

for any $\delta Z_N \in B_1$ and some small $\delta > 0$. As for the last term on the right-hand side of (6.3.20), we use (6.3.10) and (6.3.12) in place of Lemma 5.3.1. Then, a slight modification of (5.5.7) yields

$$\|e^{-\lambda t}\mathrm{I}_3\|_{X^{s_1}(T)} \le C(T)\lambda^{-\theta}K_0\big(R^3\|\delta Z_N\|_{\mathcal{Z}_\lambda^{s_1,s_2,s_3}(T)} + K_0^4\big) \tag{6.3.22}$$

for any $\delta Z_N \in B_1$.

Next, we study the second equation in (6.3.16). As in (6.3.20), we can write

$$e^{-\lambda t}\Psi_2(\delta X_N, \delta Y_N, \delta\mathfrak{R}_N)(t) = e^{-\lambda t}\mathrm{II}_1(t) + e^{-\lambda t}\mathrm{II}_2(t) + e^{-\lambda t}\mathrm{II}_3(t), \tag{6.3.23}$$

where (i) II_1 contains the difference of one of the elements in the enhanced data sets Ξ and Ξ_N, (ii) II_2 contains the terms with the high frequency projection $\pi_N^\perp = \mathrm{Id} - \pi_N$ onto the frequencies $\{|n| \gtrsim N\}$, and (iii) II_3 consists of the rest, which contains at least one of the differences δX_N, δY_N, or $\delta\mathfrak{R}_N$ (other than those in $Z = \delta Z_N + Z_N$). As for the first two terms on the right-hand side of (6.3.23), we can proceed as in (5.5.9) with (6.3.3), (6.3.4), (6.3.5), (6.3.8), and (6.3.19), and obtain

$$\|e^{-\lambda t}\mathrm{II}_1 + e^{-\lambda t}\mathrm{II}_2\|_{Y^{s_2}(T)} \le C(T)(\kappa + N^{-\delta})(R^5 + K_0^5) \tag{6.3.24}$$

for any $\delta Z_N \in B_1$ and some small $\delta > 0$. Before we proceed to study the last term $e^{-\lambda t}\mathrm{II}_3(t)$, let us make a preliminary computation. By the fractional Leibniz rule (Lemma 2.1.3 (i)) and Sobolev's inequality, we have

$$\begin{aligned}\|\langle\nabla\rangle^{s_2-\frac12}(fg)\|_{L^{\frac32}} &\lesssim \|\langle\nabla\rangle^{s_2-\frac12}f\|_{L^{r_1}}\|g\|_{L^{r_2}} + \|f\|_{L^{r_2}}\|\langle\nabla\rangle^{s_2-\frac12}g\|_{L^{r_1}}\\ &\lesssim \|\langle\nabla\rangle^{s_1-\frac14}f\|_{L^{\frac83}}\|\langle\nabla\rangle^{s_1-\frac14}g\|_{L^{\frac83}},\end{aligned} \tag{6.3.25}$$

provided that $\frac{1}{r_1} + \frac{1}{r_2} = \frac23$ with $1 < r_1, r_2 \le \infty$,

$$\frac{s_1 - s_2 + \frac14}{3} \ge \frac38 - \frac{1}{r_1} \quad\text{and}\quad \frac{s_1 - \frac14}{3} \ge \frac38 - \frac{1}{r_2}. \tag{6.3.26}$$

This condition is easily satisfied by taking $s_1 < \frac12 < s_2$ both sufficiently close to $\frac12$ and $r_1 = r_2 = 3$. By (6.3.13), (6.3.25), and Lemma 2.1.3 (i), we have

$$\begin{aligned}&\big\|e^{-\lambda t}\mathcal{I}\big((X_1 + Y_1 + \Xi_0)(X_2 + Y_2 + \Xi_0)\big)\big\|_{Y^{s_2}(T)}\\ &\le C(T)\lambda^{-\theta}\big\|e^{-\lambda t}\langle\nabla\rangle^{s_2-\frac12}\big((X_1 + Y_1 + \Xi_0)(X_2 + Y_2 + \Xi_0)\big)\big\|_{L_T^{1+\delta}L_x^{\frac32}}\\ &\le C(T)\lambda^{-\theta}\big(\|\langle\nabla\rangle^{s_1-\frac14}X_1\|_{L_T^8L_x^{\frac83}} + \|\langle\nabla\rangle^{s_2-\frac12}Y_1\|_{L^4_{T,x}} + \|\langle\nabla\rangle^{s_2-\frac12}\Xi_0\|_{L^\infty_{T,x}}\big)\\ &\quad\times\big(\|e^{-\lambda t}\langle\nabla\rangle^{s_1-\frac14}X_2\|_{L_T^8L_x^{\frac83}} + \|e^{-\lambda t}\langle\nabla\rangle^{s_2-\frac12}Y_2\|_{L^4_{T,x}} + \|\langle\nabla\rangle^{s_2-\frac12}\Xi_0\|_{L^\infty_{T,x}}\big),\end{aligned} \tag{6.3.27}$$

provided that $s_1 < \frac{1}{2} < s_2$ are both sufficiently close to $\frac{1}{2}$. Compare this with (5.5.8). Then, from (6.3.10), (6.3.12), and (6.3.27) with (6.3.3), (6.3.4), (6.3.8), and (6.3.19), a slight modification of (5.5.9) yields

$$\|e^{-\lambda t}\mathrm{II}_3\|_{Y^{s_2}(T)} \le C(T)\lambda^{-\theta}\big(R^4\|\delta Z_N\|_{\mathcal{Z}_\lambda^{s_1,s_2,s_3}(T)} + K_0^5\big) \tag{6.3.28}$$

for any $\delta Z_N \in B_1$.

Finally, we study the third equation in (6.3.16). As in (6.3.20) and (6.3.23), we can write

$$e^{-\lambda t}\Psi_3(\delta X_N, \delta Y_N, \delta\mathfrak{R}_N)(t) = e^{-\lambda t}\mathrm{III}_1(t) + e^{-\lambda t}\mathrm{III}_2(t) + e^{-\lambda t}\mathrm{III}_3(t), \tag{6.3.29}$$

where (i) III_1 contains the difference of one of the elements in the enhanced data sets Ξ and Ξ_N, (ii) III_2 contains the terms with the high frequency projection $\pi_N^\perp = \mathrm{Id} - \pi_N$ onto the frequencies $\{|n| \gtrsim N\}$, and (iii) III_3 consists of the rest, which contains at least one of the differences δX_N, δY_N, or $\delta\mathfrak{R}_N$ (other than those in $Z = \delta Z_N + Z_N$). Proceeding as in (5.5.10) with (6.3.3), (6.3.4), (6.3.5), (6.3.8), and (6.3.19), we have

$$\|e^{-\lambda t}\mathrm{III}_1 + e^{-\lambda t}\mathrm{III}_2\|_{L_T^3 H_x^{s_3}} \le C(T)(\kappa + N^{-\delta})K_0(R^4 + K_0^4) \tag{6.3.30}$$

for any $\delta Z_N \in B_1$ and some small $\delta > 0$. As for the last term on the right-hand side of (6.3.29), let us fist consider the terms with the random operator $\mathfrak{I}_{\circledless,\circleddash}$. By (6.3.8) and (6.3.9), we have

$$\begin{aligned}
&\big\|e^{-\lambda t}\mathfrak{I}_{\circledless,\circleddash}\big(X_1 + Y_1 + \Xi_0\big)(t) - e^{-\lambda t}\mathfrak{I}_{\circledless,\circleddash}\big(X_2 + Y_2 + \Xi_0\big)(t)\big\|_{L_T^3 H_x^{s_3}} \\
&\quad\le K_0\Big\|e^{-\lambda t}\big\|e^{\lambda t'}(e^{-\lambda t'}(X_1 + Y_1 - X_2 - Y_2))\big\|_{L_{t'}^{\frac{3}{2}}([0,t];L_x^2)}\Big\|_{L_T^3} \\
&\quad\le C(T)\lambda^{-\theta}K_0\big(\|e^{-\lambda t}(X_1 - X_2)\|_{L_T^\infty H_x^{s_1}} + \|e^{-\lambda t}(Y_1 - Y_2)\|_{L_T^\infty H_x^{s_2}}\big)
\end{aligned}$$

for some $\theta > 0$. The other terms can be estimated in a similar manner and thus we obtain

$$\|e^{-\lambda t}\mathrm{III}_3\|_{L_T^3 H_x^{s_3}} \le C(T)\lambda^{-\theta}K_0\big(R^3\|\delta Z_N\|_{\mathcal{Z}_\lambda^{s_1,s_2,s_3}(T)} + K_0^4\big) \tag{6.3.31}$$

for any $\delta Z_N \in B_1$.

Hence, putting (6.3.21), (6.3.22), (6.3.24), (6.3.28), (6.3.30), and (6.3.31) together, we obtain

$$\begin{aligned}
\|\vec{\Psi}(\delta Z_N)\|_{\mathcal{Z}_\lambda^{s_1,s_2,s_3}(T)} &\le C(T, K_0, R)\lambda^{-\theta}\|\delta Z_N\|_{\mathcal{Z}_\lambda^{s_1,s_2,s_3}(T)} \\
&\quad + C(T, K_0, R)(\kappa + N^{-\delta})
\end{aligned} \tag{6.3.32}$$

for any $\delta Z_N \in B_1$, where $\vec{\Psi}$ is as in (6.3.18). By a similar computation, we also obtain the difference estimate:

$$\begin{aligned}\|\vec{\Psi}(\delta Z_N^{(1)}) - \vec{\Psi}(\delta Z_N^{(2)})\|_{\mathcal{Z}_\lambda^{s_1,s_2,s_3}(T)} \\ \leq C(T, K_0, R)\lambda^{-\theta}\|\delta Z_N^{(1)} - \delta Z_N^{(2)}\|_{\mathcal{Z}_\lambda^{s_1,s_2,s_3}(T)}\end{aligned} \tag{6.3.33}$$

for any $\delta Z_N^{(1)}, \delta Z_N^{(2)} \in B_1$. We now introduce small $r = r(T, \lambda) > 0$ such that, in view of (6.3.7), we have

$$\|\delta Z_N\|_{\mathcal{Z}^{s_1,s_2,s_3}(T)} \leq e^{\lambda T}\|\delta Z_N\|_{\mathcal{Z}_\lambda^{s_1,s_2,s_3}(T)} \leq e^{\lambda T} r \leq 1 \tag{6.3.34}$$

for any $\delta Z_N \in B_r^\lambda$, where $B_r^\lambda \subset \mathcal{Z}_\lambda^{s_1,s_2,s_3}(T)$ is the closed ball of radius r (with respect to the $\mathcal{Z}_\lambda^{s_1,s_2,s_3}(T)$-norm) centered at the origin. From (6.3.34), we see that both (6.3.32), (6.3.33) hold on B_r^λ. Therefore, by choosing large $\lambda = \lambda(T, K_0, R) \gg 1$, small $\kappa = \kappa(T, K_0, R) > 0$, and large $N_0 = N_0(T, K_0, R) \in \mathbb{N}$, we conclude that $\vec{\Psi}$ is a contraction on B_r^λ for any $N \geq N_0$. Hence, there exists a unique solution $\delta Z_N \in B_r^\lambda$ to the fixed point problem $\delta Z_N = \vec{\Psi}(\delta Z_N)$. We need to check that by setting $Z = \delta Z_N + Z_N$, Z satisfies the Duhamel formulation (5.5.2) of the full system (5.2.27) with the zero initial data and the enhanced data set $\Xi = \Xi(\vec{u}_0, \omega_2)$. From (6.3.16) and (6.3.14), we have

$$\begin{aligned} Z = \delta Z_N + Z_N &= \vec{\Psi}(\delta Z_N) + \vec{\Phi}_N(Z_N) \\ &= \vec{\Phi}(\delta Z_N + Z_N) = \vec{\Phi}(Z), \end{aligned}$$

where $\vec{\Phi}_N = (\Phi_{1,N}, \Phi_{2,N}, \Phi_{3,N})$. This shows that Z indeed satisfies the Duhamel formulation (5.5.2) with the zero initial data and the enhanced data set $\Xi = \Xi(\vec{u}_0, \omega_2)$. Lastly, we point out that from (6.3.8) and (6.3.19), we have $K_0 = K + 1$ and $R = C_0 + 1$ and thus the parameters λ, κ, and N_0 depend on T, K, and C_0.

As for the second claim in this proposition, we write $Z_N = Z - (Z - Z_N)$ and study the system for $\delta Z_N = Z - Z_N$:

$$\delta Z_N = \vec{\Psi}^N(\delta Z_N)$$

where $\vec{\Psi}^N = (\Psi_1^N, \Psi_2^N, \Psi_3^N)$ and Ψ_j^N, $j = 1, 2, 3$, is given by

$$\begin{aligned}&\Psi_j^N(\delta X_N, \delta Y_N, \delta \mathfrak{R}_N) \\ &\quad = \Phi_j(X, Y, \mathfrak{R}) - \Phi_{j,N}(X - \delta X_N, Y - \delta Y_N, \mathfrak{R} - \delta \mathfrak{R}_N).\end{aligned}$$

Here, we view $Z = (X, Y, Z)$, Ξ_N, and Ξ as given source terms. By a slight modification of the computation presented above, we obtain

$$\begin{aligned}\|\vec{\Psi}^N(\delta Z_N)\|_{\mathcal{Z}_\lambda^{s_1,s_2,s_3}(T)} \leq C(T, K_0, R)\lambda^{-\theta}\|\delta Z_N\|_{\mathcal{Z}_\lambda^{s_1,s_2,s_3}(T)} \\ + C(T, K_0, R)(\kappa + N^{-\delta})\end{aligned} \tag{6.3.35}$$

and

$$\begin{aligned}&\|\vec{\Psi}^N(\delta Z_N^{(1)}) - \vec{\Psi}^N(\delta Z_N^{(2)})\|_{\mathcal{Z}_\lambda^{s_1,s_2,s_3}(T)}\\&\quad\le C(T,K_0,R)\lambda^{-\theta}\|\delta Z_N^{(1)} - \delta Z_N^{(2)}\|_{\mathcal{Z}_\lambda^{s_1,s_2,s_3}(T)}\end{aligned}$$

for any $\delta Z_N, \delta Z_N^{(1)}, \delta Z_N^{(2)} \in B_1$. This shows that there exists a solution

$$Z_N = Z - \delta Z_N = \Phi(Z) - \vec{\Psi}^N(\delta Z_N) = \vec{\Phi}_N(Z_N)$$

to the truncated system (6.3.2) on $[0,T]$. Furthermore, from equation (6.3.35) with $\lambda = \lambda(T,K_0,R) \gg 1$, we have

$$\begin{aligned}\|Z - Z_N\|_{Z^{s_1,s_2,s_3}(T)} &\le e^{\lambda T}\|\vec{\Psi}^N(\delta Z_N)\|_{\mathcal{Z}_\lambda^{s_1,s_2,s_3}(T)}\\&\le C(T,K_0,R)e^{\lambda T}(\kappa + N^{-\delta}) \to 0,\end{aligned}$$

as $N \to \infty$ and $\kappa \to 0$. This proves (6.3.6). This concludes the proof of Proposition 6.3.1. ∎

Next, we prove that the solution $(X_N, Y_N, \mathfrak{R}_N)$ to the truncated system (6.3.2) has a uniform bound with a large probability. The proof is based on the invariance of the truncated Gibbs measure $\vec{\rho}_N$ under the truncated hyperbolic Φ^3_3-model (6.1.2) (Lemma 6.2.3) and a discrete Gronwall argument.

Proposition 6.3.2. *Let $T > 0$. Then, given any $\delta > 0$, there exists $C_0 = C_0(T,\delta) \gg 1$ such that*

$$\vec{\rho}_N \otimes \mathbb{P}_2\big(\|(X_N, Y_N, \mathfrak{R}_N)\|_{Z^{s_1,s_2,s_3}(T)} > C_0\big) < \delta, \tag{6.3.36}$$

uniformly in $N \in \mathbb{N}$, where $(X_N, Y_N, \mathfrak{R}_N)$ is the solution to the truncated system (6.3.2) on $[0,T]$ with the truncated enhanced data set $\Xi_N(\vec{u}_0, \omega_2)$ in (6.1.10).

Proof. Let $(u_N, \partial_t u) = \Phi^N(t)(\vec{u}_0, \omega_2)$ be a global solution to (6.1.2) constructed in Lemma 6.2.3, where $\Phi^N(t)(\vec{u}_0, \omega_2)$ is as in (6.2.8). Then, by the invariance of the truncated Gibbs measure $\vec{\rho}_N$ (Lemma 6.2.3), we have

$$\int F(\Phi^N(t)(\vec{u}_0,\omega_2))d(\vec{\rho}_N \otimes \mathbb{P}_2)(\vec{u}_0,\omega_2) = \int F(\vec{u}_0)d\rho_N(\vec{u}_0) \tag{6.3.37}$$

for any bounded continuous function $F : \mathcal{C}^{-100}(\mathbb{T}^3) \times \mathcal{C}^{-100}(\mathbb{T}^3) \to \mathbb{R}$ and $t \in \mathbb{R}_+$. By Minkowski's integral inequality, (6.3.37), (1.3.2), and Proposition 6.2.4, we have, for any finite $p \ge 1$,

$$\begin{aligned}&\Big\|\int_0^T |M(:(\pi_N u_N)^2:)(t)|dt\Big\|_{L^p_{\vec{u}_0,\omega_2}(\vec{\rho}_N\otimes\mathbb{P}_2)}\\&\quad\le \int_0^T \|M(:(\pi_N u_0)^2:)\|_{L^p_{\vec{u}_0,\omega_2}(\vec{\rho}_N\otimes\mathbb{P}_2)}dt\\&\quad\le C(T,p) < \infty,\end{aligned} \tag{6.3.38}$$

for any $0 \le t \le T$ and $p \ge 1$, uniformly in $N \in \mathbb{N}$. By defining

$$v_N := u_N - \uparrow,$$

we see that v_N satisfies the equation

$$(\partial_t^2 + \partial_t + 1 - \Delta)v_N = \sigma\pi_N\big(:(\pi_N u_N)^2:\big) - M\big(:(\pi_N u_N)^2:\big)\pi_N u_N$$

with the zero initial data, or equivalently

$$v_N(t) = \int_0^t e^{-\frac{t-t'}{2}} \frac{\sin((t-t')[\![\nabla]\!])}{[\![\nabla]\!]} \times \big(\sigma\pi_N(:(\pi_N u_N)^2:) - M(:(\pi_N u_N)^2:)\pi_N u_N\big)(t')dt'.$$

Thus, we have

$$\|v_N(t)\|_{W_x^{-\varepsilon,\infty}} \le \int_0^t \bigg(\bigg\| \frac{\sin((t-t')[\![\nabla]\!])}{[\![\nabla]\!]} \sigma\pi_N(:(\pi_N u_N)^2:)(t') \bigg\|_{W_x^{-\varepsilon,\infty}} + \bigg\| M(:(\pi_N u_N)^2:)(t') \frac{\sin((t-t')[\![\nabla]\!])}{[\![\nabla]\!]} \pi_N u_N(t') \bigg\|_{W_x^{-\varepsilon,\infty}} \bigg) dt'$$

for any $t > 0$. Then, by using Minkowski's integral inequality, (6.3.37), and Proposition 6.2.4 once again, we have

$$\begin{aligned} &\big\| \|v_N(t)\|_{W_x^{-\varepsilon,\infty}} \big\|_{L^p_{\vec{u}_0,\omega_2}(\vec{\rho}_N \otimes \mathbb{P}_2)} \\ &\lesssim \int_0^t \bigg(\bigg\| \frac{\sin(\tau[\![\nabla]\!])}{[\![\nabla]\!]} \pi_N(:(\pi_N u_0)^2:) \bigg\|_{L^p_{\vec{u}_0,\omega_2}(\vec{\rho}_N \otimes \mathbb{P}_2; W_x^{-\varepsilon,\infty})} \\ &\qquad + \bigg\| M(:(\pi_N u_0)^2:) \frac{\sin(\tau[\![\nabla]\!])}{[\![\nabla]\!]} \pi_N u_0 \bigg\|_{L^p_{\vec{u}_0,\omega_2}(\vec{\rho}_N \otimes \mathbb{P}_2; W_x^{-\varepsilon,\infty})} \bigg) d\tau \\ &\le C(T,p) < \infty \end{aligned} \tag{6.3.39}$$

for any $0 \le t \le T$, $p \ge 1$, and $\varepsilon > 0$, uniformly in $N \in \mathbb{N}$.

We rewrite the system (6.3.2) as

$$\begin{aligned} &(\partial_t^2 + \partial_t + 1 - \Delta)X_N = 2\sigma\pi_N(v_N \olessthan \uparrow_N) - M(:(\pi_N u_N)^2:)\uparrow_N, \\ &(\partial_t^2 + \partial_t + 1 - \Delta)Y_N \\ &\quad = \sigma\pi_N\big(v_N(X_N + Y_N + \sigma\Upsilon_N) + 2(\mathfrak{R}_N + Y_N \circleddash \uparrow_N + \sigma \Upsilon^{\circleddash}_N) \\ &\qquad + 2(X_N + Y_N + \sigma\Upsilon_N) \ogreaterthan \uparrow_N\big) \\ &\qquad - M(:(\pi_N u_N)^2:)(X_N + Y_N + \sigma\Upsilon_N), \\ &\mathfrak{R}_N = 2\sigma\widetilde{\mathfrak{I}}^{(1),N}_{\olessthan}(X_N + Y_N + \sigma\Upsilon_N) \circleddash \uparrow_N + 2\sigma\widetilde{\mathfrak{I}}^N_{\olessthan,\circleddash}(X_N + Y_N + \sigma\Upsilon_N) \\ &\qquad - \int_0^t M(:(\pi_N u_N)^2:)(t')\mathbb{A}_N(t,t')dt', \end{aligned} \tag{6.3.40}$$

where we used (5.2.16) (with the frequency truncations and extra σ's in appropriate places) and $v_N = \sigma \curlyvee + X_N + Y_N$ so the right-hand side is linear in $(X_N, Y_N, \mathfrak{R}_N)$.

Let $\delta > 0$. In view of Proposition 6.2.4, we choose $K = K(T, \delta) \gg 1$ such that

$$\vec{\rho}_N \otimes \mathbb{P}_2\big(\|\Xi_N(\vec{u}_0, \omega_2)\|_{\mathcal{X}^\varepsilon_T} > K\big) < \frac{\delta}{3}, \tag{6.3.41}$$

uniformly in $N \in \mathbb{N}$. We also define $L(t)$ by

$$L(t) = 1 + \|v_N(t)\|_{W_x^{-\varepsilon,\infty}} + |M(:(\pi_N u_N)^2:)(t)|. \tag{6.3.42}$$

In view of (6.3.38) and (6.3.39), we choose $L_1 = L_1(T, \delta) \gg 1$ such that

$$\vec{\rho}_N \otimes \mathbb{P}_2\big(\|L\|_{L^3_T} > L_1\big) < \frac{\delta}{3}. \tag{6.3.43}$$

In the following, we work on the set

$$\|\Xi_N(\vec{u}_0, \omega_2)\|_{\mathcal{X}^\varepsilon_T} \le K \quad \text{and} \quad \|L\|_{L^3_T} \le L_1. \tag{6.3.44}$$

By applying Lemma 5.3.1 with (5.5.1) and Lemma 2.1.2 to (6.3.40) and using (5.5.3), (6.3.42), and (6.3.44), we have

$$\begin{aligned}
\|X_N\|_{X^{s_1}(T)} &\lesssim \int_0^T \Big(\|v_N \circledless \uparrow_N(t)\|_{H_x^{s_1-1}} \\
&\qquad + |M(:(\pi_N u_N)^2:)(t)| \cdot \|\uparrow_N(t)\|_{H_x^{s_1-1}}\Big) dt \\
&\lesssim K \int_0^T L(t) dt.
\end{aligned} \tag{6.3.45}$$

Since $s_2 < 1$, we can choose sufficiently small $\varepsilon > 0$ such that Lemma 2.1.3 (ii) yields

$$\begin{aligned}
&\|v_N(X_N + Y_N + \curlyvee_N)\|_{H_x^{s_2-1}} \\
&\quad \lesssim \|v_N\|_{W_x^{-\varepsilon,\infty}} \|X_N + Y_N + \curlyvee_N\|_{H_x^\varepsilon} \\
&\quad \lesssim \|v_N\|_{W_x^{-\varepsilon,\infty}} \big(\|X_N\|_{H_x^{s_1}} + \|Y_N\|_{H_x^{s_2}} + \|\Xi_N(\vec{u}_0, \omega_2)\|_{\mathcal{X}^\varepsilon_T}\big).
\end{aligned}$$

Hence, by (6.3.40), Lemma 5.3.1 with (5.5.1), Lemma 2.1.2 (see also (5.5.9)), (6.3.42), and (6.3.44), we have

$$\begin{aligned}
\|Y_N\|_{Y^{s_2}(T)} \lesssim \int_0^T &\Big(\|v_N(t)(X_N(t) + Y_N(t) + \curlyvee_N(t))\|_{H_x^{s_2-1}} \\
&+ \|\mathfrak{R}_N(t) + Y_N(t) \circleddash \uparrow_N(t) + \sigma \curlyvee\!\!\!\!\circ_N(t)\|_{H_x^{s_2-1}} \\
&+ \|(X_N(t) + Y_N(t) + \sigma \curlyvee_N(t)) \circledgtr \uparrow_N(t)\|_{H_x^{s_2-1}} \\
&+ |M(:(\pi_N u_N)^2:)(t)| \\
&\times \|X_N(t) + Y_N(t) + \sigma \curlyvee_N(t)\|_{H_x^{s_2-1}}\Big) dt
\end{aligned}$$

$$
\begin{aligned}
&\leq C(T)K^2 + K\int_0^T L(t)\big(1 + \|X_N\|_{X^{s_1}(T)} + \|Y_N\|_{Y^{s_2}(t)}\big)dt \\
&\quad + \int_0^T \|\mathfrak{R}_N(t)\|_{H_x^{s_3}} dt.
\end{aligned}
\tag{6.3.46}
$$

Fix $0 < \tau < 1$ and set

$$
L^q_{I_k} = L^q(I_k), \quad \text{where } I_k = [k\tau, (k+1)\tau].
$$

By a computation analogous to that in (5.5.10), we obtain

$$
\begin{aligned}
\|\mathfrak{R}_N\|_{L^3_{I_k} H_x^{s_3}} &\lesssim \big\|\widetilde{\mathfrak{I}}^{(1),N}_{\circledless}\big(X_N + Y_N + \sigma \curlyvee_N\big) \circleddash \,\mathsf{l}_N \big\|_{L^3_{I_k} H_x^{s_3}} \\
&\quad + \big\|\widetilde{\mathfrak{I}}^{N}_{\circledless, \circleddash}\big(X_N + Y_N + \sigma \curlyvee_N\big)\big\|_{L^3_{I_k} H_x^{s_3}} \\
&\quad + \int_0^T |M(:(\pi_N u_N)^2:)(t')| \cdot \|\mathbb{A}_N(t,t')\|_{L_t^3([t',T];H_x^{s_3})} dt' \\
&\leq C(T)K^2\big(K + \|X_N\|_{X^{s_1}(T)} + \|Y_N\|_{Y^{s_2}((k+1)\tau)}\big) \\
&\quad + K\int_0^T L(t)dt.
\end{aligned}
\tag{6.3.47}
$$

Given $0 < t \leq T$, let $k_*(t)$ be the largest integer such that $k_*(t)\tau \leq t$. Then, from (6.3.46) and (6.3.47), we have

$$
\begin{aligned}
&\|Y_N\|_{Y^{s_2}(t)} \leq \|Y_N\|_{Y^{s_2}((k_*(t)+1)\tau)} \\
&\quad \leq C(T)K^2 + C_1(T)K^3 \sum_{k=0}^{k_*(t)} \tau^{\frac{2}{3}}\big(1 + \|L(t)\|_{L^3_{I_k}}\big)\big(1 + \|X_N(t)\|_{X^{s_1}(T)}\big) \\
&\quad + C_2 K T \sum_{k=0}^{k_*(t)} \tau^{\frac{1}{3}} \|L(t)\|_{L^3_{I_k}} \\
&\quad + C_3 K^2 \sum_{k=0}^{k_*(t)} \tau^{\frac{2}{3}}\big(1 + \|L(t)\|_{L^3_{I_k}}\big)\|Y_N\|_{Y^{s_2}((k+1)\tau)}.
\end{aligned}
\tag{6.3.48}
$$

Now, choose $\tau = \tau(K, L_1) = \tau(T, \delta) > 0$ sufficiently small such that

$$
C_3 K^2 \tau^{\frac{2}{3}} L_1 \ll 1.
\tag{6.3.49}
$$

In view of (6.3.38) and (6.3.39), and define $L_2 = L_2(T, \delta) \gg 1$ such that

$$
\vec{\rho}_N \otimes \mathbb{P}_2\Bigg(\sum_{k=0}^{k_*(T)} \tau^{\frac{1}{3}}\big(1 + \|L(t)\|_{L^3_{I_k}}\big) > L_2\Bigg) < \frac{\delta}{3}.
\tag{6.3.50}
$$

In the following, we work on the set

$$\sum_{k=0}^{k_*(T)} \tau^{\frac{1}{3}}\big(1 + \|L(t)\|_{L^3_{I_k}}\big) \le L_2. \tag{6.3.51}$$

It follows from (6.3.48) with (6.3.44), (6.3.45), (6.3.49), and (6.3.51) that

$$\begin{aligned}
&\|Y_N\|_{Y^{s_2}((k_*(t)+1)\tau)} \\
&\quad \le C(T)K^4L_1L_2 + C_4K^2 \sum_{k=0}^{k_*(t)-1} \tau^{\frac{2}{3}} \|L(t)\|_{L^3_{I_k}} \|Y_N\|_{Y^{s_2}((k+1)\tau)}.
\end{aligned}$$

By applying the discrete Gronwall inequality with (6.3.51), we then obtain

$$\begin{aligned}
\|Y_N\|_{Y^{s_2}(t)} &\le \|Y_N\|_{Y^{s_2}((k_*(t)+1)\tau)} \\
&\le C(T)K^4L_1L_2 \exp\Bigg(C_4K^2 \sum_{k=0}^{k_*(t)-1} \tau^{\frac{2}{3}} \|L(t)\|_{L^3_{I_k}}\Bigg) \\
&\le C(T)K^4L_1L_2 \exp\big(C_4K^2L_2\big).
\end{aligned} \tag{6.3.52}$$

Therefore, from (6.3.45) and (6.3.52), we have

$$\|X_N\|_{X^{s_1}(T)} + \|Y_N\|_{Y^{s_2}(T)} \le C(T)KL_1 + C(T)K^4L_1L_2 \exp\big(C_4K^2L_2\big).$$

Together with (6.3.47), we then obtain

$$\|(X_N, Y_N, \mathfrak{R}_N)\|_{Z^{s_1,s_2,s_3}(T)} \le C_5(T, K, L_1, L_2)$$

under the conditions (6.3.44) and (6.3.51). Hence, by choosing $C_0 = C_0(T, \delta) > 0$ in (6.3.36) such that $C_0 > C_5(T, K, L_1, L_2)$, we have

$$\begin{aligned}
\vec{\rho}_N \otimes \mathbb{P}_2\Bigg(&\big\{\|(X_N, Y_N, \mathfrak{R}_N)\|_{Z^{s_1,s_2,s_3}(T)} > C_0\big\} \cap \big\{\|\Xi_N(\vec{u}_0, \omega_2)\|_{\mathcal{X}^\varepsilon_T} \le K\big\} \\
&\cap \big\{\|L\|_{L^3_T} \le L_1\big\} \cap \Bigg\{\sum_{k=0}^{k_*(T)} \tau^{\frac{1}{3}} \|L(t)\|_{L^3_{I_k}} \le L_2\Bigg\}\Bigg) = 0.
\end{aligned} \tag{6.3.53}$$

Finally, the bound (6.3.36) follows from (6.3.41), (6.3.43) (6.3.50), and (6.3.53). ∎

Given a map S from a measure space (X, μ) to a space Y, we use $S_\#\mu$ to denote the image measure (the pushforward) of μ under S. Fix $T > 0$ and we set

$$\nu_N = (\Xi_N)_\#(\vec{\rho}_N \otimes \mathbb{P}_2) \quad \text{and} \quad \nu = \Xi_\#(\vec{\rho} \otimes \mathbb{P}_2), \tag{6.3.54}$$

where we view $\Xi_N = \Xi_N(\vec{u}_0, \omega_2)$ in (6.1.10) and $\Xi = \Xi(\vec{u}_0, \omega_2)$ in (6.1.11) as maps from $\mathcal{H}^{-\frac{1}{2}-\varepsilon}(\mathbb{T}^3) \times \Omega_2$ to $\mathcal{X}^\varepsilon_T$ defined in (5.5.3). In view of the weak convergence of $\vec{\rho}_N \otimes \mathbb{P}_2$ to $\vec{\rho} \otimes \mathbb{P}_2$ (Theorem 1.2.1 (i)) and the $\vec{\rho} \otimes \mathbb{P}_2$-almost sure convergence of $\Xi_N(\vec{u}_0, \omega_2)$ to $\Xi(\vec{u}_0, \omega_2)$ (Corollary 6.2.6), we see that ν_N converges weakly to ν. Indeed, given a bounded continuous function $F : \mathcal{X}_T^\varepsilon \to \mathbb{R}$, by the dominated convergence theorem, we have

$$\begin{aligned}
&\left| \int F(\Xi) d\nu_N - \int F(\Xi) d\nu \right| \\
&\quad = \left| \int F(\Xi_N(\vec{u}_0, \omega_2)) d(\vec{\rho}_N \otimes \mathbb{P}_2) - \int F(\Xi(\vec{u}_0, \omega_2)) d(\vec{\rho} \otimes \mathbb{P}_2) \right| \\
&\quad \le \|F\|_{L^\infty} \left| \int 1 d\big((\vec{\rho}_N \otimes \mathbb{P}_2) - (\vec{\rho} \otimes \mathbb{P}_2)\big) \right| \\
&\qquad + \left| \int \big(F(\Xi_N(\vec{u}_0, \omega_2)) - F(\Xi(\vec{u}_0, \omega_2))\big) d(\vec{\rho} \otimes \mathbb{P}_2) \right| \\
&\quad \to 0,
\end{aligned}$$

as $N \to \infty$.

Next, we prove that $\nu_N = (\Xi_N)_\#(\vec{\rho}_N \otimes \mathbb{P}_2)$ converges to $\nu = \Xi_\#(\vec{\rho} \otimes \mathbb{P}_2)$ in the Wasserstein-1 metric. We view this problem as of Kantorovich's mass optimal transport problem and study the dual problem under the Kantorovich duality, using the Boué–Dupuis variational formula. This proposition plays a crucial role in the proof of almost sure global well-posedness and invariance of the Gibbs measure $\vec{\rho}$ presented at the end of this chapter.

Proposition 6.3.3. *Fix $T > 0$. Then, there exists a sequence $\{\mathfrak{p}_N\}_{N \in \mathbb{N}}$ of probability measures on $\mathcal{X}_T^\varepsilon \times \mathcal{X}_T^\varepsilon$ with the first and second marginals ν and ν_N on $\mathcal{X}_T^\varepsilon$, respectively, namely,*

$$\int_{\Xi^2 \in \mathcal{X}_T^\varepsilon} d\mathfrak{p}_N(\Xi^1, \Xi^2) = d\nu(\Xi^1) \quad \textit{and} \quad \int_{\Xi^1 \in \mathcal{X}_T^\varepsilon} d\mathfrak{p}_N(\Xi^1, \Xi^2) = d\nu_N(\Xi^2), \tag{6.3.55}$$

such that

$$\int_{\mathcal{X}_T^\varepsilon \times \mathcal{X}_T^\varepsilon} \min(\|\Xi^1 - \Xi^2\|_{\mathcal{X}_T^\varepsilon}, 1) d\mathfrak{p}_N(\Xi^1, \Xi^2) \to 0,$$

as $N \to \infty$. Namely, the total transportation cost associated to $\mathfrak{p}_N$ tends to 0 as $N \to \infty$.

Remark 6.3.4. In view of the weak convergence of the truncated Gibbs measure $\vec{\rho}_N$ to $\vec{\rho}$ (Theorem 1.2.1) and the almost sure convergence of the truncated enhanced data set Ξ_N to Ξ with respect to $\vec{\rho} \otimes \mathbb{P}_2$ (Corollary 6.2.6), it suffices to define $\mathfrak{p}_N = (\Xi, \Xi_N)_\#(\vec{\rho} \otimes \mathbb{P}_2)$. In the following, however, we present the full proof of Proposition 6.3.3, using the Kantorovich duality and the variational approach since we believe that such an argument is of general interest.

Proof of Proposition 6.3.3. Define a cost function $c(\Xi^1, \Xi^2)$ on $\mathcal{X}_T^\varepsilon \times \mathcal{X}_T^\varepsilon$ by setting

$$c(\Xi^1, \Xi^2) = \min(\|\Xi^1 - \Xi^2\|_{\mathcal{X}_T^\varepsilon}, 1).$$

Then, define the Lipschitz norm for a function $F : \mathcal{X}_T^\varepsilon \to \mathbb{R}$ by

$$\|F\|_{\mathrm{Lip}} = \sup_{\substack{\Xi^1, \Xi^2 \in \mathcal{X}_T^\varepsilon \\ \Xi^1 \neq \Xi^2}} \frac{|F(\Xi^1) - F(\Xi^2)|}{c(\Xi^1, \Xi^2)}.$$

Note that $\|F\|_{\mathrm{Lip}} \leq 1$ implies that F is bounded and Lipschitz continuous. From the Kantorovich duality (the Kantorovich–Rubinstein theorem [78, Theorem 1.14]), we have

$$\begin{aligned} &\inf_{\mathfrak{p} \in \Gamma(\nu, \nu_N)} \int_{\mathcal{X}_T^\varepsilon \times \mathcal{X}_T^\varepsilon} c(\Xi^1, \Xi^2) d\mathfrak{p}(\Xi^1, \Xi^2) \\ &\quad = \sup_{\|F\|_{\mathrm{Lip}} \leq 1} \bigg(\int F(\Xi) d\nu_N(\Xi) - \int F(\Xi) d\nu(\Xi) \bigg), \end{aligned} \tag{6.3.56}$$

where $\Gamma(\nu, \nu_N)$ is the set of probability measures on $\mathcal{X}_T^\varepsilon \times \mathcal{X}_T^\varepsilon$ with the first and second marginals ν and ν_N on $\mathcal{X}_T^\varepsilon$, respectively.

For a function F with $\|F\|_{\mathrm{Lip}} \leq 1$, let

$$G := F - \inf F + 1.$$

Then, we have

$$\int F(\Xi) d\nu_N(\Xi) - \int F(\Xi) d\nu(\Xi) = \int G(\Xi) d\nu_N(\Xi) - \int G(\Xi) d\nu(\Xi). \tag{6.3.57}$$

Note that $\|G\|_{\mathrm{Lip}} = \|F\|_{\mathrm{Lip}} \leq 1$ and $1 \leq G \leq 2$. Moreover, the mean value theorem yields that

$$\frac{1}{e} \leq \frac{\log x - \log y}{x - y} \leq 1 \tag{6.3.58}$$

for any $x, y \in [1, e]$ with $x \neq y$. Set $\{a\}_+ = \max(a, 0)$ for any $a \in \mathbb{R}$. By (6.3.57) and (6.3.58), we obtain

$$\begin{aligned} &\int F(\Xi) d\nu_N(\Xi) - \int F(\Xi) d\nu(\Xi) \\ &\quad \lesssim \bigg\{ - \log\bigg(\int G(\Xi) d\nu(\Xi) \bigg) + \log\bigg(\int G(\Xi) d\nu_N(\Xi) \bigg) \bigg\}_+ \end{aligned} \tag{6.3.59}$$

for any $N \in \mathbb{N}$.

Finally, define $H = \log G$. Then, from (6.3.58) and $1 \le G \le 2$, we have $\|H\|_{\mathrm{Lip}} \lesssim 1$. Hence, it follows from (6.3.56), (6.3.57), and (6.3.59) that

$$\begin{aligned}
&\inf_{\mathfrak{p}\in\Gamma(\nu,\nu_N)} \int_{\mathcal{X}_T^\varepsilon\times\mathcal{X}_T^\varepsilon} c(\Xi^1,\Xi^2)\,d\mathfrak{p}(\Xi^1,\Xi^2)\\
&\lesssim \sup_{\substack{0\le H\le 1\\ \|H\|_{\mathrm{Lip}}\lesssim 1}} \bigg\{-\log\bigg(\int \exp(H(\Xi))\,d\nu(\Xi)\bigg) + \log\bigg(\int \exp(H(\Xi))\,d\nu_N(\Xi)\bigg)\bigg\}_+ .
\end{aligned}$$

Our goal is to show that the right-hand side tends 0 as $N\to\infty$. Since $\|H\|_{\mathrm{Lip}} \lesssim 1$, H is bounded and Lipschitz continuous. Then, by the weak convergence of $\{\nu_N\}_{N\in\mathbb{N}}$ to ν, it suffices to show that

$$\begin{aligned}
\limsup_{N\to\infty} \sup_{\substack{0\le H\le 1\\ \|H\|_{\mathrm{Lip}}\lesssim 1}} \sup_{M\ge N} \bigg\{&-\log\bigg(\int \exp(H(\Xi))\,d\nu_M(\Xi)\bigg)\\
&+ \log\bigg(\int \exp(H(\Xi))\,d\nu_N(\Xi)\bigg)\bigg\}_+ \le 0. \qquad (6.3.60)
\end{aligned}$$

From (6.3.54), (6.1.1), and (6.3.58) with $0 \le H \le 1$, we have

$$\begin{aligned}
&\bigg\{-\log\bigg(\int \exp(H(\Xi))\,d\nu_M(\Xi)\bigg) + \log\bigg(\int \exp(H(\Xi))\,d\nu_N(\Xi)\bigg)\bigg\}_+\\
&\quad = \bigg\{-\log\bigg(\iiint \exp(H(\Xi_M(\vec u_0,\omega_2)))\,d\rho_M(u_0)\,d\mu_0(u_1)\,d\mathbb{P}_2(\omega_2)\bigg)\\
&\qquad\quad + \log\bigg(\iiint \exp(H(\Xi_N(\vec u_0,\omega_2)))\,d\rho_N(u_0)\,d\mu_0(u_1)\,d\mathbb{P}_2(\omega_2)\bigg)\bigg\}_+\\
&\quad \lesssim \bigg\{-\iiint \exp(H(\Xi_M(\vec u_0,\omega_2)))\,d\rho_M(u_0)\,d\mu_0(u_1)\,d\mathbb{P}_2(\omega_2)\\
&\qquad\quad + \iiint \exp(H(\Xi_N(\vec u_0,\omega_2)))\,d\rho_N(u_0)\,d\mu_0(u_1)\,d\mathbb{P}_2(\omega_2)\bigg\}_+\\
&\quad \lesssim \iint \bigg[\bigg\{-\int \exp(H(\Xi_M(\vec u_0,\omega_2)))\,d\rho_M(u_0)\\
&\qquad\quad + \int \exp(H(\Xi_N(\vec u_0,\omega_2)))\,d\rho_N(u_0)\bigg\}_+\bigg]\,d\mu_0(u_1)\,d\mathbb{P}_2(\omega_2)\\
&\quad \lesssim \iint \bigg[\bigg\{-\log\bigg(\int \exp(H(\Xi_M(\vec u_0,\omega_2)))\,d\rho_M(u_0)\bigg)\\
&\qquad\quad + \log\bigg(\int \exp(H(\Xi_N(\vec u_0,\omega_2)))\,d\rho_N(u_0)\bigg)\bigg\}_+\bigg]\,d\mu_0(u_1)\,d\mathbb{P}_2(\omega_2).
\end{aligned}$$

(6.3.61)

In the following, we study the integrand of the (u_1,ω_2)-integral. Thus, we fix u_1 and ω_2 and write $\Xi_N(\vec u_0,\omega_2) = \Xi_N(u_0,u_1,\omega_2)$ as $\Xi_N(u_0)$ for simplicity of notation. By the Boué–Dupuis variational formula (Lemma 3.1.1) with the change of variables (3.2.4), we have

$$
\begin{aligned}
&-\log\Big(\int \exp(H(\Xi_M(u_0)))d\rho_M(u_0)\Big) + \log\Big(\int \exp(H(\Xi_N(u_0)))d\rho_N(u_0)\Big)\\
&= \inf_{\dot\Upsilon^M \in \mathbb{H}^1_a} \mathbb{E}\Big[-H(\Xi_M(Y+\Upsilon^M+\sigma\mathfrak{Z}_M)) + \widehat{R}^\diamond_M(Y+\Upsilon^M+\sigma\mathfrak{Z}_M)\\
&\qquad + \frac{1}{2}\int_0^1 \|\dot\Upsilon^M(t)\|^2_{H^1_x}dt\Big]\\
&\quad - \inf_{\dot\Upsilon^N \in \mathbb{H}^1_a} \mathbb{E}\Big[-H(\Xi_N(Y+\Upsilon^N+\sigma\mathfrak{Z}_N)) + \widehat{R}^\diamond_N(Y+\Upsilon^N+\sigma\mathfrak{Z}_N)\\
&\qquad + \frac{1}{2}\int_0^1 \|\dot\Upsilon^N(t)\|^2_{H^1_x}dt\Big] + \log Z_M - \log Z_N,
\end{aligned}
\tag{6.3.62}
$$

where $\widehat{R}^\diamond_N$ is as in (3.2.25). Given $\delta > 0$, let $\underline{\Upsilon}^N$ be an almost optimizer, namely,

$$
\begin{aligned}
&\inf_{\dot\Upsilon^N \in \mathbb{H}^1_a} \mathbb{E}\Big[-H(\Xi_N(Y+\Upsilon^N+\sigma\mathfrak{Z}_N))\\
&\qquad + \widehat{R}^\diamond_N(Y+\Upsilon^N+\sigma\mathfrak{Z}_N) + \frac{1}{2}\int_0^1 \|\dot\Upsilon^N(t)\|^2_{H^1_x}dt\Big]\\
&\ge \mathbb{E}\Big[-H(\Xi_N(Y+\underline{\Upsilon}^N+\sigma\mathfrak{Z}_N))\\
&\qquad + \widehat{R}^\diamond_N(Y+\underline{\Upsilon}^N+\sigma\mathfrak{Z}_N) + \frac{1}{2}\int_0^1 \|\dot{\underline{\Upsilon}}^N(t)\|^2_{H^1_x}dt\Big] - \delta.
\end{aligned}
$$

Then, by choosing $\Upsilon^M = \underline{\Upsilon}^N$ and the Lipschitz continuity of H, we have

$$
\begin{aligned}
&\inf_{\dot\Upsilon^M \in \mathbb{H}^1_a} \mathbb{E}\Big[-H(\Xi_M(Y+\Upsilon^M+\sigma\mathfrak{Z}_M))\\
&\qquad + \widehat{R}^\diamond_M(Y+\Upsilon^M+\sigma\mathfrak{Z}_M) + \frac{1}{2}\int_0^1 \|\dot\Upsilon^M(t)\|^2_{H^1_x}dt\Big]\\
&\quad - \inf_{\dot\Upsilon^N \in \mathbb{H}^1_a} \mathbb{E}\Big[-H(\Xi_N(Y+\Upsilon^N+\sigma\mathfrak{Z}_N))\\
&\qquad + \widehat{R}^\diamond_N(Y+\Upsilon^N+\sigma\mathfrak{Z}_N) + \frac{1}{2}\int_0^1 \|\dot\Upsilon^N(t)\|^2_{H^1_x}dt\Big]\\
&\le \delta + \mathbb{E}\big[H(\Xi_N(Y+\underline{\Upsilon}^N+\sigma\mathfrak{Z}_N)) - H(\Xi_M(Y+\underline{\Upsilon}^N+\sigma\mathfrak{Z}_M))\\
&\qquad + \widehat{R}^\diamond_M(Y+\underline{\Upsilon}^N+\sigma\mathfrak{Z}_M) - \widehat{R}^\diamond_N(Y+\underline{\Upsilon}^N+\sigma\mathfrak{Z}_N)\big]\\
&\le \delta + \|H\|_{\mathrm{Lip}}\cdot \mathbb{E}\big[\|\Xi_M(Y+\underline{\Upsilon}^N+\sigma\mathfrak{Z}_N) - \Xi_N(Y+\underline{\Upsilon}^N+\sigma\mathfrak{Z}_M)\|_{\mathcal{X}^\varepsilon_T}\big]\\
&\quad + \mathbb{E}\big[\widehat{R}^\diamond_M(Y+\underline{\Upsilon}^N+\sigma\mathfrak{Z}_M) - \widehat{R}^\diamond_N(Y+\underline{\Upsilon}^N+\sigma\mathfrak{Z}_N)\big].
\end{aligned}
\tag{6.3.63}
$$

Proceeding as in Section 3.3 with $0 \le H \le 1$, we have (3.3.16). Then, using the computations from (3.3.7) to (3.3.18) we obtain

$$\mathbb{E}\big[\widehat{R}^{\diamond}_M(Y + \underline{\Upsilon}^N + \sigma \mathfrak{Z}_M) - \widehat{R}^{\diamond}_N(Y + \underline{\Upsilon}^N + \sigma \mathfrak{Z}_N)\big] \to 0, \tag{6.3.64}$$

as $M \ge N \to \infty$. We also note that as a consequence of (3.3.16) with (3.2.16) and Lemma 3.1.2, we have

$$\mathbb{E}\big[\|\underline{\Upsilon}^N\|^2_{H^1}\big] \lesssim 1, \tag{6.3.65}$$

uniformly in $N \in \mathbb{N}$.

Moreover, by slightly modifying (part of) the proof of Proposition 6.2.4, we can show that

$$\mathbb{E}\big[\big\|\Xi_M(Y + \underline{\Upsilon}^N + \sigma \mathfrak{Z}_N) - \Xi_N(Y + \underline{\Upsilon}^N + \sigma \mathfrak{Z}_M)\big\|_{\mathcal{X}^\varepsilon_T}\big] \to 0, \tag{6.3.66}$$

as $M \ge N \to \infty$. Here, we only consider the contribution from $\widetilde{\mathfrak{I}}^N_{\circledless, \circleddash}$. The other terms in the truncated enhanced data sets can be handled in a similar manner. With the notations (6.2.18) and (6.2.19) (recall that we suppress the dependence on u_1 and ω_2), we have

$$\begin{aligned}
&\widetilde{\mathfrak{I}}^M_{\circledless, \circleddash}[\upharpoonleft(Y + \underline{\Upsilon}^N + \sigma \mathfrak{Z}_M)] - \widetilde{\mathfrak{I}}^N_{\circledless, \circleddash}[\upharpoonleft(Y + \underline{\Upsilon}^N + \sigma \mathfrak{Z}_N)] \\
&\quad = \widetilde{\mathfrak{I}}^M_{\circledless, \circleddash}[\upharpoonleft(Y + \underline{\Upsilon}^N + \sigma \mathfrak{Z}_M), \upharpoonleft(\sigma(\mathfrak{Z}_M - \mathfrak{Z}_N))] \\
&\qquad + \widetilde{\mathfrak{I}}^M_{\circledless, \circleddash}[\upharpoonleft(\sigma(\mathfrak{Z}_M - \mathfrak{Z}_N)), \upharpoonleft(Y + \underline{\Upsilon}^N + \sigma \mathfrak{Z}_N)] \\
&\qquad + \big(\widetilde{\mathfrak{I}}^M_{\circledless, \circleddash}[\upharpoonleft(Y + \underline{\Upsilon}^N + \sigma \mathfrak{Z}_N)] - \widetilde{\mathfrak{I}}^N_{\circledless, \circleddash}[\upharpoonleft(Y + \underline{\Upsilon}^N + \sigma \mathfrak{Z}_N)]\big) \\
&\quad =: \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned} \tag{6.3.67}$$

It follows from (6.2.28), (6.2.29), and (6.2.32) together with Remark 5.4.2 that there exists small $\delta_0 > 0$ such that

$$\begin{aligned}
&\|\mathrm{I}\|_{\mathcal{L}_2(q,T)} + \|\mathrm{II}\|_{\mathcal{L}_2(q,T)} \\
&\quad \le C(T)\big(\|Y\|_{L^\infty_T W_x^{-\frac12-\varepsilon,\infty}} + \|\underline{\Upsilon}^N\|_{H^1} + \|\mathfrak{Z}_N\|_{W^{1-\varepsilon,\infty}}\big)\|\mathfrak{Z}_N - \mathfrak{Z}_M\|_{W^{1-\varepsilon,\infty}} \\
&\quad \le C(T)N^{-\delta_0}\big(\|Y\|_{L^\infty_T W_x^{-\frac12-\varepsilon,\infty}} + \|\underline{\Upsilon}^N\|_{H^1} + \|\mathfrak{Z}_N\|_{W^{1-\varepsilon,\infty}}\big)^2 \\
&\qquad + N^{\delta_0}\|\mathfrak{Z}_N - \mathfrak{Z}_M\|^2_{W^{1-\varepsilon,\infty}}
\end{aligned} \tag{6.3.68}$$

and

$$\mathbb{E}\big[N^{\delta_0}\|\mathfrak{Z}_N - \mathfrak{Z}_M\|^2_{W^{1-\varepsilon,\infty}}\big] \to 0, \tag{6.3.69}$$

as $M \ge N \to \infty$. From (6.2.16) and (6.2.18), we have

$$\widetilde{\mathfrak{I}}^N_{\circledless, \circleddash}[\psi_1, \psi_2](w) = \mathcal{I}\big(\pi_N(\mathcal{K}^\theta(w, \pi_N \psi_1))\big) \circleddash (\pi_N \psi_2).$$

Hence, when we consider the difference in III, we see that one of the factors comes with $\pi_M - \pi_N$, from which we can gain a small negative power of N. Hence, by repeating the calculation above with this observation, we obtain

$$\begin{aligned}\big\|\mathrm{III} - (\widetilde{\mathfrak{I}}^N_{\circledcirc,\circleddash}[\upharpoonleft(Y)] - \widetilde{\mathfrak{I}}^M_{\circledcirc,\circleddash}[\upharpoonleft(Y)])\big\|_{\mathcal{L}_2(q,T)} \\ \lesssim N^{-\delta_0}\big(\|Y\|_{L^\infty_T W_x^{-\frac{1}{2}-\varepsilon,\infty}} + \|\Upsilon^N\|_{H^1} + \|\mathfrak{Z}_N\|_{W^{1-\varepsilon,\infty}}\big)^2\end{aligned} \tag{6.3.70}$$

for any $M \geq N \geq 1$. Lastly, from (6.2.35) and (6.2.23), there exists $\delta > 0$ such that

$$\big\|\widetilde{\mathfrak{I}}^N_{\circledcirc,\circleddash}[\upharpoonleft(Y)] - \widetilde{\mathfrak{I}}^M_{\circledcirc,\circleddash}[\upharpoonleft(Y)]\big\|_{\mathcal{L}_2(q,T)} \leq N^{-\delta_0}\widetilde{K}(Y, u_1, \omega_2) \tag{6.3.71}$$

for any $M \geq N \geq 1$, where, in view of (6.2.36), $\mathbb{E}[\widetilde{K}(Y, u_1, \omega_2)] \leq C(u_1, \omega_2) < \infty$ for almost every u_1 and ω_2. Therefore, from (6.3.67), (6.3.68), (6.3.69), (6.3.70), and (6.3.71) with the bound (6.3.65), we obtain

$$\mathbb{E}\Big[\big\|\widetilde{\mathfrak{I}}^M_{\circledcirc,\circleddash}[Y + \Upsilon^N_\delta + \sigma\mathfrak{Z}_M] - \widetilde{\mathfrak{I}}^N_{\circledcirc,\circleddash}[Y + \Upsilon^N_\delta + \sigma\mathfrak{Z}_N]\big\|_{\mathcal{L}_2(q,T)}\Big] \to 0,$$

as $M \geq N \to \infty$.

Note that $\{Z_N\}_{N\in\mathbb{N}}$ is a convergent sequence and $\delta > 0$ was arbitrary. Hence, it follows from (6.3.62), (6.3.63), (6.3.64), and (6.3.66) that

$$\begin{aligned}\limsup_{N\to\infty} \sup_{\substack{0\leq H\leq 1\\ \|H\|_{\mathrm{Lip}}\lesssim 1}} \sup_{M\geq N} \bigg\{ -\log\bigg(\int \exp(H(\Xi_M(u_0, u_1, \omega_2)))d\rho_M(u_0)\bigg) \\ + \log\bigg(\int \exp(H(\Xi_N(u_0, u_1, \omega_2)))d\rho_N(u_0)\bigg)\bigg\}_+ \leq 0,\end{aligned} \tag{6.3.72}$$

for almost every u_1 and ω_2, where the supremum in H was trivially dropped in the last step of (6.3.63). Therefore, (6.3.60) follows from (6.3.61) and (6.3.72) with Fatou's lemma. This concludes the proof of Proposition 6.3.3. ∎

Finally, we present the proof of Theorem 1.3.2.

Proof of Theorem 1.3.2. We break the proof into two parts.

Part 1. We first prove almost sure global well-posedness of the hyperbolic Φ^3_3-model. As in [5, 9, 19], it suffices to prove "almost" almost sure global well-posedness. More precisely, it suffices to prove that given any $T > 0$ and small $\delta > 0$, there exists $\Sigma_{T,\delta} \subset \mathcal{H}^{-\frac{1}{2}-\varepsilon}(\mathbb{T}^3) \times \Omega_2$ with $\vec{\rho} \otimes \mathbb{P}_2(\Sigma^c_{T,\delta}) < \delta$ such that for each $(\vec{u}_0, \omega_2) \in \Sigma_{T,\delta}$, the solution $(X, Y, \mathfrak{R})$ to (5.2.27), with the zero initial data and the enhanced data $\Xi(\vec{u}_0, \omega)$ in (6.1.11), exists on the time interval $[0, T]$.

We assume this "almost" almost sure global well-posedness claim for the moment. Denote by $(X_N, Y_N, \mathfrak{R}_N)$ the solution to the truncated system (6.3.2) with the truncated enhanced data $\Xi_N(\vec{u}_0, \omega)$ in (6.1.10) and set

$$u_N(\vec{u}_0, \omega_2) = \uparrow(\vec{u}_0, \omega_2) + \sigma \curlyvee_N(\vec{u}_0, \omega_2) + X_N + Y_N, \tag{6.3.73}$$

which is the solution to the truncated hyperbolic Φ^3_3-model (6.1.2) with the initial data $(u_N, \partial_t u_N)|_{t=0} = \vec{u}_0 = (u_0, u_1)$ and the noise $\xi = \xi(\omega_2)$. Here, we used the uniqueness of the solution u_N to (6.1.2); see Remark 6.2.2. Then, we conclude from Corollary 6.2.6 (on the almost sure convergence of $\Xi_N(\vec{u}_0, \omega)$ to $\Xi(\vec{u}_0, \omega)$) and the second part of Proposition 6.3.1 that $(u_N, \partial_t u_N)(\vec{u}_0, \omega_2)$ in (6.3.73) converges to $(u, \partial_t u)(\vec{u}_0, \omega_2)$ in $C([0, T]; \mathcal{H}^{-\frac{1}{2}-\varepsilon}(\mathbb{T}^3))$ for each $(\vec{u}_0, \omega_2) \in \Sigma_{T,\delta}$, where $u(\vec{u}_0, \omega_2)$ is defined by

$$u(\vec{u}_0, \omega_2) = \uparrow(\vec{u}_0, \omega_2) + \sigma \curlyvee(\vec{u}_0, \omega_2) + X + Y. \tag{6.3.74}$$

Now, we define

$$\Sigma = \bigcup_{k=1}^{\infty} \bigcap_{j=1}^{\infty} \Sigma_{2^j, 2^{-j}k^{-1}}.$$

Then, we have $\vec{\rho} \otimes \mathbb{P}_2(\Sigma) = 1$ and, for each $(\vec{u}_0, \omega_2) \in \Sigma$, the solution $(u_N, \partial_t u_N)(\vec{u}_0, \omega_2)$ to the truncated hyperbolic Φ^3_3-model (6.1.2) converges to $(u, \partial_t u)(\vec{u}_0, \omega_2)$ in (6.3.74) in $C(\mathbb{R}_+; \mathcal{H}^{-\frac{1}{2}-\varepsilon}(\mathbb{T}^3))$ (endowed with the compact-open topology in time). This proves the almost sure global well-posedness claim in Theorem 1.3.2, assuming "almost" almost sure global well-posedness.

We now prove "almost" almost sure global well-posedness. Fix $T > 0$ and small $\delta > 0$. Given $\Xi = (\Xi_1, \dots, \Xi_6) \in \mathcal{X}^\varepsilon_T$, let $Z(\Xi) = (X, Y, \mathfrak{R})(\Xi)$ be the solution to (5.2.27) with the zero initial data and the enhanced data set given by Ξ, namely, Ξ_j replacing the jth element in (5.2.28). Note that Ξ here denotes a general element in $\mathcal{X}^\varepsilon_T$ and is not associated with any specific $(\vec{u}_0, \omega_2) \in \mathcal{H}^{-\frac{1}{2}-\varepsilon}(\mathbb{T}^3) \times \Omega_2$. Similarly, given $N \in \mathbb{N}$ and $\Xi \in \mathcal{X}^\varepsilon_T$, let $Z_N(\Xi) = (X_N, Y_N, \mathfrak{R}_N)(\Xi)$ be the solution to (6.3.2) with the enhanced data set Ξ, namely, Ξ_j replacing the jth element of $\Xi_N(\vec{u}_0, \omega_2)$ in (6.1.10).

Given $C_0 > 0$, define the set $\Sigma_{C_0} \subset \mathcal{X}^\varepsilon_T$ such that, for each $\Xi \in \Sigma_{C_0}$, the solution $Z(\Xi)$ to (5.2.27), with the zero initial data and the enhanced data Ξ, exists on the time interval $[0, T]$, satisfying the bound

$$\|Z(\Xi)\|_{Z^{s_1,s_2,s_3}(T)} \le C_0 + 1. \tag{6.3.75}$$

Let $N \in \mathbb{N}$. Given $K, C_0 > 0$, we set

$$A_{N,K,C_0} = \big\{\Xi' \in \mathcal{X}^\varepsilon_T : \|\Xi'\|_{\mathcal{X}^\varepsilon_T} \le K, \|Z_N(\Xi')\|_{Z^{s_1,s_2,s_3}(T)} \le C_0\big\} \tag{6.3.76}$$

and

$$B_{N,K,C_0} = \big\{(\Xi, \Xi') \in \mathcal{X}^\varepsilon_T \times \mathcal{X}^\varepsilon_T : \|\Xi - \Xi'\|_{\mathcal{X}^\varepsilon_T} \le \kappa,\ \Xi' \in A_{N,K,C_0}\big\}, \tag{6.3.77}$$

where $\kappa > 0$ is a small number to be chosen later. Then, from the stability result (the first claim in Proposition 6.3.1) with (6.3.75), (6.3.76), and (6.3.77), there exists small $\kappa(T, K, C_0) \in (0,1)$ and $N_0 = N_0(T, K, C_0) \in \mathbb{N}$ such that

$$\Sigma_{C_0} \times \mathcal{X}_T^\varepsilon \supset B_{N,K,C_0} \tag{6.3.78}$$

for any $N \geq N_0$.

Let $C_0 = C_0(T, \delta) \gg 1$ be as in Proposition 6.3.2 and let $\mathfrak{p}_N$, $N \in \mathbb{N}$, be as in Proposition 6.3.3. Then, from (6.3.54), (6.3.55), and (6.3.78), we have

$$\begin{aligned}
&\vec{\rho} \otimes \mathbb{P}_2\big(\Xi(\vec{u}_0, \omega_2) \in \Sigma_{C_0}\big) \\
&\quad = \int \mathbf{1}_{\Xi \in \Sigma_{C_0}}(\Xi, \Xi') d\mathfrak{p}_N(\Xi, \Xi') \\
&\quad \geq \int \mathbf{1}_{B_{N,K,C_0}}(\Xi, \Xi') d\mathfrak{p}_N(\Xi, \Xi') \\
&\quad \geq 1 - \int \mathbf{1}_{\{\|\Xi - \Xi'\|_{\mathcal{X}_T^\varepsilon} > \kappa\}} d\mathfrak{p}_N(\Xi, \Xi') - \int \mathbf{1}_{A^c_{N,K,C_0}}(\Xi') d\mathfrak{p}_N(\Xi, \Xi') \\
&\quad \geq 1 - \frac{1}{\kappa} \int \min(\|\Xi - \Xi'\|_{\mathcal{X}_T^\varepsilon}, 1) d\mathfrak{p}_N(\Xi, \Xi') \\
&\qquad - \vec{\rho}_N \otimes \mathbb{P}_2(\{\Xi_N(\vec{u}_0', \omega_2') \in A^c_{N,K,C_0}\}) \\
&\quad > 1 - \frac{1}{\kappa} \int \min(\|\Xi - \Xi'\|_{\mathcal{X}_T^\varepsilon}, 1) d\mathfrak{p}_N(\Xi, \Xi') - 2\delta,
\end{aligned} \tag{6.3.79}$$

where the last step follows from Proposition 6.2.4 by choosing $K = K(\delta) \gg 1$, together with Proposition 6.3.2. By Proposition 6.3.3, we have

$$\frac{1}{\kappa} \int \min(\|\Xi - \Xi_N'\|_{\mathcal{X}_T^\varepsilon}, 1) d\mathfrak{p}_N(\Xi, \Xi_N') \to 0, \tag{6.3.80}$$

as $N \to \infty$. Therefore, we conclude from (6.3.79) and (6.3.80) that

$$\vec{\rho} \otimes \mathbb{P}_2\big(\Xi(\vec{u}_0, \omega_2) \in \Sigma_{C_0}\big) > 1 - 2\delta.$$

This proves "almost" almost sure global well-posedness with

$$\Sigma_{T,\delta} = \{(\vec{u}_0, \omega_2) \in \mathcal{H}^{-\frac{1}{2}-\varepsilon}(\mathbb{T}^3) \times \Omega_2 : \Xi(\vec{u}_0, \omega_2) \in \Sigma_{C_0}\},$$

and hence almost sure global well-posedness of the hyperbolic Φ^3_3-model, namely, the unique limit $u = u(\vec{u}_0, \omega_2)$ in (6.3.74) exists globally in time almost surely with respect to $\vec{\rho} \otimes \mathbb{P}_2$.

Part 2. Next, we prove invariance of the Gibbs measure $\vec{\rho} = \rho \otimes \mu_0$ under the limiting hyperbolic Φ^3_3-dynamics. In the following, we prove

$$\int F(\Phi(t)(\vec{u}_0, \omega_2)) d(\vec{\rho} \otimes \mathbb{P}_2)(\vec{u}_0, \omega_2) = \int F(\vec{u}_0) d\vec{\rho}(\vec{u}_0) \tag{6.3.81}$$

for any bounded Lipschitz functional $F : \mathcal{C}^{-100}(\mathbb{T}^3) \times \mathcal{C}^{-100}(\mathbb{T}^3) \to \mathbb{R}$ and $t \in \mathbb{R}_+$, where $\Phi(\vec{u}_0, \omega_2)$ is the limit of the solution $(u_N, \partial_t u_N) = \Phi^N(\vec{u}_0, \omega_2)$ to the truncated hyperbolic Φ^3_3-model defined in (6.2.8).

As in Part 1, we use the notation $(X, Y, \mathfrak{R}) = (X, Y, \mathfrak{R})(\Xi)$, etc. Also, let $\mathfrak{p}_N$, $N \in \mathbb{N}$, be as in Proposition 6.3.3. Then, by the decomposition (6.3.1) (also for $N = \infty$), (6.3.54), (6.3.55), and the invariance of $\vec{\rho}_N$ under the truncated hyperbolic Φ^3_3-model (6.1.2) (Lemma 6.2.3), we have

$$\begin{aligned}
&\int F(\Phi(t)(\vec{u}_0, \omega_2)) d(\vec{\rho} \otimes \mathbb{P}_2)(\vec{u}_0, \omega_2) \\
&\quad = \int F(\Phi(t)(\Xi)) d\mathfrak{p}_N(\Xi, \Xi') \\
&\quad = \int F(\Phi^N(t)(\Xi')) d\mathfrak{p}_N(\Xi, \Xi') \\
&\qquad + \int \big[F(\Phi(t)(\Xi)) - F(\Phi^N(t)(\Xi'))\big] d\mathfrak{p}_N(\Xi, \Xi') \\
&\quad = \int F(\vec{u}_0) d\vec{\rho}_N(\vec{u}_0) + \int \big[F(\Phi(t)(\Xi)) - F(\Phi^N(t)(\Xi'))\big] d\mathfrak{p}_N(\Xi, \Xi').
\end{aligned}$$

By the weak convergence of $\vec{\rho}_N$ to $\vec{\rho}$, we have

$$\lim_{N\to\infty} \int F(\vec{u}_0) d\vec{\rho}_N(\vec{u}_0) = \int F(\vec{u}_0) d\vec{\rho}(\vec{u}_0).$$

Hence, since F is bounded and Lipschitz, (6.3.81) is reduced to showing that

$$\int \min\big(\|\Phi(t)(\Xi) - \Phi^N(t)(\Xi')\|_{\mathcal{C}^{-100}\times\mathcal{C}^{-100}}, 1\big) d\mathfrak{p}_N(\Xi, \Xi') \to 0, \tag{6.3.82}$$

as $N \to \infty$.

As in (6.2.8), we write

$$\Phi(t)(\Xi) = \big(\Phi_1(t)(\Xi), \Phi_2(t)(\Xi)\big) \quad \text{and} \quad \Phi^N(t)(\Xi') = \big(\Phi^N_1(t)(\Xi'), \Phi^N_2(t)(\Xi')\big),$$

where $\Xi = (\Xi_1, \dots, \Xi_6)$ and $\Xi' = (\Xi'_1, \dots, \Xi'_6)$ (see also (6.1.10) and (6.1.11)). With the decomposition as in (6.3.1), we have

$$\begin{aligned}
\Phi_1(t)(\Xi) &= \Xi_1 + \sigma\Xi_3 + X(\Xi) + Y(\Xi), \\
\Phi^N_1(t)(\Xi') &= \Xi'_1 + \sigma\Xi'_3 + X_N(\Xi') + Y_N(\Xi'),
\end{aligned} \tag{6.3.83}$$

and $\Phi_2(t)(\Xi) = \partial_t\Phi_1(t)(\Xi)$ and $\Phi^N_2(t)(\Xi') = \partial_t\Phi^N_1(t)(\Xi')$ are given by term-by-term differentiation of the terms on the right-hand sides of (6.3.83). From the definition (5.5.3) of the $\mathcal{X}^\varepsilon_T$-norm, we clearly have

$$\begin{aligned}
&\|(\Xi_1 + \sigma\Xi_3)(t) - (\Xi'_1 + \sigma\Xi'_3)(t)\|_{\mathcal{C}^{-100}} \\
&\quad + \|(\partial_t\Xi_1 + \sigma\partial_t\Xi_3)(t) - (\partial_t\Xi'_1 + \sigma\partial_t\Xi'_3)(t)\|_{\mathcal{C}^{-100}} \lesssim \|\Xi - \Xi'\|_{\mathcal{X}^\varepsilon_T}.
\end{aligned}$$

Hence, in view of (5.2.31) with (5.5.1), (6.3.82) is reduced to showing that

$$\int \min\big(\|Z(\Xi) - Z_N(\Xi')\|_{\mathcal{Z}^{s_1,s_2,s_3}(T)}, 1\big) d\mathfrak{p}_N(\Xi, \Xi') \to 0, \tag{6.3.84}$$

as $N \to \infty$, where $Z(\Xi) = (X, Y, \mathfrak{R})(\Xi)$ and $Z_N(\Xi') = (X_N, Y_N, \mathfrak{R}_N)(\Xi')$ as in Part 1.

It follows from the second part of Proposition 6.3.1 (with $\kappa = \|\Xi - \Xi'\|_{\mathcal{X}_T^\varepsilon}$) and Proposition 6.3.3 that

$$\begin{aligned}
&\int \min\big(\|Z(\Xi) - Z_N(\Xi')\|_{\mathcal{Z}^{s_1,s_2,s_3}(T)}, 1\big) d\mathfrak{p}_N(\Xi, \Xi') \\
&\le A\big(T, \|\Xi\|_{\mathcal{X}_T^\varepsilon}, \|Z(\Xi)\|_{\mathcal{Z}^{s_1,s_2,s_3}(T)}\big) \\
&\quad \times \int \min\big(\|\Xi - \Xi'\|_{\mathcal{X}_T^\varepsilon} + N^{-\delta}, 1\big) d\mathfrak{p}_N(\Xi, \Xi') \to 0,
\end{aligned}$$

as $N \to \infty$. This proves (6.3.84) and therefore, we conclude (6.3.81), which proves invariance of the Gibbs measure $\vec{\rho}$ under the limiting hyperbolic Φ_3^3-model. ∎