# Appendix A

## Absolute continuity with respect to the shifted measure

### A.1 Preliminary lemmas

In this appendix, we prove that the $\Phi^3_3$-measure $\rho$ in the weakly nonlinear regime ($|\sigma| \ll 1$), constructed in Theorem 1.2.1 (i), is absolutely continuous with respect to the shifted measure $\mathrm{Law}(Y(1) + \sigma\mathfrak{Z}(1) + \mathcal{W}(1))$, where $Y$ is as in (3.1.2), $\mathfrak{Z}$ is defined as the limit of the antiderivative of $\dot{\mathfrak{Z}}^N$ in (3.2.3) as $N \to \infty$, and the auxiliary process $\mathcal{W}$ is defined by

$$\mathcal{W}(t) = (1-\Delta)^{-1} \int_0^t \langle \nabla \rangle^{-\frac{1}{2}-\varepsilon} \big( \langle \nabla \rangle^{-\frac{1}{2}-\varepsilon} Y(t') \big)^5 dt' \tag{A.1.1}$$

for some small $\varepsilon > 0$. For the proof, we construct a drift as in the discussion in [4, Section 3]. See also [54, Appendix C]. The coercive term $\mathcal{W}$ is introduced to guarantee global existence of a drift on the time interval $[0,1]$. See Lemma A.1.2 below. We closely follow the presentation in Appendix C of our previous work [54].

First, we recall the following general lemma, giving a criterion for absolute continuity. See [54, Lemma C.1] for the proof.

**Lemma A.1.1.** *Let $\mu_n$ and $\rho_n$ be probability measures on a Polish space $X$. Suppose that $\mu_n$ and $\rho_n$ converge weakly to $\mu$ and $\rho$, respectively. Furthermore, suppose that for every $\varepsilon > 0$, there exist $\delta(\varepsilon) > 0$ and $\eta(\varepsilon) > 0$ with $\delta(\varepsilon), \eta(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that for every continuous function $F: X \to \mathbb{R}$ with $0 < \inf F \le F \le 1$ satisfying*

$$\mu_n(\{F \le \varepsilon\}) \ge 1 - \delta(\varepsilon)$$

*for any $n \ge n_0(F)$, we have*

$$\limsup_{n\to\infty} \int F(u) d\rho_n(u) \le \eta(\varepsilon).$$

*Then, $\rho$ is absolutely continuous with respect to $\mu$.*

By regarding $\dot{\mathfrak{Z}}^N$ in (3.2.3) and $\mathcal{W}$ in (A.1.1) as functions of $Y$, we write them as

$$\begin{aligned} \dot{\mathfrak{Z}}^N(Y)(t) &= (1-\Delta)^{-1} :Y_N^2(t):, \\ \mathcal{W}(Y)(t) &= (1-\Delta)^{-1} \int_0^t \langle \nabla \rangle^{-\frac{1}{2}-\varepsilon} \big( \langle \nabla \rangle^{-\frac{1}{2}-\varepsilon} Y(t') \big)^5 dt' \end{aligned} \tag{A.1.2}$$

and we set $\dot{\mathfrak{Z}}_N(Y) = \pi_N \dot{\mathfrak{Z}}^N(Y)$. Then, from (A.1.2), we have

$$\dot{\mathfrak{Z}}_N(Y+\Theta) - \dot{\mathfrak{Z}}_N(Y) = (1-\Delta)^{-1}\pi_N(2\Theta_N Y_N + \Theta_N^2), \tag{A.1.3}$$

where $\Theta_N = \pi_N \Theta$. We also define $\mathcal{W}_N(Y)(t)$ by

$$\mathcal{W}_N(Y)(t) = (1-\Delta)^{-1}\pi_N \int_0^t \langle \nabla \rangle^{-\frac{1}{2}-\varepsilon}\big(\langle \nabla \rangle^{-\frac{1}{2}-\varepsilon} Y_N(t')\big)^5 dt'. \tag{A.1.4}$$

Next, we state a lemma on the construction of a drift $\Theta$.

**Lemma A.1.2.** *Let $\sigma \in \mathbb{R}$ and $\dot{\Upsilon} \in L^2([0,1]; H^1(\mathbb{T}^3))$. Then, given any $N \in \mathbb{N}$, the Cauchy problem for $\Theta$:*

$$\begin{cases} \dot{\Theta} + \sigma(1-\Delta)^{-1}\pi_N(2\Theta_N Y_N + \Theta_N^2) + \dot{\mathcal{W}}_N(Y+\Theta) - \dot{\Upsilon} = 0 \\ \Theta(0) = 0 \end{cases} \tag{A.1.5}$$

*is almost surely globally well-posed on the time interval $[0,1]$ such that a solution $\Theta$ belongs to $C([0,1]; H^1(\mathbb{T}^3))$. Moreover, if $\|\dot{\Upsilon}\|^2_{L^2([0,\tau];H^1_x)} \le M$ for some $M > 0$ and for some stopping time $\tau \in [0,1]$, then, for any $1 \le p < \infty$, there exists $C = C(M,p) > 0$ such that*

$$\mathbb{E}\big[\|\dot{\Theta}\|^p_{L^2([0,\tau];H^1_x)}\big] \le C(M,p), \tag{A.1.6}$$

*where $C(M,p)$ is independent of $N \in \mathbb{N}$.*

## A.2 Absolute continuity

In this section, we prove the absolute continuity of the $\Phi^3_3$-measure $\rho$ with respect to $\text{Law}(Y(1) + \sigma\mathfrak{Z}(1) + \mathcal{W}(1))$ by assuming Lemma A.1.2. We present the proof of Lemma A.1.2 at the end of this appendix. For simplicity, we use the same shorthand notations as in Chapters 3 and 4, for instance, $Y = Y(1)$, $\mathfrak{Z} = \mathfrak{Z}(1)$, $\mathcal{W} = \mathcal{W}(1)$, and $\mathcal{W}_N = \mathcal{W}_N(1)$.

Given $L \gg 1$, let $\delta(L)$ and $R(L)$ satisfy $\delta(L) \to 0$ and $R(L) \to \infty$ as $L \to \infty$, which will be specified later. In view of Lemma A.1.1, it suffices to show that if $G : \mathcal{C}^{-100}(\mathbb{T}^3) \to \mathbb{R}$ is a bounded continuous function with $G > 0$ and

$$\mathbb{P}\big(\{G(Y + \sigma\mathfrak{Z}_N + \mathcal{W}_N) \ge L\}\big) \ge 1 - \delta(L), \tag{A.2.1}$$

then we have

$$\limsup_{N\to\infty} \int \exp(-G(u)) d\rho_N(u) \le \exp(-R(L)), \tag{A.2.2}$$

where $\rho_N$ denotes the truncated $\Phi^3_3$-measure defined in (1.2.11). Here, think of $\text{Law}(Y + \sigma\mathfrak{Z}_N + \mathcal{W}_N)$ as the measure $\mu_N$, weakly converging to $\mu = \text{Law}(Y + \sigma\mathfrak{Z} + \mathcal{W})$.

By the Boué–Dupuis variational formula (Lemma 3.1.1) and the change of variables (3.2.4), we have

$$
\begin{aligned}
&-\log\Big(\int \exp(-G(u) - R_N^{\diamond}(u))d\mu(u)\Big)\\
&\quad = \inf_{\dot{\Upsilon}^N \in \mathbb{H}_a^1} \mathbb{E}\Big[G(Y + \Upsilon^N + \sigma \mathfrak{Z}_N) + \widehat{R}_N^{\diamond}(Y + \Upsilon^N + \sigma \mathfrak{Z}_N)\\
&\qquad\qquad + \frac{1}{2}\int_0^1 \|\dot{\Upsilon}^N(t)\|_{H_x^1}^2 dt\Big],
\end{aligned}
$$

where $\widehat{R}_N^{\diamond}$ is as in (3.2.25). We proceed as in Section 3.2, using Lemmas 3.2.2 and 3.2.3 with Lemma 3.1.2, (3.2.17), and the smallness of $|\sigma|$. See (3.2.9), (3.2.16), and (3.2.19). Thus, we have

$$
\begin{aligned}
&-\log\Big(\int \exp(-G(u) - R_N^{\diamond}(u))d\mu(u)\Big)\\
&\geq \inf_{\dot{\Upsilon}^N \in \mathbb{H}_a^1} \mathbb{E}\Big[G(Y + \Upsilon^N + \sigma \mathfrak{Z}_N) + \frac{1}{20}\int_0^1 \|\dot{\Upsilon}^N(t)\|_{H_x^1}^2 dt\Big] - C_1 \qquad \text{(A.2.3)}
\end{aligned}
$$

for some constant $C_1 > 0$. For $\dot{\Upsilon}^N \in \mathbb{H}_a^1$, let $\Theta^N$ be the solution to (A.1.5) with $\dot{\Upsilon}$ replaced by $\dot{\Upsilon}^N$. For any $M > 0$, define the stopping time $\tau_M$ as

$$
\begin{aligned}
\tau_M = \min\Big(1, \min\Big\{\tau : \int_0^\tau \|\dot{\Upsilon}^N(t)\|_{H_x^1}^2 dt = M\Big\},\\
\min\Big\{\tau : \int_0^\tau \|\dot{\Theta}^N(t)\|_{H_x^1}^2 dt = 2C(M,2)\Big\}\Big), \qquad \text{(A.2.4)}
\end{aligned}
$$

where $C(M,2)$ is the constant appearing in (A.1.6) with $p = 2$. Let

$$
\Theta_M^N(t) := \Theta^N(\min(t, \tau_M)). \qquad \text{(A.2.5)}
$$

From (3.1.2), we have $Y(0) = 0$, while $\mathfrak{Z}^N(0) = 0$ by definition. Then, from the change of variables (3.2.4) with $\Theta(0) = 0$, we see that $\Upsilon^N(0) = 0$. We also have $\mathcal{W}_N(0) = 0$ from (A.1.4). Then, substituting (A.1.3) into (A.1.5) and integrating from $t = 0$ to 1 gives

$$
Y + \Upsilon^N + \sigma \mathfrak{Z}_N = Y + \Theta_M^N + \sigma \mathfrak{Z}_N(Y + \Theta_M^N) + \mathcal{W}_N(Y + \Theta_M^N) \qquad \text{(A.2.6)}
$$

on the set $\{\tau_M = 1\}$.

From the definition (A.2.5) with (A.2.4), we have

$$
\|\dot{\Theta}_M^N\|_{L_t^2([0,1];H_x^1)}^2 \leq 2C(M,2) \qquad \text{(A.2.7)}
$$

and thus the Novikov condition is satisfied. Then, Girsanov's theorem [21, Theorem 10.14] yields that $\mathrm{Law}(Y + \Theta_M^N)$ is absolutely continuous with respect to

Law$(Y)$; see (A.2.10) below. Let $\mathbb{Q} = \mathbb{Q}^{\dot{\Theta}_M^N}$ the probability measure whose Radon–Nikodym derivative with respect to $\mathbb{P}$ is given by the following stochastic exponential:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-\int_0^1 \langle \dot{\Theta}_M^N(t), dY(t)\rangle_{H_x^1} - \frac{1}{2}\int_0^1 \|\dot{\Theta}_M^N(t)\|_{H_x^1}^2 dt} \tag{A.2.8}$$

such that, under this new measure $\mathbb{Q}$, the process

$$W^{\dot{\Theta}_M^N}(t) = W(t) + \langle\nabla\rangle\dot{\Theta}_M^N(t) = \langle\nabla\rangle(Y + \dot{\Theta}_M^N)(t)$$

is a cylindrical Wiener process on $L^2(\mathbb{T}^3)$. By setting

$$Y^{\dot{\Theta}_M^N}(t) = \langle\nabla\rangle^{-1} W^{\dot{\Theta}_M^N}(t),$$

we have

$$Y^{\dot{\Theta}_M^N}(t) = Y(t) + \Theta_M^N(t). \tag{A.2.9}$$

Moreover, from Cauchy–Schwarz inequality with (A.2.8) and the bound (A.2.7), and then (A.2.9), we have

$$\begin{aligned}\mathbb{P}\big(\{Y + \Theta_M^N \in E\}\big) &= \int \mathbf{1}_{\{Y+\Theta_M^N \in E\}} \frac{d\mathbb{P}}{d\mathbb{Q}} d\mathbb{Q} \le C_M \Big(\mathbb{Q}\big(\{Y^{\dot{\Theta}_M^N} \in E\}\big)\Big)^{\frac{1}{2}} \\ &= C_M \big(\mathbb{P}\big(\{Y \in E\}\big)\big)^{\frac{1}{2}} \end{aligned} \tag{A.2.10}$$

for any measurable set $E$.

From (A.2.3), (A.2.6), and the non-negativity of $G$, we have

$$\begin{aligned}\text{(A.2.3)} \ge \inf_{\dot{\Upsilon}^N \in \mathbb{H}_a^1} \mathbb{E}\bigg[&\Big(G\big(Y + \Theta_M^N + \sigma\mathfrak{Z}_N(Y + \Theta_M^N) + \mathcal{W}_N(Y + \Theta_M^N)\big) \\ &+ \frac{1}{20}\int_0^1 \|\dot{\Upsilon}^N(t)\|_{H_x^1}^2 dt\Big)\mathbf{1}_{\{\tau_M = 1\}} \\ &+ \Big(G(Y + \Upsilon^N + \sigma\mathfrak{Z}_N) \\ &\quad + \frac{1}{20}\int_0^1 \|\dot{\Upsilon}^N(t)\|_{H_x^1}^2 dt\Big)\mathbf{1}_{\{\tau_M < 1\}}\bigg] - C_1 \\ \ge \inf_{\dot{\Upsilon}^N \in \mathbb{H}_a^1} \mathbb{E}\bigg[&G\big(Y + \Theta_M^N + \sigma\mathfrak{Z}_N(Y + \Theta_M^N) + \mathcal{W}_N(Y + \Theta_M^N)\big)\cdot\mathbf{1}_{\{\tau_M = 1\}} \\ &+ \frac{1}{20}\int_0^1 \|\dot{\Upsilon}^N(t)\|_{H_x^1}^2 dt \cdot \mathbf{1}_{\{\tau_M < 1\}}\bigg] - C_1.\end{aligned}$$

Then, using the definition (A.2.4) of the stopping time $\tau_M$ and applying (A.2.10) and (A.2.1), we have

$$\begin{aligned}(\text{A.2.3}) \geq \inf_{\dot{\Upsilon}^N \in \mathbb{H}_a^1} \mathbb{E}\Big[ & L \cdot \mathbf{1}_{\{\tau_M = 1\} \cap \{G(Y+\Theta_M^N + \sigma \mathfrak{Z}_N(Y+\Theta_M^N)) + \mathcal{W}_N(Y+\Theta_M^N)) \geq L\}} \\ & + \frac{M}{20} \cdot \mathbf{1}_{\{\tau_M < 1\} \cap \{\int_0^1 \|\dot{\Theta}_M^N(t)\|_{H_x^1}^2 dt < 2C(M,2)\}} \Big] - C_1 \\ \geq \inf_{\dot{\Upsilon}^N \in \mathbb{H}_a^1} \Big\{ & L\big(\mathbb{P}(\{\tau_M = 1\}) - C_M \delta(L)^{\frac{1}{2}}\big) \\ & + \frac{M}{20} \mathbb{P}\bigg(\{\tau_M < 1\} \cap \bigg\{\int_0^1 \|\dot{\Theta}_M^N(t)\|_{H_x^1}^2 dt < 2C(M,2)\bigg\}\bigg)\Big\} - C_1. \end{aligned} \tag{A.2.11}$$

In view of (A.1.6) with (A.2.4) and (A.2.5), Markov's inequality gives

$$\mathbb{P}\bigg(\int_0^1 \|\dot{\Theta}_M^N(t)\|_{H_x^1}^2 dt = \int_0^{\tau_M} \|\dot{\Theta}_M^N(t)\|_{H_x^1}^2 dt \geq 2C(M,2)\bigg) \leq \frac{1}{2},$$

which yields

$$\mathbb{P}\bigg(\{\tau_M < 1\} \cap \bigg\{\int_0^1 \|\dot{\Theta}_M^N(t)\|_{H_x^1}^2 dt < 2C(M,2)\bigg\}\bigg) \geq \mathbb{P}(\{\tau_M < 1\}) - \frac{1}{2}. \tag{A.2.12}$$

Now, we set $M = 20L$. Note from (A.2.4) that $\mathbb{P}(\{\tau_M = 1\}) + \mathbb{P}(\{\tau_M < 1\}) = 1$. Then, from (A.2.11) and (A.2.12), we obtain

$$\begin{aligned} & -\log\bigg(\int \exp(-G(u) - R_N^{\diamond}(u)) d\mu(u)\bigg) \\ & \geq \inf_{\dot{\Upsilon}^N \in \mathbb{H}_a^1} \bigg\{ L\big(\mathbb{P}(\{\tau_M = 1\}) - C_L' \delta(L)^{\frac{1}{2}}\big) + L\bigg(\mathbb{P}(\{\tau_M < 1\}) - \frac{1}{2}\bigg)\bigg\} - C_1 \\ & = L\bigg(\frac{1}{2} - C_L' \delta(L)^{\frac{1}{2}}\bigg) - C_1. \end{aligned}$$

Therefore, by choosing $\delta(L) > 0$ such that $C_L' \delta(L)^{\frac{1}{2}} \to 0$ as $L \to \infty$, this shows (A.2.2) with

$$R(L) = L\bigg(\frac{1}{2} - C_L' \delta(L)^{\frac{1}{2}}\bigg) - C_1 + \log Z,$$

where $Z = \lim_{N \to \infty} Z_N$ denotes the limit of the partition functions for the truncated $\Phi_3^3$-measures $\rho_N$.

## A.3 Proof of Lemma A.1.2

We conclude this appendix by presenting the proof of Lemma A.1.2.

*Proof of Lemma* A.1.2. By Lemma 2.1.3 (ii) and Sobolev's inequality, we have

$$\begin{aligned}
&\|(1-\Delta)^{-1}(2\Theta_N Y_N + \Theta_N^2)(t)\|_{H_x^1}\\
&\quad\lesssim \|(2\Theta_N Y_N + \Theta_N^2)(t)\|_{H_x^{-1}}\\
&\quad\lesssim \|\Theta_N(t)\|_{H_x^{\frac{1}{2}+\varepsilon}}\|Y_N(t)\|_{W_x^{-\frac{1}{2}-\varepsilon,\infty}} + \|\Theta_N^2(t)\|_{L_x^{\frac{6}{5}}}\\
&\quad\lesssim \|\Theta_N(t)\|_{H_x^1}\|Y_N(t)\|_{W_x^{-\frac{1}{2}-\varepsilon,\infty}} + \|\Theta_N(t)\|_{H_x^1}^2
\end{aligned} \tag{A.3.1}$$

for small $\varepsilon > 0$. Moreover, from (A.1.1), we have

$$\begin{aligned}
\|\dot{\mathcal{W}}_N(Y(t)+\Theta(t))\|_{H_x^1} &\lesssim \|\langle\nabla\rangle^{-\frac{1}{2}-\varepsilon}Y_N(t)\|_{L_x^\infty}^5 + \|\langle\nabla\rangle^{-\frac{1}{2}-\varepsilon}\Theta_N(t)\|_{L_x^\infty}^5\\
&\lesssim \|Y_N(t)\|_{W_x^{-\frac{1}{2}-\varepsilon,\infty}}^5 + \|\Theta_N(t)\|_{H_x^1}^5.
\end{aligned} \tag{A.3.2}$$

Therefore, by studying the integral formulation of (A.1.5), a contraction argument in $L^\infty([0,T];H^1(\mathbb{T}^3))$ for small $T>0$ with (A.3.1) and (A.3.2) yields local well-posedness. Here, the local existence time $T$ depends on $\|\Theta(0)\|_{H^1}$, $\|\dot{\Upsilon}\|_{L_T^2 H_x^1}$, and $\|Y_N\|_{L_T^6 W_x^{-\frac{1}{2}-\varepsilon,\infty}}$, where the last term is almost surely bounded in view of Lemma 3.1.2 and (2.1.4).

Next, we prove global existence on $[0,1]$ by establishing an a priori bound on the $H^1$-norm of a solution. From (A.1.5) with (A.1.4), we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\Theta(t)\|_{H^1}^2 &= -\sigma\int_{\mathbb{T}^3}(2\Theta_N(t)Y_N(t)+\Theta_N^2(t))\Theta_N(t)dx\\
&\quad-\int_{\mathbb{T}^3}\big(\langle\nabla\rangle^{-\frac{1}{2}-\varepsilon}(Y_N(t)+\Theta_N(t))\big)^5\cdot\langle\nabla\rangle^{-\frac{1}{2}-\varepsilon}\Theta_N(t)dx\\
&\quad+\int_{\mathbb{T}^3}\langle\nabla\rangle\Theta(t)\cdot\langle\nabla\rangle\dot{\Upsilon}(t)dx.
\end{aligned} \tag{A.3.3}$$

The second term on the right-hand side of (A.3.3), coming from $\mathcal{W}$ is a coercive term, allowing us to hide part of the first term on the right-hand side.

From Lemma 2.1.1 and Young's inequality, we have

$$\begin{aligned}
&\bigg|\int_{\mathbb{T}^3}(2\Theta_N(t)Y_N(t)+\Theta_N^2(t))\Theta_N(t)dx\bigg|\\
&\qquad\lesssim \|\Theta_N(t)\|_{H^1}^2 + \|\Theta_N(t)\|_{L^3}^3 + \|Y_N(t)\|_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}}^c
\end{aligned} \tag{A.3.4}$$

for small $\varepsilon > 0$ and some $c > 0$. We now estimate the second term on the right-hand side of (A.3.4). By (2.1.3), we have

$$\begin{aligned}
\|\Theta_N(t)\|_{L^3}^3 &\lesssim \|\Theta_N(t)\|_{H^1}^{\frac{3+6\varepsilon}{3+2\varepsilon}}\|\Theta_N(t)\|_{W^{-\frac{1}{2}-\varepsilon,6}}^{\frac{6}{3+2\varepsilon}}\\
&\le \|\Theta_N(t)\|_{H^1}^2 + \varepsilon_0\|\Theta_N(t)\|_{W^{-\frac{1}{2}-\varepsilon,6}}^6 + C_{\varepsilon_0}
\end{aligned} \tag{A.3.5}$$

for small $\varepsilon, \varepsilon_0 > 0$. As for the coercive term, from (3.2.32) and Young's inequality, we have

$$\begin{aligned}
&\int_{\mathbb{T}^3} \big(\langle\nabla\rangle^{-\frac{1}{2}-\varepsilon}(Y_N(t)+\Theta_N(t))\big)^5 \langle\nabla\rangle^{-\frac{1}{2}-\varepsilon}\Theta_N(t)dx \\
&\quad \geq \frac{1}{2}\int_{\mathbb{T}^3} (\langle\nabla\rangle^{-\frac{1}{2}-\varepsilon}\Theta_N(t))^6 dx \\
&\qquad - c\int_{\mathbb{T}^3} \big|(\langle\nabla\rangle^{-\frac{1}{2}-\varepsilon}Y_N(t))^5 \langle\nabla\rangle^{-\frac{1}{2}-\varepsilon}\Theta_N(t)\big| dx \\
&\quad \geq \frac{1}{2}\|\Theta_N(t)\|^6_{W^{-\frac{1}{2}-\varepsilon,6}} - c\|Y_N(t)\|^5_{W^{-\frac{1}{2}-\varepsilon,6}}\|\Theta_N(t)\|_{W^{-\frac{1}{2}-\varepsilon,6}} \\
&\quad \geq \frac{1}{4}\|\Theta_N(t)\|^6_{W^{-\frac{1}{2}-\varepsilon,6}} - c\|Y_N(t)\|^6_{W^{-\frac{1}{2}-\varepsilon,6}}.
\end{aligned} \tag{A.3.6}$$

Therefore, putting (A.3.3), (A.3.4), (A.3.5), and (A.3.6) together we obtain

$$\frac{d}{dt}\|\Theta(t)\|^2_{H^1} \lesssim \|\Theta(t)\|^2_{H^1} + \|\dot{\Upsilon}(t)\|^2_{H^1} + \|Y(t)\|^c_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}} + \|Y(t)\|^6_{W^{-\frac{1}{2}-\varepsilon,6}} + 1.$$

By Gronwall's inequality, we then obtain

$$\begin{aligned}
\|\Theta(t)\|^2_{H^1} \lesssim \|\dot{\Upsilon}\|^2_{L^2([0,t];H^1_x)} &+ \|Y_N\|^c_{L^c([0,1];\mathcal{C}_x^{-\frac{1}{2}-\varepsilon})} \\
&+ \|Y\|^6_{L^6([0,1];W_x^{-\frac{1}{2}-\varepsilon,6})} + 1,
\end{aligned} \tag{A.3.7}$$

uniformly in $0 \leq t \leq 1$. The a priori bound (A.3.7) together with Lemma 3.1.2 allows us to iterate the local well-posedness argument, guaranteeing existence of the solution $\Theta$ on $[0,1]$.

Lastly, we prove the bound (A.1.6). From (A.3.1), (A.3.2), and (A.3.7), we have

$$\begin{aligned}
&\|\sigma(1-\Delta)^{-1}(2\Theta_N Y_N + \Theta_N^2) + \dot{\mathcal{W}}_N(Y+\Theta)\|_{L^2([0,\tau];H^1_x)} \\
&\quad \lesssim \|\dot{\Upsilon}\|^5_{L^2([0,\tau];H^1_x)} + \|Y_N\|^{c_0}_{L^q([0,1];\mathcal{C}_x^{-\frac{1}{2}-\frac{1}{2}\varepsilon})} + 1
\end{aligned} \tag{A.3.8}$$

for some finite $q, c_0 \geq 1$ and for any $0 \leq \tau \leq 1$. Then, using the equation (A.1.5), the bound (A.1.6) follows from (A.3.8), the bound on $\dot{\Upsilon}$, and the following corollary to Lemma 3.1.2:

$$\mathbb{E}\Big[\|Y_N\|^p_{L^q([0,1];\mathcal{C}_x^{-\frac{1}{2}-\frac{1}{2}\varepsilon})}\Big] < \infty$$

for any finite $p, q \geq 1$, uniformly in $N \in \mathbb{N}$. ∎

**Remark A.3.1.** A slight modification of the argument presented above shows that the tamed $\Phi^3_3$-measure $\nu_\delta$ constructed in Proposition 4.1.1 is absolutely continuous with respect to the shifted measure $\mathrm{Law}(Y(1) + \sigma\mathfrak{Z}(1) + \mathcal{W}(1))$. In this setting, we can use the analysis in Section 4.2 (Step 1 of the proof of Proposition 4.1.1) to arrive at (A.2.3). The rest of the argument remains unchanged. As a consequence, the $\sigma$-finite version $\overline{\rho}_\delta$ of the $\Phi^3_3$-measure defined in (4.1.9) is also absolutely continuous with respect to the shifted measure $\mathrm{Law}(Y(1) + \sigma \mathfrak{Z}(1) + \mathcal{W}(1))$ for any $\delta > 0$.