Chapter 1

Introduction

1.1 Background

The Seiberg–Witten equations [51] have been an important tool in the study of 4-manifolds since their introduction. Soon after these equations appeared, Kronheimer–Mrowka [28] extended their study to define the monopole Floer homology of 3-manifolds, and established its relationship with the 4-manifold invariant; the resulting theory has since had many applications in low-dimensional topology.

In both gauge theory and symplectic geometry, certain Floer homology theories have since been shown to arise as the homology of well-defined *Floer spectra* as envisioned by Cohen, Jones and Segal [11], and some invariants, obtained by counting solutions of certain PDEs, are now either known or conjectured to come from the degree of certain maps between spectra. One of the first examples of such a construction is the Bauer–Furuta invariant [8,21], which associates an element in stable homotopy $\pi^{\mathrm{st}}(S^0)$ to certain 4-manifolds, refining the ordinary Seiberg–Witten invariant. Building on the finite-dimensional approximation technique introduced by Furuta, Manolescu [35] constructed an S^1-equivariant stable homotopy type $SWF(Y,\mathfrak{s})$ associated to rational homology 3-spheres with spin^c structure $(Y,\mathfrak{s})$.

It is natural to want to extend Manolescu's construction to 3-manifolds with $b_1(Y) > 0$. In the case $b_1(Y) = 1$, Kronheimer–Manolescu [30] constructed a *periodic pro-spectrum* for pairs $(Y,\mathfrak{s})$. Later, together with T. Khandhawit and J. Lin, the first author constructed the *unfolded* Seiberg–Witten Floer spectrum for arbitrary closed, oriented $(Y,\mathfrak{s})$ in [24, 25].

The *unfolded* spectrum comes in multiple flavors. For now, we consider only the type-A unfolded invariant $\underline{\mathrm{swf}}^A(Y,\mathfrak{s})$, which depends on $(Y,\mathfrak{s})$ as well as some additional data we suppress. This invariant is a directed system in the S^1-equivariant stable homotopy category. In particular, it is not per se a spectrum, and tends to be very large.

Khandhawit, Lin and the first author [25] showed that the unfolded invariant allowed for gluing formulas, in a very general setting, for the calculation of the Bauer–Furuta invariant of a 4-manifold cut along 3-manifolds with $b_1 > 0$. In particular, this enables one to prove vanishing formulas for the Bauer–Furuta invariant in many situations.

However, the invariant $\underline{\mathrm{swf}}^A(Y,\mathfrak{s})$ is not expected to recover the monopole Floer homology, but is instead expected to recover a version of monopole Floer homology with fully twisted coefficients.

Here we construct a new Seiberg–Witten Floer spectrum $\mathbf{SWF}(Y, \mathfrak{s})$ for $b_1(Y) > 0$, as follows.

Theorem 1.1.1. *Let* $(Y, \mathfrak{s})$ *be a closed,* spin^c 3*-manifold which satisfies that the first Chern class* $c_1(\mathfrak{s}) \in H^2(Y; \mathbb{Z})$ *is torsion, and so that the triple-cup product on* $H^1(Y; \mathbb{Z})$ *vanishes. Associated to a* Floer framing $\mathfrak{P}$ *(see Section 3.5 for this notation), there is a well-defined parameterized,* S^1*-equivariant stable homotopy type* $\mathcal{SWF}(Y, \mathfrak{s}, \mathfrak{P})$*, over the Picard torus* $\mathrm{Pic}(Y) = H^1(Y; \mathbb{R})/H^1(Y; \mathbb{Z})$*, called the* Seiberg–Witten Floer stable homotopy type *of* $(Y, \mathfrak{s}, \mathfrak{P})$*. Moreover, there is a well-defined (unparameterized)* S^1*-equivariant connected simple system of spectra* $\mathbf{SWF}^u(Y, \mathfrak{s}, \mathfrak{P})$*, the* Seiberg–Witten Floer spectrum.

If $\mathfrak{s}$ *is self-conjugate and* $\mathfrak{P}$ *is a* Pin(2)*-equivariant Floer framing, then the equivariant, parameterized stable homotopy type* $\mathcal{SWF}(Y, \mathfrak{s}, \mathfrak{P})$ *naturally comes with the structure of a parameterized* Pin(2)*-equivariant stable homotopy type, where the Picard torus has a* Pin(2)*-action factoring through* $\pi_0(\mathrm{Pin}(2))$ *by conjugation. Similarly,* $\mathbf{SWF}^u(Y, \mathfrak{s}, \mathfrak{P})$ *has an underlying (unparameterized)* Pin(2)*-equivariant spectrum,* $\mathbf{SWF}^{u,\mathrm{Pin}(2)}(Y, \mathfrak{s}, \mathfrak{P})$.

The homotopy type $\mathcal{SWF}(Y, \mathfrak{s}, \mathfrak{P})$*, viewed without its parameterization, has the homotopy type of a finite* S^1 *(respectively* Pin(2)*)-CW complex. The Seiberg–Witten Floer spectrum* $\mathbf{SWF}^u(Y, \mathfrak{s}, \mathfrak{P})$ *(respectively* $\mathbf{SWF}^{u,\mathrm{Pin}(2)}(Y, \mathfrak{s}, \mathfrak{P})$*) has the homotopy type of a finite* S^1 *(respectively* Pin(2)*) CW-spectrum.*

If $b_1(Y) = 0$*,* $\mathcal{SWF}(Y, \mathfrak{s}, \mathfrak{P})$ *agrees with the invariant* $SWF(Y, \mathfrak{s})$ *in* [35]*, in that*

$$\mathcal{SWF}(Y, \mathfrak{s}, \mathfrak{P}) \simeq \Sigma^{n\mathbb{C}} SWF(Y, \mathfrak{s}),$$

for some $n \in \mathbb{Q}$*, depending only on* $\mathfrak{P}$.

For the notion of the parameterized spaces that we use, ex-spaces, we refer to Appendix A, as well as for the notion of a connected simple system. In particular, see Definition A.1.9 for the notion of a parameterized equivariant stable homotopy type.

The collection of Floer framings of $(Y, \mathfrak{s})$, should any exist, is an affine space over $K(\mathrm{Pic}(Y)) \cong \mathbb{Z}^{2^{b_1(Y)-1}}$. Moreover, there is an explicit relationship between the Floer spectra constructed for different spectral sections; see Corollary 3.6.3.

In order to explain the context of Theorem 1.1.1, and its apparent difference from the unfolded invariant, we review below the process of finite-dimensional approximation, introduced by Furuta, and used by Manolescu [35] in his construction of the 3-manifold invariant for rational homology 3-sphere input, as well as in [24, 25, 30].

1.2 Finite-dimensional approximation

There are two main approaches to refining the construction of Floer-theoretic invariants from homology theories to generalized homology theories (and, in some instances, spectra). There is the approach by constructing *framed flow categories* (or variations on this type of category) as envisioned originally by [11]. A very general version of this has just been accomplished in [1] (while the present work was in its final stages of preparation). There is also the method of finite-dimensional approximation, mentioned above, which we now summarize.

Manolescu's construction of $SWF(Y, \mathfrak{s})$ takes place inside the *Coulomb gauge slice* of the Seiberg–Witten equations. All that matters for this introduction is that, roughly speaking, the Coulomb slice is some Hilbert space on which the Seiberg–Witten equations admit a particularly simple form, as a linear operator plus a compact perturbation. For certain linear subspaces of the Coulomb slice (adapted to the linear part of the Seiberg–Witten equations), Manolescu considers an approximation of the formal L^2-gradient flow of the Seiberg–Witten equations. The approximations tend to stabilize as larger and larger finite-dimensional subspaces are chosen. Associated to suitable flows on suitable topological spaces, there is a convenient invariant, the *Conley index*, which is a well-defined homotopy type associated to the flow (along with some extra data). The invariant $SWF(Y, \mathfrak{s})$ is then taken as the Conley index of these approximated flows.

The most pressing difficulty facing finite-dimensional approximation to other equations of gauge theory or symplectic geometry is that the configuration space in these other situations is usually not linear, so that it is not obvious which finite-dimensional submanifolds one should consider "approximations" on.

For $b_1(Y) > 0$ the gauge slice of the Seiberg–Witten equations is no longer linear, but Kronheimer–Manolescu [30], and the authors of [24, 25], avoided the problem of having a more general configuration space by considering the Seiberg–Witten equations on the universal cover (which is once again a Hilbert space) of a gauge slice to the Seiberg–Witten equations, where finite-dimensional approximation is still possible, but where the usual compactness of the space of Seiberg–Witten trajectories is lost. The loss of compactness leads to the resulting invariant $\underline{\mathrm{swf}}^A$ not being a single spectrum, but rather a system of them.

The problem of performing finite-dimensional approximation in nonlinear situations has been open for some time (though see [27]). In this memoir our objective is to resolve it in one (relatively simple) case, for the Seiberg–Witten equations. We hope that this method may be useful in other situations where one would like to apply finite-dimensional approximation for topologically complicated configuration spaces.

The main work of the present memoir is showing that there exist families of submanifolds of the configuration space of the Seiberg–Witten equations (for $b_1(Y) > 0$) on which the Seiberg–Witten equations can be approximated very accurately. This comes down to carefully controlling spectral sections of the Dirac operator, in the sense of Melrose–Piazza [40], and in particular relies on some control of spectra of Dirac operators. Once the submanifolds are constructed, there also remains the problem of showing that the approximate Seiberg–Witten equations thereon are sufficiently accurate; for this we use a refined version of the original argument of Manolescu which requires weaker assumptions than the original, but does not yield the same strength of convergence as in Manolescu's case.

A word is also in order about the hypotheses on the input in Theorem 1.1.1. Cohen–Jones–Segal conjectured that Floer spectra should exist for many of the familiar Floer homology theories – but only in the event that the *polarization* is trivial. The hypotheses in the theorem are necessary for the vanishing of the polarization (indeed, a Floer framing is the same thing as a trivialization of the polarization), as observed in [30].

However, in spite of usually having a dependence on the Floer framing, we can consider generalized homology theories applied to $\mathbf{SWF}^u(Y, \mathfrak{s}, \mathfrak{P})$ that are insensitive to the framing. In the following theorem, $n(Y, \mathfrak{s}, \mathfrak{P})$ is a certain numerical invariant of a Floer framing, introduced in Chapter 6, and MU and MU_{S^1} denote, respectively, complex cobordism and S^1-equivariant complex cobordism. For the notion of an equivariant complex orientation, see Section 3.6 (and for more detail, [12]).

Theorem 1.2.1. *Let E be a (possibly S^1-equivariant) complex-oriented (resp. S^1-equivariantly complex oriented) cohomology theory. Then*

$$E^{*-2n(Y,\mathfrak{s},\mathfrak{P})}(\mathbf{SWF}^u(Y, \mathfrak{s}, \mathfrak{P}))$$

is (canonically) independent of $\mathfrak{P}$.

In particular, the complex-cobordism theories

$$FMU^*(Y, \mathfrak{s}) = \widetilde{MU}^{*-2n(Y,\mathfrak{s},g,\mathfrak{P})}(\mathbf{SWF}^u(Y, \mathfrak{s}, \mathfrak{P})),$$
$$FMU^*_{S^1}(Y, \mathfrak{s}) = \widetilde{MU}^{*-2n(Y,\mathfrak{s},g,\mathfrak{P})}_{S^1}(\mathbf{SWF}^u(Y, \mathfrak{s}, \mathfrak{P})),$$

are invariants of the pair $(Y, \mathfrak{s})$, which we call the Floer (equivariant) complex cobordism *of $(Y, \mathfrak{s})$.*

As MU, MU_{S^1} are the universal complex-oriented cohomology theories, in some sense $FMU^*(Y, \mathfrak{s})$ and $FMU^*_{S^1}(Y, \mathfrak{s})$ might be interpreted as the universal monopole Floer-type invariants that are independent of the framing.

More speculatively, we remark that the independence of FMU^* on the framing suggests that its definition could be extended to pairs $(Y, \mathfrak{s})$ which do not admit a Floer framing. We plan to pursue this in future work.

It would also be desirable to relate the (generalized) homology theories of the Seiberg–Witten Floer spectrum $\mathbf{SWF}^u(Y, \mathfrak{s}, \mathfrak{P})$ to the monopole-Floer homology of Kronheimer–Mrowka. In particular, we conjecture the following.

Conjecture 1.2.2. For $(Y, \mathfrak{s})$ a pair as in Theorem 1.1.1,

$$\begin{aligned} H^{S^1}_{\bullet - 2n(Y,\mathfrak{s},\mathfrak{P})}(\mathbf{SWF}^u(Y,\mathfrak{s},\mathfrak{P})) &\cong HM_\bullet(Y,\mathfrak{s}), \\ cH^{S^1}_{\bullet - 2n(Y,\mathfrak{s},\mathfrak{P})}(\mathbf{SWF}^u(Y,\mathfrak{s},\mathfrak{P})) &\cong \widehat{HM}_\bullet(Y,\mathfrak{s}), \\ tH^{S^1}_{\bullet - 2n(Y,\mathfrak{s},\mathfrak{P})}(\mathbf{SWF}^u(Y,\mathfrak{s},\mathfrak{P})) &\cong \overline{HM}(Y,\mathfrak{s}), \\ H_{\bullet - 2n(Y,\mathfrak{s},\mathfrak{P})}(\mathbf{SWF}^u(Y,\mathfrak{s},\mathfrak{P})) &\cong \widetilde{HM}_\bullet(Y,\mathfrak{s}). \end{aligned}$$

Note that ordinary homology is (equivariantly) complex-orientable, and so the homology theories on each left-hand side are independent of the choice of spectral section (and we have been somewhat imprecise about the gradings on the right). Here, H^{S^1}, cH^{S^1}, tH^{S^1} are, respectively, Borel, coBorel and Tate homology.

This conjecture has already been established by Lidman–Manolescu in the case that Y is a rational-homology sphere [32].

We note that there is a natural generalization of Conjecture 1.2.2 to include the case of local coefficient systems on monopole Floer homology HM°; this involves using other parameterized cohomology theories (as in [39, Section 20.3]) applied to $\mathcal{SWF}(Y,\mathfrak{s},\mathfrak{P})$. There is also a further generalization of the conjecture to relate the Pin(2)-equivariant cohomology of $\mathbf{SWF}^u(Y,\mathfrak{s},\mathfrak{P})$, for $(Y,\mathfrak{s})$ admitting a Pin(2)-equivariant Floer framing, to the equivariant monopole Floer homology defined by Lin [34].

We remark that Theorem 1.1.1 should yield a well-defined connected simple system $\mathbf{SWF}(Y,\mathfrak{s},\mathfrak{P})$ of equivariant, parameterized spectra. Indeed, this would follow if the *parameterized* Conley index of a dynamical system were known to be well defined as a connected simple system (rather than as a homotopy type; the ordinary Conley index is known [47] to be a connected simple system). We hope to return to this point, and other improvements to naturality, in future work.

1.3 Four-manifolds

In this memoir we also define a Bauer–Furuta invariant associated to a spin^c 4-manifold with boundary.

Let $(Y,\mathfrak{s})$ be a closed spin^c 3-manifold and $\mathfrak{P}$ be a Floer framing of $(Y,\mathfrak{s})$. Recall that, in the parameterized setting, we only define the ex-space $\mathcal{SWF}(Y,\mathfrak{s},\mathfrak{P})$ up to stable homotopy equivalence. To fix notation, define a *map class* of maps $P \to Q$ between two spaces P, Q, themselves only well defined up to homotopy equivalence, to mean just a homotopy class, up to the action of self-homotopy-equivalences on P or Q.

For an S^1-equivariant virtual vector bundle V over a base B, let S_B^V denote the corresponding sphere bundle over B. We then construct a Bauer–Furuta invariant $\mathcal{BF}$ as follows.

Theorem 1.3.1. *Let* $(X, \mathfrak{t})$ *be a smooth, compact,* spin^c 4*-manifold with boundary* $(Y, \mathfrak{s})$*, and fix a Floer framing* $\mathfrak{P}$ *of* $(Y, \mathfrak{s})$*. Then there is a well-defined (parameterized,* S^1*-equivariant, stable) map class*

$$\mathcal{BF}(X, \mathfrak{t}) \colon S_{\mathrm{Pic}(Y)}^{\mathrm{ind}(D_X, \mathfrak{P})} \to \mathcal{SWF}(Y, \mathfrak{s}, \mathfrak{P}).$$

For the definition of the index $\mathrm{ind}(D_X, \mathfrak{P})$, see Chapter 5. There is also a version of Theorem 1.3.1 at the spectrum level, which is more complicated to state; see Corollary 5.2.7.

As a by-product of our proof of well-definedness of $\mathcal{SWF}(Y, \mathfrak{s}, \mathfrak{P})$, we also obtain an invariant of families.

Theorem 1.3.2. *Let* $\mathcal{F}$ *be a Floer-framed family of* spin^c 3*-manifolds, with compact base* B *and fibers denoted by* $\mathcal{F}_b$ *for* $b \in B$*. Let* $\mathrm{Pic}(\mathcal{F})$ *denote the bundle over* B *with fiber* $\mathrm{Pic}(\mathcal{F}_b)$*. There is a well-defined parameterized,* S^1*-equivariant stable-homotopy type* $\mathcal{SWF}(\mathcal{F})$*, which is parameterized over* $\mathrm{Pic}(\mathcal{F})$*.*

A similar families invariant exists for the Bauer–Furuta invariant, but we omit its discussion, as we do not have need of it in the present memoir.

As an application of our construction, we construct Frøyshov-type invariants associated to the Seiberg–Witten Floer stable homotopy type. In particular, we define a generalization of Manolescu's κ-invariant, from Pin(2)-equivariant K-theory of 3-manifolds with $b_1(Y) = 0$, to Y with $b_1(Y) > 0$. We show the following theorem.

Theorem 1.3.3. *Let* $(X, \mathfrak{t})$ *be a compact, spin* 4*-manifold with boundary* $-Y_0 \coprod Y_1$*. Assume that* Y_0 *is a rational homology* 3*-sphere and the index* $\mathrm{ind}\, D$ *for* $(Y_1, \mathfrak{t}|_{Y_1})$ *is zero in* $KQ^1(\mathrm{Pic}(Y_1))$*. Here,* KQ^1 *stands for the quaternionic* K*-theory. (See* [19,33]*.) Then we have*

$$-\frac{\sigma(X)}{8} + \kappa(Y_0, \mathfrak{t}|_{Y_0}) - 1 \le b^+(X) + \kappa(Y_1, \mathfrak{t}|_{Y_1}).$$

See Remark 6.2.13 for the reason why we assume $b_1(Y_0) = 0$ in this theorem.

We also define invariants associated to the S^1-equivariant monopole Floer homology, corresponding roughly to the generalized d-invariants introduced by Levine–Ruberman [31] in Heegaard Floer homology.

We also calculate the Seiberg–Witten Floer homotopy-type invariant in some relatively simple situations; see Chapter 4.

1.4 Further directions

We do not prove any gluing theorems for the Bauer–Furuta invariant, or for its families analog, and this is a natural point of departure, remaining within Seiberg–Witten theory. In this direction, we expect the surgery exact triangles [28, Section 42] (and variations) to hold for homology theories other than ordinary homology. For this, it would be particularly desirable to obtain a description of the map on FMU^* induced by the Bauer–Furuta invariant, independent of choices like the Floer framing. It is also natural to ask how the unfolded spectrum $\underline{\mathrm{swf}}^A(Y,\mathfrak{s})$ is related to the folded spectrum $\mathcal{SWF}(Y,\mathfrak{s})$.

A technical problem that may make the invariant $\mathcal{SWF}(Y,\mathfrak{s},\mathfrak{P})$ more wieldy is to establish a natural topological description (on Y) of the set of Floer framings. We hope to address some of these points in the future.

Furthermore, we expect that it should be possible to consider more detailed applications to the question of when a family of 3-manifolds extends to a family of 4-manifolds with boundary. Compare with recent work by Konno–Taniguchi [26] in the case that the boundary family of 3-manifolds is the trivial family of a rational homology sphere.

Finally, given an extension of $FMU^*(Y,\mathfrak{s})$ to 3-manifolds that do not admit a Floer framing, it seems likely that the excision argument of [29] should apply, in which case we would expect there to exist generalizations of sutured monopole Floer homology to various generalized homology theories.

1.5 Organization

This memoir is organized as follows. We first construct special families of spectral sections to the Dirac operator in Chapter 2, and show that certain subsets of the (approximate) Seiberg–Witten configuration space are isolating neighborhoods in the sense of Conley index theory. In Chapter 3 we show that the resulting invariant is well defined, as a consequence of this process we establish a Seiberg–Witten Floer homotopy type for families. This consists of showing that all of the possible choices for different approximations to the Seiberg–Witten equations are compatible. In Chapter 4 we give various example calculations of $\mathcal{SWF}(Y,\mathfrak{s},\mathfrak{P})$. In Chapter 5 we construct a relative Bauer–Furuta invariant, and show that it is well defined. Finally, in Chapter 6 we establish various Frøyshov-type inequalities that are a consequence of the existence of the new relative Bauer–Furuta invariant.

There is one appendix, Appendix A, on homotopy-theoretic background, as well as an afterword on potential further applications outside of Seiberg–Witten theory.