Chapter 2

Finite-dimensional approximation on 3-manifolds

2.1 Spectral sections

In order to define Seiberg–Witten Floer spectra, we will make use of spectral sections of a family of Dirac operators introduced by Melrose–Piazza [40]. We will recall definitions and basic things on spectral sections in this section.

Suppose that we have a closed, oriented $(2n-1)$-manifold Y and that we have a fiber bundle

$$\mathcal{Y} \to B$$

with fiber Y. Here, B is a compact Hausdorff space. Also suppose that we are given a finite-dimensional vector bundle

$$F_Y \to \mathcal{Y}$$

with metric. We consider an infinite-dimensional vector bundle on B defined by

$$\mathcal{E}_{Y,\infty} := \bigcup_{z \in B} \Gamma(F_Y|_{\mathcal{Y}_z}).$$

Let

$$D_Y \colon \mathcal{E}_{Y,\infty} \to \mathcal{E}_{Y,\infty}$$

be a family of first-order elliptic, self-adjoint differential operators. That is, D_Y preserves the fibers of $\mathcal{E}_{Y,\infty}$ and for each $z \in B$,

$$D_{Y,z} \colon \mathcal{E}_{Y,\infty,z} \to \mathcal{E}_{Y,\infty,z}$$

is a first-order, elliptic, self-adjoint differential operator. Here, $\mathcal{E}_{Y,\infty,z}$ is the fiber of $\mathcal{E}_{Y,\infty}$ over z.

We assume that for each $z \in B$, there is an open neighborhood U of z such that we have a trivialization

$$F_Y|_{\mathcal{Y}_U} \cong U \times F_{Y,z}, \tag{2.1.1}$$

where $\mathcal{Y}_U$ is the restriction of the bundle $\mathcal{Y}$ to U, and we can write

$$D_{Y,w} = D_{Y,z} + A_{Y,w}$$

for $w \in U$ through the isomorphism $\mathcal{E}_{Y,\infty,z} \cong \mathcal{E}_{Y,\infty,w}$ induced by (2.1.1). Here, $A_{Y,w}$ is the operator acting on $\mathcal{E}_{Y,\infty,w}$ induced by a fiberwise linear bundle map $F_Y|_{\mathcal{Y}_w} \to F_Y|_{\mathcal{Y}_w}$ which continuously depends on w.

For $k \geq 0$, define the L^2_k-inner product on $\mathcal{E}_{Y,\infty}$ by

$$\langle \phi_1, \phi_2 \rangle_k = \int_{\mathcal{Y}_z} \langle \phi_1, \phi_2 \rangle + \langle |D_{Y,z}|^k \phi_1, |D_{Y,z}|^k \phi_2 \rangle \, d\mu.$$

Here, $|D_{Y,z}|$ denotes the absolute value of $D_{Y,z}$ defined as in [46, Chapter VIII, §9]. We write $\mathcal{E}_{Y,k}$ for the completions with respect to the L^2_k-norm. The operator D_Y extends to a bounded operator

$$D_Y \colon \mathcal{E}_{Y,k} \to \mathcal{E}_{Y,k-1}.$$

For $w \in U$, the algebraic operator $A_{Y,w}$ extends to a bounded operator $\mathcal{E}_{Y,k,w} \to \mathcal{E}_{Y,k,w}$ which continuously depends on w with respect to the operator norm, and $D_{Y,w} = D_{Y,z} + A_{Y,w}$ as operators $\mathcal{E}_{Y,k,w} \to \mathcal{E}_{Y,k-1,w}$ through the local trivialization (2.1.1).

We now recall the definition of a spectral section from [40].

Definition 2.1.1 ([40]). A *spectral section* for $D_Y \colon \mathcal{E}_{Y,k} \to \mathcal{E}_{Y,k-1}$ over a compact base B is a family of self-adjoint projections $P \colon \mathcal{E}_{Y,0} \to \mathcal{E}_{Y,0}$ so that there is a constant $C > 0$ such that the following holds. Suppose that $z \in B$, $u \in \mathcal{E}_{Y,\infty,z}$, $D_{Y,z} u = \lambda u$ for some $\lambda \in \mathbb{R}$. Then $P_z u = u$ if $\lambda > C$ and $P_z u = 0$ if $\lambda < -C$. Here, a *family* is meant to be a continuous family in the L^2-operator norm topology, parameterized by B.

We note that the condition that P be continuous families in the L^2-norm topology is equivalent to P being continuous families in any L^2_k-norm topology with $k > 0$, using the interaction of P with the spectrum of D_Y. Also note that since P is self-adjoint, P is an orthogonal projection onto its image with respect to the L^2-inner product. In fact, for $\phi_1, \phi_2 \in \mathcal{E}_{Y,\infty,z}$, we have

$$\langle P\phi_1, (1-P)\phi_2 \rangle_0 = \langle \phi_1, P(1-P)\phi_2 \rangle_0 = 0.$$

Here we have used the fact that P is self-adjoint and $P^2 = P$.

Melrose and Piazza proved the following about the existence of a spectral section.

Theorem 2.1.2 ([40, Proposition 1]). *There exists a spectral section of D_Y if and only if the index* $\operatorname{ind} D_Y$ *is zero in $K^1(B)$. Here,* $\operatorname{ind} D_Y$ *is the index defined in* [6].

Using a spectral section, we can define the Atiyah–Patodi–Singer index for a family of differential operators on a manifold with boundary. Let X be a compact, oriented $2n$-manifold with boundary Y. Suppose that we have a fiber bundle

$$\mathcal{X} \to B$$

with fiber X, such that the family obtained by taking the boundary of each fiber of $\mathcal{X}$ is $\mathcal{Y}$. Also suppose that we have finite-dimensional vector bundles

$$F^0_X, F^1_X \to \mathcal{X}$$

and that isomorphisms

$$F_X^0|_{\mathcal{Y}} \cong F_X^1|_{\mathcal{Y}} \cong F_Y$$

are given. Define infinite-dimensional vector bundles over B by

$$\mathcal{E}_{X,\infty}^0 = \bigcup_{z\in B} \Gamma(F_X^0|_{\mathcal{X}_z}), \quad \mathcal{E}_{X,\infty}^1 = \bigcup_{z\in B} \Gamma(F_X^1|_{\mathcal{X}_z}).$$

We consider a family of first-order elliptic differential operators

$$D_X \colon \mathcal{E}_{X,\infty}^0 \to \mathcal{E}_{X,\infty}^1$$

such that

$$D_X = \frac{\partial}{\partial t} + D_Y$$

near the boundary $\mathcal{Y}$. Here, t is the coordinate of the first component of a neighborhood of $\mathcal{Y}$ in $\mathcal{X}$ which is diffeomorphic to $[0,1] \times \mathcal{Y}$. As before, we assume that for $z \in B$, there is an open neighborhood U of z and we can write $D_{X,w} = D_{X,z} + A_{X,w}$ for $w \in U$ through local trivializations of F_X^0, F_X^1. Here, $A_{X,w}$ is an algebraic operator induced by a linear bundle map $F_X^0|_{\mathcal{X}_w} \to F_X^1|_{\mathcal{X}_w}$ depending on w continuously.

We define Hilbert bundles $\mathcal{E}_{X,k}^0$, $\mathcal{E}_{X,k}^1$ over B for $k \geq 0$ using D_X as before. Note that $\operatorname{ind} D_Y = 0$ in $K^1(B)$ because of the cobordism invariance of the index. Hence there is a spectral section of D_Y.

Let $(\mathcal{E}_{Y,k-\frac{1}{2}})_{-\infty}^0$ be the subspace spanned by nonpositive eigenvectors of D_Y and p^0 be the $L^2_{k-\frac{1}{2}}$-orthogonal projection onto $(\mathcal{E}_{Y,k-\frac{1}{2}})_{-\infty}^0$. Let us consider the family of operators with the APS boundary condition. That is, we consider the family of operators

$$(D_X, p^0 \circ r) \colon \mathcal{E}_{X,k}^0 \to \mathcal{E}_{X,k-1}^1 \oplus (\mathcal{E}_{Y,k-\frac{1}{2}})_{-\infty}^0.$$

Here, r is the restriction to $\mathcal{Y}$. Note that this family is not continuous because of the spectral flow of D_Y. Hence we cannot use this family to define the index. A spectral section enables us to avoid this issue. Since our sign convention is different from that of [40], taking a spectral section of $-D_Y$ rather than D_Y is more convenient for us.

Proposition 2.1.3. *Fix $k \geq 1$. Let P be a spectral section of $-D_Y$. We also denote by P the image of P in $\mathcal{E}_{Y,0}$, which is a Hilbert subbundle. Let π_P be the $L^2_{k-\frac{1}{2}}$-projection onto $P \cap \mathcal{E}_{Y,k-\frac{1}{2}}$. Then*

$$(D_X, \pi_P \circ r) \colon \mathcal{E}_{X,k}^0 \to \mathcal{E}_{X,k-1}^1 \oplus (P \cap \mathcal{E}_{Y,k-\frac{1}{2}})$$

is a continuous family of Fredholm operators and we can define the index $\operatorname{ind}(D_X, P) \in K(B)$. *The index* $\operatorname{ind}(D_X, P)$ *is independent of the choice of k.*

Let P be a spectral section of $-D_Y$. We write P for the image of P in $\mathcal{E}_{Y,0}$ too. Then we can take other spectral sections Q, R of $-D_Y$ such that

$$Q \subset P \subset R.$$

See our construction of spectral sections in Section 2.4. Define a family of operators

$$D'_Y := QD_YQ + (1-R)D_Y(1-R) - (1-Q)P + R(1-P).$$

We can see that D'_Y is injective and that P is equal to the subspace spanned by negative eigenvectors of D'_Y. Also we see that the operator $\mathbb{A} = D'_Y - D_Y$ is a family of smoothing operators acting on $\mathcal{E}_{Y,k}$. In fact, the image of $\mathbb{A}$ is included in the subspace spanned by finitely many eigenvectors of D_Y.

Take a smooth function $f\colon \mathcal{X} \to [0,1]$ such that

$$f(x) = \begin{cases} 1 & \text{for } x \in [\frac{1}{2}, 1] \times \mathcal{Y}, \\ 0 & \text{for } x \in \mathcal{X} \setminus ([0,1] \times \mathcal{Y}). \end{cases}$$

Define $D'_X\colon \mathcal{E}^0_{X,k} \to \mathcal{E}^1_{X,k-1}$ by

$$D'_X = D_X + f\mathbb{A}.$$

Then

$$D'_X = \frac{\partial}{\partial t} + D'_Y$$

near $\mathcal{Y}$ and there is no spectral flow of D'_Y. Therefore, the family of operators D'_X with the APS boundary condition defines the index $\operatorname{ind} D'_X \in K(B)$, and

$$\operatorname{ind} D'_X = \operatorname{ind}(D_X, P).$$

2.2 Connections on Hilbert bundles

Since we will consider a connection on a Hilbert bundle later, we give the definition of a connection on a Hilbert bundle.

Let M be a connected, smooth n-manifold and H be a Hilbert space. We write $\operatorname{Aut} H$ and $\operatorname{End} H$ for the group of bounded linear isomorphisms $H \to H$ and the ring of bounded operators $H \to H$ respectively.

Take a coordinate chart (U, φ) of M. For a map

$$f\colon U \to H,$$

we define the partial derivative $\frac{\partial f}{\partial x^i}(x)$ at $x \in U$ by

$$\frac{\partial f}{\partial x^i}(x) = \lim_{h\to 0} \frac{1}{h}\big(f \circ \varphi^{-1}(\varphi(x) + he_i) - f(x)\big)$$

if the limit exists in H. Here, e_i is the ith standard basis of $\mathbb{R}^n$. For $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{Z}_{\geq 0})^n$, we define $\frac{\partial^\alpha f}{\partial x^\alpha}$ to be $\left(\frac{\partial}{\partial x^1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x^n}\right)^{\alpha_n} f$. We say that f is smooth if the derivatives $\frac{\partial^\alpha f}{\partial x^\alpha}$ exist and are continuous on U for all $\alpha \in (\mathbb{Z}_{\geq 0})^n$.

Let $p\colon \mathcal{E} \to M$ be a smooth Hilbert bundle on M with fiber H. By a smooth Hilbert bundle we mean that for each small open set U in M, we have a local trivialization

$$\psi\colon \mathcal{E}|_U \to U \times H$$

such that if $\psi'\colon \mathcal{E}|_{U'} \to U' \times H$ is another local trivialization with $U \cap U' \neq \emptyset$, we can write

$$\psi' \circ \psi^{-1}(x, v) = (x, g(x)v)$$

for $x \in U \cap U'$ and $v \in H$, and g is a map $U \cap U' \to \operatorname{Aut} H$ which is smooth with respect to the operator norm. We always assume that Hilbert bundles are smooth.

A section $s\colon M \to \mathcal{E}$ is said to be smooth if for each local trivialization $\psi\colon \mathcal{E}|_U \to U \times H$, the map

$$\psi \circ s|_U\colon U \to U \times H$$

is smooth. We denote by $\Gamma(\mathcal{E})$ the space of smooth sections of $\mathcal{E}$.

A connection ∇ on $\mathcal{E}$ is defined to be a map

$$\nabla\colon \Gamma(\mathcal{E}) \to \Gamma(T^*M \otimes \mathcal{E})$$

having the following properties:

(i) For any sections $s_1, s_2 \in \Gamma(\mathcal{E})$,

$$\nabla(s_1 + s_2) = \nabla s_1 + \nabla s_2.$$

(ii) For any section $s \in \Gamma(\mathcal{E})$, vector fields $X_1, X_2 \in \Gamma(TM)$ and smooth functions $f_1, f_2 \in C^\infty(M)$,

$$\nabla_{f_1 X_1 + f_2 X_2} s = f_1 \nabla_{X_1} s + f_2 \nabla_{X_2} s.$$

(iii) For any section $s \in \Gamma(\mathcal{E})$ and function $f \in C^\infty(M)$,

$$\nabla(fs) = df \otimes s + f \nabla s.$$

We define a connection ∇ on the dual Hilbert bundle $\mathcal{E}^*$ by

$$(\nabla_X \alpha)(s) := X(\alpha(s)) - \alpha(\nabla_X s).$$

Here, $s \in \Gamma(\mathcal{E})$, $\alpha \in \Gamma(\mathcal{E}^*)$, $X \in \Gamma(TM)$.

For connections ∇_1, ∇_2 on Hilbert bundles $\mathcal{E}_1$, $\mathcal{E}_2$ over M, we define connections $\nabla_1 \oplus \nabla_2$, $\nabla_1 \otimes \nabla_2$ on $\mathcal{E}_1 \oplus \mathcal{E}_2$, $\mathcal{E}_1 \otimes \mathcal{E}_2$ by

$$(\nabla_1 \oplus \nabla_2)(s_1 \oplus s_2) := (\nabla_1 s_1) \oplus (\nabla_2 s_2),$$

$$(\nabla_1 \otimes \nabla_2)(s_1 \otimes s_2) := (\nabla_1 s_1) \otimes s_2 + s_1 \otimes (\nabla_2 s_2).$$

Write $\Omega^i(M;\mathcal{E})$ for the space of i-forms on M with values in $\mathcal{E}$:

$$\Omega^i(M;\mathcal{E}) := \Gamma(\Lambda^i T^*M \otimes \mathcal{E}).$$

For a connection ∇ on $\mathcal{E}$, we have the exterior derivative

$$d_\nabla\colon \Omega^i(M;\mathcal{E}) \to \Omega^{i+1}(M;\mathcal{E})$$

defined by

$$\begin{aligned} d_\nabla(\eta s) &= (d\eta)s + \eta \wedge (\nabla s), \\ d_\nabla(\eta_1 + \eta_2) &= d_\nabla \eta_1 + d_\nabla \eta_2. \end{aligned}$$

Here, $s \in \Gamma(\mathcal{E})$, $\eta \in \Omega^i(M)$, $\eta_1, \eta_2 \in \Omega^i(M;\mathcal{E})$.

We will make an assumption on the smoothness of ∇. Take a local trivialization $\psi\colon \mathcal{E}|_U \to U \times H$. We can write

$$\psi \nabla_X s = X(\psi s) + \omega(X)(\psi s) \tag{2.2.1}$$

for $s \in \Gamma(\mathcal{E}|_U)$ and $X \in \Gamma(TU)$. Here, for each $x \in U$ and $X \in T_x U$, $\omega(X)$ is a linear map $H \to H$. The assumption is that $\omega(X)$ is bounded and the map $\omega\colon TU \to \operatorname{End} H$ is smooth with respect to the operator norm. In particular, for a compact set K in U, the restriction $\omega(X)|_K$ is a Lipschitz continuous map $K \to \operatorname{End} H$.

Under the above assumption, for any smooth curve $c\colon [-\varepsilon, \varepsilon] \to U$ and $e \in \mathcal{E}_{c(0)}$, where $\varepsilon > 0$, we have a unique smooth section s of $\mathcal{E}$ along c which solves the ordinary differential equation in the Hilbert space:

$$\frac{d}{dt}\psi(s(t)) + \omega\Big(\frac{dc}{dt}(t)\Big)(\psi s(t)) = 0, \quad s(0) = e.$$

We call s a parallel section of $\mathcal{E}$ along c or a horizontal lift of c. See [18] for the existence and uniqueness of solutions to the equation.

Take $x \in U$ and let $x^1, \dots, x^n$ be local coordinates around x. For $i = 1, \dots, n$, let c_i be a smooth curve $[-\varepsilon, \varepsilon] \to U$ such that

$$c_i(0) = x, \quad \frac{dc_i}{dt}(0) = \frac{\partial}{\partial x^i}.$$

For $e \in \mathcal{E}_x$, we define the horizontal component $(T_e\mathcal{E})_H$ of $T_e\mathcal{E}$ to be the subspace spanned by $\{ds_i(\frac{\partial}{\partial t})\}_{i=1,\dots,n}$. Here, s_i is the parallel section of $\mathcal{E}$ along c_i with $s_i(0) = e$. We can show that $(T_e\mathcal{E})_H$ is independent of the choice of the local coordinates $x^1, \dots, x^n$. The connection ∇ defines a decomposition

$$T\mathcal{E} = (T\mathcal{E})_H \oplus p^*\mathcal{E}.$$

Note that we have a natural isomorphism

$$(T\mathcal{E})_H \cong p^*TM.$$

As usual, there is a unique 2-form $F_\nabla \in \Omega^2(M; \operatorname{End}\mathcal{E})$ such that

$$d_\nabla \circ d_\nabla \eta = F_\nabla \wedge \eta$$

for $\eta \in \Omega^i(M;\mathcal{E})$. We can write

$$\psi F_\nabla = d\omega + \omega \wedge \omega$$

on U, where ω is the 1-form with values in $\operatorname{End} H$ in (2.2.1). We call F_∇ the curvature of ∇. We say that ∇ is flat if $F_\nabla = 0$.

We can associate a flat connection to a representation

$$\rho\colon \pi_1(M) \to \operatorname{Aut}(H)$$

in the usual way. Let $\mathcal{E}$ be the Hilbert bundle on M defined by

$$\mathcal{E} := \widetilde{M} \times_\rho H,$$

where $\widetilde{M}$ is the universal cover of M. A smooth section $s\colon M \to \mathcal{E}$ corresponds to a smooth map $\tilde{s}\colon \widetilde{M} \to H$ such that

$$\tilde{s}(\gamma \cdot x) = \rho(\gamma)\tilde{s}(x)$$

for $x \in \widetilde{M}$, $\gamma \in \pi_1(M)$. Taking the exterior derivative, we have

$$d\tilde{s}(\gamma \cdot x) = \rho(\gamma)\, d\tilde{s}(x)$$

and hence $d\tilde{s}$ descends to a section of $T^*M \otimes \mathcal{E}$, which we denote by ∇s. We can show that the map

$$\nabla\colon \Gamma(\mathcal{E}) \to \Gamma(T^*M \otimes \mathcal{E})$$

is a flat connection on $\mathcal{E}$.

2.3 Notation and main statements

Let Y be a connected, closed, oriented 3-manifold and take a Riemannian metric g and spin^c structure $\mathfrak{s}$ with $c_1(\mathfrak{s})$ torsion on Y. We denote the spinor bundle over Y by $\mathbb{S}$. Fix a spin^c connection A_0 on Y with $F_{A_0} = 0$. For a 1-form $a \in \Omega^1(Y)$, we write D_a for the Dirac operator D_{A_0+ia} which acts on the space $C^\infty(\mathbb{S})$ of smooth sections of $\mathbb{S}$. The family $\{D_a\}_{a\in\mathcal{H}^1(Y)}$ parameterized by the harmonic 1-forms on Y induces an operator D acting on the vector bundle

$$\mathcal{E}_\infty = \mathcal{H}^1(Y) \times_{H^1(Y;\mathbb{Z})} C^\infty(\mathbb{S})$$

over the Picard torus $\mathrm{Pic}(Y) = H^1(Y;\mathbb{R})/H^1(Y;\mathbb{Z})$. The action of $H^1(Y;\mathbb{Z})$ is defined by

$$h(a,\phi) = (a - h, u_h \phi)$$

for $h \in H^1(Y;\mathbb{Z})$, $a \in \mathcal{H}^1(Y)$, $\phi \in C^\infty(\mathbb{S})$, where u_h is the harmonic gauge transformation $Y \to U(1)$ with $-iu_h^{-1}\,du_h = h$ in $\mathcal{H}^1(Y)$.

For $k \in \mathbb{R}_{\geq 0}$, define a Hilbert bundle on $\mathrm{Pic}(Y)$ by

$$\mathcal{E}_k := \mathcal{H}^1(Y) \times_{H^1(Y;\mathbb{Z})} L^2_k(\mathbb{S}).$$

For $k \geq 1$, the operator D on $\mathcal{E}_\infty$ extends to a bounded operator

$$D\colon \mathcal{E}_k \to \mathcal{E}_{k-1}.$$

We have a canonical flat connection ∇ on $\mathcal{E}_k$ corresponding to the representation

$$\begin{aligned} \pi_1(B) = H^1(Y;\mathbb{Z}) &\to \mathrm{Aut}(L^2_k(\mathbb{S})), \\ h &\mapsto u_h, \end{aligned}$$

where $B = \mathrm{Pic}(Y)$, $\mathrm{Aut}(L^2_k(\mathbb{S}))$ is the group of bounded linear automorphisms on $L^2_k(\mathbb{S})$. See Section 2.2.

A smooth section $s\colon B \to \mathcal{E}_k$ can be considered to be a smooth map

$$\tilde{s}\colon \mathcal{H}^1(Y) \to L^2_k(\mathbb{S})$$

such that

$$\tilde{s}(a-h) = u_h \tilde{s}(a)$$

for $h \in \mathrm{im}(H^1(Y;\mathbb{Z}) \to \mathcal{H}^1(Y))$. The covariant derivative ∇s corresponds to the usual exterior derivative $d\tilde{s}$ of $\tilde{s}$.

Denote by $\langle \cdot,\cdot \rangle_{a,k}$ the L^2_k-inner product with respect to D_a:

$$\langle \phi_1, \phi_2 \rangle_{a,k} = \langle \phi_1, \phi_2 \rangle_0 + \langle |D_a|^k \phi_1, |D_a|^k \phi_2 \rangle_0,$$

where $\langle \cdot,\cdot \rangle_0$ is the $L^2(Y)$-inner product. Here we write $|D_a|$ for the absolute value of the Dirac operator D_a, defined using the spectral theorem (see e.g. [46, Chapter VIII, §9]). Then the family $\{\langle \cdot,\cdot \rangle_{a,k}\}_{a \in \mathcal{H}^1(Y)}$ of L^2_k-inner products induces a fiberwise inner product $\langle \cdot,\cdot \rangle_k$ on $\mathcal{E}_k$. To see this, take sections $s_1, s_2\colon B \to \mathcal{E}_k$ and $h \in \mathrm{im}(H^1(Y;\mathbb{Z}) \to \mathcal{H}^1(Y))$. Let $\tilde{s}_1, \tilde{s}_2\colon \mathcal{H}^1(Y) \to L^2_k(\mathbb{S})$ be the maps corresponding to s_1, s_2. Note that

$$\tilde{s}_i(a-h) = u_h \tilde{s}_i(a), \quad D_{a-h} = u_h D_a u_h^{-1}.$$

Therefore,

$$\begin{aligned}
&\langle \tilde{s}_1(a-h), \tilde{s}_2(a-h)\rangle_{a-h,k} \\
&\quad = \langle \tilde{s}_1(a-h), \tilde{s}_2(a-h)\rangle_0 + \langle |D_{a-h}|^k \tilde{s}_1(a-h), |D_{a-h}|^k \tilde{s}_2(a-h)\rangle_0 \\
&\quad = \langle u_h \tilde{s}_1(a), u_h \tilde{s}_2(a)\rangle_0 + \langle (u_h |D_a|^k u_h^{-1}) u_h \tilde{s}_1(a), (u_h |D_a|^k u_h^{-1}) u_h \tilde{s}_2(a)\rangle_0 \\
&\quad = \langle u_h \tilde{s}_1(a), u_h \tilde{s}_2(a)\rangle_0 + \langle u_h |D_a|^k \tilde{s}_1(a), u_h |D_a|^k \tilde{s}_2(a)\rangle_0 \\
&\quad = \langle \tilde{s}_1(a), \tilde{s}_2(a)\rangle_{a,k}.
\end{aligned}$$

This implies that the family $\{\langle \cdot, \cdot \rangle_{a,k}\}_{a \in \mathcal{H}^1(Y)}$ descends to a fiberwise inner product $\langle \cdot, \cdot \rangle_k$ on $\mathcal{E}_k$. We write $\| \cdot \|_k$ for the fiberwise norm on $\mathcal{E}_k$ induced by $\langle \cdot, \cdot \rangle_k$.

The flat connection ∇, with respect to $k = 0$, defines a decomposition

$$T\mathcal{E}_0 = p^* TB \oplus p^* \mathcal{E}_0, \tag{2.3.1}$$

where $p\colon \mathcal{E}_0 \to B$ is the projection, p^*TB is the horizontal component and $p^*\mathcal{E}_0$ is the vertical component. See Section 2.2. Note that the flat connection ∇ is not compatible with the inner product $\langle \cdot, \cdot \rangle_k$ on $\mathcal{E}_k$ for $k > 0$.

Put

$$\mathcal{W}_k = B \times L^2_k(\operatorname{im} d^*),$$

where $d^*\colon i\Omega^2(Y) \to i\Omega^1(Y)$ is the adjoint of the exterior derivative. We consider $\mathcal{W}_k$ to be a trivial Hilbert bundle on B. The Seiberg–Witten equations on $Y \times \mathbb{R}$ are equations for $\gamma = (\phi, a, \omega)\colon \mathbb{R} \to L^2_k(\mathbb{S}) \times \mathcal{H}^1(Y) \times L^2_k(\operatorname{im} d^*)$, written as

$$\begin{aligned}
\frac{d\phi}{dt} &= -D_a \phi(t) - c_1(\gamma(t)), \\
\frac{da}{dt} &= -X_H(\phi), \\
\frac{d\omega}{dt} &= - * d\omega - c_2(\gamma(t)).
\end{aligned} \tag{2.3.2}$$

The terms $X_H(\phi)$, $c_1(\gamma(t))$, $c_2(\gamma(t))$ are defined by

$$\begin{aligned}
q(\phi) &= \rho^{-1}\Big(\phi \otimes \phi^* - \frac{1}{2}|\phi|^2 \mathrm{id}\Big) \in \Omega^1(Y), \\
X_H(\phi) &= q(\phi)_{\mathcal{H}} \in \mathcal{H}^1(Y), \\
c_1(\gamma(t)) &= \big(\rho(\omega(t)) - i\xi(\phi(t))\big)\phi(t), \\
c_2(\gamma(t)) &= \pi_{\operatorname{im} d^*}\big(q(\phi(t))\big),
\end{aligned} \tag{2.3.3}$$

where ρ is the Clifford multiplication which defines an isomorphism

$$T^*Y \otimes \mathbb{C} \to \mathfrak{sl}(\mathbb{S}),$$

$q(\phi)_{\mathcal{H}}$ is the harmonic component of $q(\phi)$, $\pi_{\mathrm{im}\, d^*}$ is the L^2-projection on $\mathcal{W}_k$ and $\xi(\phi)$ is the function $Y \to \mathbb{R}$ satisfying

$$d\xi(\phi) = i\pi_{\mathrm{im}\, d}(q(\phi)), \qquad \int_Y \xi(\phi)\,\mathrm{vol} = 0.$$

The equations (2.3.2) do not correspond to the Seiberg–Witten equations in Coulomb gauge in $Y \times \mathbb{R}$ (that is, solutions of the equations are not Seiberg–Witten trajectories in Coulomb gauge). Instead, we use the *pseudo-temporal gauge* of [32, Definition 5.2.1] (see also [35, Section 3]). The correspondence between solutions of (2.3.2) and the Seiberg–Witten equations modulo gauge is given by [32, Proposition 5.4.2]. Note that Lidman–Manolescu work in the setting of $b_1 = 0$; however, the argument is local in the configuration space and passes over without change to the $b_1 > 0$ case. We will, however, call solutions of (2.3.2) *Seiberg–Witten trajectories.*

The equations descend to equations for $\gamma = (\phi, \omega)\colon \mathbb{R} \to \mathcal{E}_k \oplus \mathcal{W}_k$:

$$\begin{aligned} \left(\frac{d\phi}{dt}(t)\right)_V &= -D\phi(t) - c_1(\gamma(t)), \\ \left(\frac{d\phi}{dt}(t)\right)_H &= -X_H(\phi(t)), \\ \frac{d\omega}{dt}(t) &= -*d\omega(t) - c_2(\gamma(t)). \end{aligned} \tag{2.3.4}$$

Here, $(\frac{d\phi}{dt})_V$, $(\frac{d\phi}{dt})_H$ are the vertical component and horizontal component of $\frac{d\phi}{dt}$ respectively, and we have suppressed the subscript from D.

Assume that the family index of the family of Dirac operators D over $\mathrm{Pic}(Y)$ vanishes, that is,

$$\mathrm{ind}\, D = 0 \in K^1(B).$$

Then we can choose a spectral section P_0 of $-D$, and using P_0, we can define a self-adjoint (with respect to the L^2) operator

$$\mathbb{A}\colon C^\infty(\mathbb{S}) \to C^\infty(\mathbb{S})$$

such that the image of $\mathbb{A}$ is included in a subspace spanned by finitely many eigenvectors of D, and so that $\ker(D + \mathbb{A}) = 0$. Put $D' = D + \mathbb{A}$. The L^2-closure of the subspace spanned by the negative eigenvectors of D' is exactly the image of P_0, acting on L^2 (see [40] and Section 2.1 for all of these assertions). In the future, for a spectral section P, we will also often write P to refer to the image of P. We have a decomposition

$$\mathcal{E}_\infty = \mathcal{E}_\infty^+ \oplus \mathcal{E}_\infty^-,$$

where $\mathcal{E}_\infty^+$ and $\mathcal{E}_\infty^-$ are the subbundles of $\mathcal{E}$ spanned by positive eigenvectors and negative eigenvectors of D'.

For positive numbers k_+, k_- and $s_1, s_2 \in C^\infty(\mathbb{S})$, we define an inner product $\langle s_1, s_2\rangle_{a,k_+,k_-}$ by

$$\langle s_1, s_2\rangle_{a,k_+,k_-} := \langle |D'_a|^{k_+} s_1^+, |D'_a|^{k_+} s_2^+\rangle_0 + \langle |D'_a|^{k_-} s_1^-, |D'_a|^{k_-} s_2^-\rangle_0, \quad (2.3.5)$$

where $s_j = s_j^+ + s_j^-$ and $s_j^+ \in \mathcal{E}_\infty^+$, $s_j^- \in \mathcal{E}_\infty^-$. Note that we do not need the term $\langle s_1, s_2\rangle_0$, since the kernel of D'_a is zero. We call this inner product the $L^2_{k_+,k_-}$-inner product.

As before, the family $\{\langle\cdot,\cdot\rangle_{a,k_+,k_-}\}_{a\in\mathcal{H}^1(Y)}$ induces a fiberwise inner product on $\mathcal{E}_\infty$ and we denote by $\mathcal{E}_{k_+,k_-}$ the completion of $\mathcal{E}_\infty$ with respect to the norm $\|\cdot\|_{k_+,k_-}$.

On the space $\operatorname{im} d^* \cap \Omega^1(Y)$, we define an inner product $\langle\cdot,\cdot\rangle_{k_+,k_-}$ by

$$\langle \omega_1, \omega_2\rangle_{k_+,k_-} = \langle |*d|^{k_+}\omega_1^+, |*d|^{k_+}\omega_2^+\rangle_0 + \langle |*d|^{k_-}\omega_1^-, |*d|^{k_-}\omega_2^-\rangle_0,$$

where $\omega_j = \omega_j^+ + \omega_j^-$ and ω_j^+ is in the subspace spanned by positive eigenvectors of the operator $*d$ and ω_j^- is in the negative one. We denote by $\mathcal{W}_{k_+,k_-}$ the completion of the vector bundle $B \times \operatorname{im} d^*$ over B with respect to $\|\cdot\|_{k_+,k_-}$. We will use the $L^2_{k-\frac{1}{2},k}$-norm in Chapter 5 to define the relative Bauer–Furuta invariant. See Remark 5.1.4 for the reason why we use the $L^2_{k-\frac{1}{2},k}$-norm.

We recall the definition of finite-type trajectories (from e.g. [35, Definition 1]).

Definition 2.3.1. A Seiberg–Witten trajectory $\gamma(t) = (\phi(t), a(t), \omega(t))$ is *finite-type* if $CSD(\gamma(t))$ and $\|\phi(t)\|_{C^0}$ are bounded functions of t, where CSD is the Chern–Simons–Dirac functional.

The following is a direct consequence of a standard argument in Seiberg–Witten theory; see e.g. [35, Proposition 1].

Proposition 2.3.2. *For positive numbers $k_+, k_- > 0$, there is a positive constant $R_{k_+,k_-} > 0$ such that for any finite-type solution $\gamma\colon \mathbb{R} \to \mathcal{E}_2 \times \mathcal{W}_2$ to (2.3.4), we have*

$$\|\gamma(t)\|_{k_+,k_-} \le R_{k_+,k_-}$$

for all $t \in \mathbb{R}$.

Write $\mathcal{E}_0(D)_{b'}^b$ for the span of eigenvectors of D with eigenvalue in $(b', b]$, as a space over $\mathcal{H}^1(Y)$ (note that it will not usually be a bundle). For a spectral section P of D, we also write P for the image of the projection P. By Theorem 2.4.1 below, we can take sequences of smooth spectral sections P_n, Q_n, of $-D$, D, respectively, such that

$$\begin{aligned}(\mathcal{E}_0(D))_{-\infty}^{\mu_{n,-}} \subset P_n \subset (\mathcal{E}_0(D))_{-\infty}^{\mu_{n,+}},\\ (\mathcal{E}_0(D))_{\lambda_{n,+}}^{\infty} \subset Q_n \subset (\mathcal{E}_0(D))_{\lambda_{n,-}}^{\infty},\end{aligned} \quad (2.3.6)$$

with

$$\begin{aligned} &\mu_{n,-} + 10 < \mu_{n,+} < \mu_{n,+} + 10 < \mu_{n+1,-}, \\ &\lambda_{n+1,+} < \lambda_{n,-} - 10 < \lambda_{n,-} < \lambda_{n,+} - 10, \\ &\mu_{n,+} - \mu_{n,-} < \delta, \\ &\lambda_{n,+} - \lambda_{n,-} < \delta. \end{aligned} \tag{2.3.7}$$

Here, $\delta > 0$ is a positive constant independent of n, and a smooth spectral section means a spectral section which depends smoothly on the base space B.

We define a finite rank subbundle F_n in $\mathcal{E}_\infty$ by

$$F_n = P_n \cap Q_n.$$

Define a connection ∇_{F_n} on F_n by

$$\nabla_{F_n} = \pi_{F_n} \nabla,$$

where π_{F_n} is the $L^2_{k_+,k_-}$-projection on F_n. The connection ∇_{F_n} defines a decomposition

$$TF_n = (TF_n)_{H,\nabla_{F_n}} \oplus (TF_n)_V \cong p^*TB \oplus p^*F_n. \tag{2.3.8}$$

A calculation shows that the horizontal component $(T_\phi F_n)_{H,\nabla_{F_n}}$ of TF_n at $\phi \in F_n$ is given by

$$\left\{(v, (\nabla_v \pi_{F_n})\phi) : v \in T_a B\right\} \subset (p^*TB \oplus p^*\mathcal{E}_0)_\phi = T_\phi \mathcal{E}_0. \tag{2.3.9}$$

Here, $a = p(\phi) \in B$.

Let W_n be the finite-dimensional subbundle of the Hilbert bundle $\mathcal{W}_k$ spanned by the eigenvectors of the operator $*d$ whose eigenvalues are in the interval $(\lambda_{n,-}, \mu_{n,+}]$:

$$W_n = (\mathcal{W}_k)_{\lambda_{n,-}}^{\mu_{n,+}} = B \times L^2_k(\operatorname{im} d^*)_{\lambda_{n,-}}^{\mu_{n,+}}.$$

Fix a positive number R' with $R' \geq 100 R_{k_+,k_-}$ and a smooth function

$$\chi \colon \mathcal{E}_{k_+,k_-} \oplus \mathcal{W}_{k_+,k_-} \to [0,1]$$

with compact support such that $\chi(\phi,\omega) = 1$ if $\|(\phi,\omega)\|_{k_+,k_-} \leq R'$. We consider the following equations for $\gamma = (\phi,\omega) \colon \mathbb{R} \to F_n \oplus W_n$, which we call the *finite-dimensional approximation* of (2.3.4):

$$\begin{aligned} \left(\frac{d\phi}{dt}(t)\right)_V &= -\chi\left\{(\nabla_{X_H}\pi_{F_n})\phi(t) + \pi_{F_n}\bigl(D\phi(t) + c_1(\gamma(t))\bigr)\right\}, \\ \left(\frac{d\phi}{dt}(t)\right)_H &= -\chi X_H(\phi(t)), \\ \frac{d\omega}{dt}(t) &= -\chi\left\{*d\omega(t) + \pi_{W_n} c_2(\gamma(t))\right\}. \end{aligned} \tag{2.3.10}$$

Here, $\left(\frac{d\phi}{dt}\right)_V$, $\left(\frac{d\phi}{dt}\right)_H$ are the vertical component and the horizontal component with respect to the fixed decomposition (2.3.1) rather than (2.3.8). It follows from (2.3.9) that the right-hand side of (2.3.10) is a tangent vector on $F_n \oplus W_n$. Hence, equations (2.3.10) define a flow

$$\varphi_n = \varphi_{n,k_+,k_-} : (F_n \oplus W_n) \times \mathbb{R} \to F_n \oplus W_n.$$

(This flow depends on k_+, k_- because π_{F_n} does.)

We have decompositions

$$F_n = F_n^+ \oplus F_n^-, \quad W_n = W_n^+ \oplus W_n^-,$$

where F_n^+, W_n^+ are the positive eigenvalue components of D', $*d$, and F_n^-, W_n^- are the negative eigenvalue components. In the remainder of Chapter 2, we will prove the following.

Theorem 2.3.3. *Let k_+, k_- be half-integers (that is, $k_+, k_- \in \frac{1}{2}\mathbb{Z}$) with $k_+, k_- > 5$ and with $|k_+ - k_-| \leq \frac{1}{2}$. Fix a positive number R with $R_{k_+,k_-} < R < \frac{1}{10}R'$, where R_{k_+,k_-} is the constant of Proposition 2.3.2. Then*

$$(B_{k_+}(F_n^+; R) \times_B B_{k_-}(F_n^-; R)) \times_B (B_{k_+}(W_n^+; R) \times_B B_{k_-}(W_n^-; R))$$

is an isolating neighborhood of the flow φ_{n,k_+,k_-} for $n \gg 0$. Here, $B_{k_\pm}(F_n^\pm; R)$ are the disk bundle of $F_n^\pm$ of radius R in $L^2_{k_\pm}$ and $B_{k_+}(F_n^+; R) \times_B B_{k_-}(F_n^-; R)$ is the fiberwise product. Similarly for $B_{k_\pm}(W_n^\pm; R)$.

The general strategy to prove Theorem 2.3.3 is as follows: once we have Theorem 2.4.1 in hand, we must control the gradient term $(\nabla_{X_H} \pi_{F_n})\phi(t)$ appearing in the approximate Seiberg–Witten equations (2.3.10); a number of bounds for this are obtained in Sections 2.5 and 2.6. The proof proper is in Section 2.7, where Theorem 2.3.3 follows from establishing that, for sufficiently large approximations, the linear term in the approximate Seiberg–Witten equations (2.3.10) tends to dominate the other terms with respect to appropriate norms.

We also note that the total space $B_{n,R}$ appearing in Theorem 2.3.3 is an ex-space over $B = \mathrm{Pic}(Y)$ in the sense of Appendix A.1, with projection given by restricting $p: \mathcal{E}_k \to B$ to $B_{n,R}$, and with a section $s_B: \mathrm{Pic}(Y) \to B_{n,R}$ given by the zero-section.

2.4 Construction of spectral sections

We will prove the following.

Theorem 2.4.1. *Assume that* ind $D = 0$ *in $K^1(B)$. Take a sequence μ_n of positive numbers $\mu_n \ll \mu_{n+1}$, where $\mu_n \to \infty$ as $n \to \infty$. There is a sequence of spectral sections P_n of $-D$ with the following properties:*

(i) *We have*

$$\mathcal{E}_0(D)^{\mu_n}_{-\infty} \subset P_n \subset \mathcal{E}_0(D)^{\mu_n+\delta}_{-\infty},$$

where δ is a positive constant independent of n.

(ii) *We can write*

$$P_{n+1} = P_n \oplus \langle f_1^{(n)}, \dots, f_{r_n}^{(n)} \rangle,$$

where $\{f_1^{(n)}, \dots, f_{r_n}^{(n)}\}$ is a frame of $P_n^{\perp}$ (where $P_n^{\perp}$ is the L^2-orthogonal complement of P_n inside of P_{n+1}). In particular,

$$P_{n+1} \cong P_n \oplus \underline{\mathbb{C}}^{r_n},$$

where $\underline{\mathbb{C}}^{r_n}$ is the trivial vector bundle over B.

Before we start proving Theorem 2.4.1, we will show the following.

Proposition 2.4.2. *Take any nonnegative numbers k, l. Let P_n be a sequence of spectral sections of $-D$ having property* (i) *of Theorem 2.4.1. Let $\pi_n \colon \mathcal{E}_k \to P_n \cap \mathcal{E}_k$ be the L^2_k-projection.*

(1) *The commutators*

$$[D, \pi_n] \colon \mathcal{E}_\infty \to \mathcal{E}_\infty$$

extend to bounded operators

$$[D, \pi_n] \colon \mathcal{E}_l \to \mathcal{E}_l$$

and we have

$$\|[D, \pi_n] \colon \mathcal{E}_l \to \mathcal{E}_l\| < C,$$

where C is a positive constant independent of n. Moreover, for any $l > 0$, $\varepsilon > 0$ with $0 < \varepsilon \le l$,

$$\sup_{a \in B} \|[D_a, \pi_{n,a}] \colon L^2_l(\mathbb{S}) \to L^2_{l-\varepsilon}(\mathbb{S})\| \to 0$$

as $n \to \infty$.

(2) *The operator $\pi_n \colon \mathcal{E}_\infty \to \mathcal{E}_\infty$ extends to a bounded operator $\mathcal{E}_l \to \mathcal{E}_l$ for each nonnegative real number l. Moreover, there is a positive constant C independent of n such that*

$$\|\pi_n \colon \mathcal{E}_l \to \mathcal{E}_l\| < C.$$

Proof. Take $a \in B$ and let $\{e_j\}_j$ be an orthonormal basis of $L^2(\mathbb{S})$ with

$$D_a e_j = \eta_j e_j,$$

where $\eta_j \in \mathbb{R}$.

Let $P_{n,a}$ be the fiber of P_n over a. Take $\phi \in \mathcal{E}_\infty \cap P_{n,a}$. We can write

$$\phi = \sum_{\eta_j \le \mu_n + \delta} c_j e_j,$$

where $c_j \in \mathbb{C}$. Note that

$$\sum_{\eta_j \le \mu_n} c_j e_j \in \mathcal{E}_\infty \cap P_{n,a}, \qquad \sum_{\mu_n < \eta_j \le \mu_n + \delta} c_j e_j \in \mathcal{E}_\infty \cap P_{n,a}.$$

We have

$$\begin{aligned}
[D_a, \pi_{n,a}]\phi &= (D_a \pi_{n,a} - \pi_{n,a} D)\phi \\
&= \sum_{\eta_j \le \mu_n + \delta} \eta_j c_j e_j - \pi_{n,a} \sum_{\eta_j \le \mu_n + \delta} \eta_j c_j e_j \\
&= (1 - \pi_{n,a}) \sum_{\mu_n < \eta_j \le \mu_n + \delta} \eta_j c_j e_j \\
&= (1 - \pi_{n,a}) \Big\{ \sum_{\mu_n < \eta_j \le \mu_n + \delta} (\eta_j - \mu_n) c_j e_j + \mu_n \sum_{\mu_n < \eta_j \le \mu_n + \delta} c_j e_j \Big\} \\
&= \sum_{\mu_n < \eta_j \le \mu_n + \delta} (\eta_j - \mu_n) c_j (1 - \pi_{n,a}) e_j. \qquad (2.4.1)
\end{aligned}$$

Since

$$\pi_n = \pi_{-\infty}^{\mu_n} + \pi_{P_n \cap \mathcal{E}(D)_{\mu_n}^{\mu_n + \delta}},$$

for j with $\mu_n < \eta_j \le \mu_n + \delta$, we have

$$(1 - \pi_{n,a}) e_j \in \mathcal{E}_0(D_a)_{\mu_n}^{\mu_n + \delta}.$$

Hence we can write

$$(1 - \pi_{n,a}) e_j = \sum_{\mu_n < \eta_p \le \mu_n + \delta} \alpha_{jp} e_p \qquad (2.4.2)$$

for j with $\mu_n < \eta_j \le \mu_n + \delta$. Here, $\alpha_{jp} \in \mathbb{C}$. Since

$$\|(1 - \pi_{n,a}) \colon L_k^2 \to L_k^2\| = 1, \quad \|e_j\|_k = (1 + |\eta_j|^{2k})^{\frac{1}{2}},$$

we have

$$\|(1 - \pi_{n,a}) e_j\|_k^2 = \sum_{\mu_n < \eta_p \le \mu_n + \delta} |\alpha_{jp}|^2 (1 + |\eta_p|^{2k}) \le (1 + |\eta_j|^{2k}).$$

For j with $\mu_n < \eta_j \le \mu_n + \delta$,

$$\begin{aligned}\sum_{\mu_n<\eta_p\le\mu_n+\delta}|\alpha_{jp}|^2 &= \sum_{\mu_n<\eta_p\le\mu_n+\delta}|\alpha_{jp}|^2(1+|\eta_p|^{2k})\frac{1}{1+|\eta_p|^{2k}}\\ &\le \frac{C_1}{1+(\mu_n+\delta)^{2k}}\sum_{\mu_n<\eta_p\le\mu_n+\delta}|\alpha_{jp}|^2(1+|\eta_p|^{2k})\\ &\le \frac{C_1(1+|\eta_j|^{2k})}{1+(\mu_n+\delta)^{2k}}\\ &\le C_1, \end{aligned} \tag{2.4.3}$$

where C_1 is a positive constant independent of j, n.

By (2.4.1), (2.4.2) and (2.4.3),

$$\begin{aligned}\|[D_a,\pi_{n,a}]\phi\|_l^2 &= \sum_{\mu_n<\eta_j\le\mu_n+\delta}\ \sum_{\mu_n<\eta_p\le\mu_n+\delta}|\eta_j-\mu_n|^2(1+|\eta_p|^{2l})|c_j|^2|\alpha_{jp}|^2\\ &\le \delta^2\sum_{\mu_n<\eta_j\le\mu_n+\delta}\ \sum_{\mu_n<\eta_p\le\mu_n+\delta}(1+|\eta_j|^{2l})|c_j|^2|\alpha_{jp}|^2\cdot\frac{1+|\eta_p|^{2l}}{1+|\eta_j|^{2l}}\\ &\le C_2\sum_{\mu_n<\eta_j\le\mu_n+\delta}(1+|\eta_j|^{2l})|c_j|^2\left(\sum_{\mu_n<\eta_p\le\mu_n+\delta}|\alpha_{jp}|^2\right)\\ &\le C_3\sum_{\mu_n<\eta_j\le\mu_n+\delta}(1+|\eta_j|^{2l})|c_j|^2\\ &\le C_3\|\phi\|_l^2.\end{aligned}$$

Here, $C_2, C_3 > 0$ are positive constants independent of n, ϕ, a. Also we have

$$\begin{aligned}&\|[D_a,\pi_{n,a}]\phi\|_{l-\varepsilon}^2\\ &= \sum_{\mu_n<\eta_j\le\mu_n+\delta}\ \sum_{\mu_n<\eta_p\le\mu_n+\delta}|\eta_j-\mu_n|^2(1+|\eta_p|^{2(l-\varepsilon)})|c_j|^2|\alpha_{jp}|^2\\ &\le \delta^2\sum_{\mu_n<\eta_j\le\mu_n+\delta}\ \sum_{\mu_n<\eta_p\le\mu_n+\delta}(1+|\eta_j|^{2(l-\varepsilon)})|c_j|^2|\alpha_{jp}|^2\cdot\frac{1+|\eta_p|^{2(l-\varepsilon)}}{1+|\eta_j|^{2(l-\varepsilon)}}\\ &\le C_4\sum_{\mu_n<\eta_j\le\mu_n+\delta}(1+|\eta_j|^{2(l-\varepsilon)})|c_j|^2\left(\sum_{\mu_n<\eta_p\le\mu_n+\delta}|\alpha_{jp}|^2\right)\\ &\le C_5\sum_{\mu_n<\eta_j\le\mu_n+\delta}(1+|\eta_j|^{2(l-\varepsilon)})|c_j|^2\\ &\le C_6(\mu_n^{-2l}+\mu_n^{-2\varepsilon})\|\phi\|_l^2.\end{aligned}$$

Here, C_4, C_5, C_6 are positive constants independent of n, ϕ, a.

On the other hand, consider $\phi \in \mathcal{E}_\infty \cap P_{n,a}^{\perp_k}$, where $P_{n,a}^{\perp_k}$ is the L^2_k-orthogonal complement of $P_{n,a} \cap L^2_k(\mathbb{S})$ in $L^2_k(\mathbb{S})$. We can write

$$\phi = \sum_{\eta_j > \mu_n} c_j e_j .$$

Note that

$$\sum_{\eta_j > \mu_n + \delta} c_j e_j \in \mathcal{E}_\infty \cap P_{n,a}^{\perp_k}, \qquad \sum_{\mu_n < \eta_j \le \mu_n + \delta} c_j e_j \in \mathcal{E}_\infty \cap P_{n,a}^{\perp_k}.$$

We have

$$\begin{aligned} [D_a, \pi_{n,a}]\phi &= \pi_{n,a} \sum_{\mu_n < \eta_j \le \mu_n + \delta} \eta_j c_j e_j \\ &= \pi_{n,a}\Bigg(\sum_{\mu_n < \eta_j \le \mu_n + \delta} (\eta_j - \mu_n) c_j e_j + \mu_n \sum_{\mu_n < \eta_j \le \mu_n + \delta} c_j e_j \Bigg) \\ &= \sum_{\mu_n < \eta_j \le \mu_n + \delta} (\eta_j - \mu_n) c_j \pi_{n,a} e_j . \end{aligned}$$

As before, using this equality, we can show that

$$\|[D_a, \pi_{n,a}]\phi\|_l \le C_7 \|\phi\|_l, \quad \|[D_a, \pi_{n,a}]\phi\|_{l-\varepsilon} \le C_8 \mu_n^{-\varepsilon} \|\phi\|_l$$

for some positive constants C_7, C_8 independent of n, ϕ, a.

Therefore $[D_a, \pi_{n,a}]$ extend to bounded maps $L^2_l \to L^2_l$ with

$$\|[D_a, \pi_{n,a}] \colon L^2_l \to L^2_l\| \le C_9,$$

for some constant C_9 independent of n, a. Also

$$\sup_{a \in B} \|[D_a, \pi_{n,a}] \colon L^2_l(\mathbb{S}) \to L^2_{l-\varepsilon}(\mathbb{S})\| \to 0$$

as $n \to \infty$. We have proved (1).

We will prove (2). It is easy to see that if $\mu_n < \eta_j \le \mu_n + \delta$, we have

$$\pi_n e_j \in (\mathcal{E}_l)_{\mu_n}^{\mu_n + \delta}.$$

So we can write

$$\pi_n e_j = \sum_{\mu_n < \eta_p \le \mu_n + \delta} \alpha_{jp} e_p .$$

Because the operator norm of $\pi_n \colon L^2_k \to L^2_k$ is 1 and $\|e_j\|_k^2 = 1 + |\eta_j|^{2k}$, we have

$$|\mu_n|^{2k} \sum_{\mu_n < \eta_p \le \mu_n + \delta} |\alpha_{jp}|^2 \le \sum_{\mu_n < \eta_p \le \mu_n + \delta} |\alpha_{jp}|^2 (1 + |\eta_p|^{2k}) \le 1 + |\eta_j|^{2k}.$$

Therefore, for j with $\mu_n < \eta_j \le \mu_n + \delta$,

$$\sum_{\mu_n < \eta_p \le \mu_n + \delta} |\alpha_{jp}|^2 \le \frac{1 + |\eta_j|^{2k}}{|\mu_n|^{2k}} \le C_9. \tag{2.4.4}$$

Here, $C_9 > 0$ is a constant independent of n, j. Take $\phi \in \mathcal{E}_\infty$. We can write

$$\phi = \sum_{\eta_j \le \mu_n} c_j e_j + \sum_{\mu_n < \eta_j \le \mu_n + \delta} c_j e_j + \sum_{\mu_n + \delta < \eta_j} c_j e_j.$$

Then

$$\pi_n \phi = \sum_{\eta_j \le \mu_n} c_j e_j + \sum_{\substack{\mu_n < \eta_j \le \mu_n + \delta \\ \mu_n < \eta_p \le \mu_n + \delta}} c_j \alpha_{jp} e_p.$$

Hence we obtain

$$\begin{aligned}
\|\pi_n \phi\|_l^2 &= \sum_{\eta_j \le \mu_n} |c_j|^2 (1 + |\eta_j|^{2l}) + \sum_{\substack{\mu_n < \eta_j \le \mu_n + \delta \\ \mu_n < \eta_p \le \mu_n + \delta}} |c_j|^2 |\alpha_{jp}|^2 (1 + |\eta_p|^{2l}) \\
&\le C_{10} \Bigg(\sum_{\eta_j \le \mu_n} |c_j|^2 (1 + |\eta_j|^{2l}) + (1 + |\mu_n|^{2l}) \sum_{\substack{\mu_n < \eta_j \le \mu_n + \delta \\ \mu_n < \eta_p \le \mu_n + \delta}} |c_j|^2 |\alpha_{jp}|^2 \Bigg) \\
&\le C_{11} \Bigg(\sum_{\eta_j \le \mu_n} |c_j|^2 (1 + |\eta_j|^{2l}) + (1 + |\mu_n|^{2l}) \sum_{\mu_n < \eta_j \le \mu_n + \delta} |c_j|^2 \Bigg) \\
&\le C_{12} \|\phi_n\|_l^2,
\end{aligned}$$

where we have used (2.4.4) and C_{10}, C_{11}, C_{12} are constant independent of n. Therefore $\|\pi_n \colon L_l^2 \to L_l^2\| \le C_{12}$. ∎

To prove Theorem 2.4.1, we need the following theorem and lemma.

Theorem 2.4.3 ([4, Theorem 1*]). *Let W be a closed, spin manifold of odd dimension. Then there is $C_* > 0$ such that each interval of length C_* contains an eigenvalue of D_A. Here, A is a connection on a complex vector bundle V over W and $D_A \colon C^\infty(\mathbb{S} \otimes V) \to C^\infty(\mathbb{S} \otimes V)$ is the twisted Dirac operator.*

Assume that ind $D = 0$. By [40], we have a spectral section P_0 of $-D$. By [40, Lemma 8], using P_0, we can construct a smoothing operator $\mathbb{A} \colon \mathcal{E}_0 \to \mathcal{E}_\infty$ whose image is included in the space spanned by finitely many eigenvectors of D such that $\ker D' = 0$ and

$$\mathcal{E}_0 (D')_{-\infty}^0 = P_0,$$

where $D' = D + \mathbb{A}$. Moreover, there is $\nu_0 \gg 0$ such that $\mathbb{A} = 0$ on $\mathcal{E}_0(D)_{-\infty}^{-\nu_0}$ and $\mathcal{E}_0(D)_{\nu_0}^{\infty}$. From the construction of $\mathbb{A}$ in the proof of [40, Lemma 8], it is easy to see that for $\lambda \ll 0$ and $\mu \gg 0$,

$$\mathcal{E}_0(D)_{-\infty}^{\mu} = \mathcal{E}_0(D')_{-\infty}^{\mu}, \quad \mathcal{E}_0(D)_{\lambda}^{\infty} = \mathcal{E}_0(D')_{\lambda}^{\infty}, \quad \mathcal{E}_0(D)_{\lambda}^{\mu} = \mathcal{E}_0(D')_{\lambda}^{\mu}.$$

Lemma 2.4.4. *There is a constant* $\delta > 0$ *such that for any* $\mu > 0$ *and* $a, a' \in B$,

$$\dim \mathcal{E}_0(D'_a)_0^{\mu} \le \dim \mathcal{E}_0(D'_{a'})_0^{\mu+\delta}.$$

Proof. Put

$$M = \max\bigl\{\|\nabla_v D' \colon L^2(\mathbb{S}) \to L^2(\mathbb{S})\| : v \in TB,\ \|v\| = 1\bigr\}.$$

Take a smooth path $\{a_t\}_{t=0}^{\ell}$ in B from a to a' with $\|\frac{d}{dt}a_t\| = 1$. Here, ℓ is the length of the path. Since B is compact, we may assume that there is a constant $C > 0$ independent of a, a' such that $\ell \le C$. Put

$$I = \bigl\{t \in [0,\ell] : \forall s \le t,\ \dim \mathcal{E}_0(D'_a)_0^{\mu} \le \dim \mathcal{E}_0(D'_{a_s})_0^{\mu+sM}\bigr\}.$$

Note that $0 \in I$ and that I is closed in $[0,\ell]$ by the continuity of the eigenvalues of D'_{a_s}. It is sufficient to prove that $\sup I = \ell$.

Put $t_0 = \sup I$ and assume that $t_0 < \ell$. Choose $t_+ \in (t_0, \ell]$ with

$$t_+ - t_0 \ll 1.$$

Let $\nu_1(t), \dots, \nu_m(t)$ be the eigenvalues of D'_{a_t} with

$$0 < \nu_1(t_0) \le \cdots \le \nu_m(t_0) \le \mu + t_0 M$$

such that $\nu_j(t)$ are continuous in $t \in [t_0, t_+]$ and $\dim \mathcal{E}(D'_{a_{t_0}})_0^{\mu+t_0 M} = m$. Note that $t_0 \in I$ since I is closed in $[0,\ell]$ and that

$$\dim \mathcal{E}_0(D'_a)_0^{\mu} \le m$$

by the definition of I. Let ν' be the smallest eigenvalue of $D'_{a_{t_0}}$ with $\nu' > \nu_m(t_0)$. We may assume that

$$M(t_+ - t_0) \ll \nu' - \nu_m(t_0). \tag{2.4.5}$$

By [22, Theorem 4.10, p. 291], we have

$$\mathrm{dist}(\nu_j(t), \Sigma(D'_{a_{t_0}})) \le M(t - t_0)$$

for $t \in [t_0, t_+]$. Here, $\Sigma(D'_{a_{t_0}})$ is the set of eigenvalues of $D'_{a_{t_0}}$. It follows from this inequality and (2.4.5) that

$$0 < \nu_j(t) \le \nu_m(t_0) + M(t - t_0) \le \mu + Mt$$

for $t \in [t_0, t_+]$ and $j \in \{1, \dots, m\}$. This implies that

$$\dim \mathcal{E}_0(D'_a)_0^{\mu} \leq m \leq \dim \mathcal{E}_0(D'_{a_t})_0^{\mu+tM}$$

for $t \in [t_0, t_+]$. This is a contradiction and we obtain $t_0 = \ell$. ∎

Proof of Theorem 2.4.1. For some $\mu \gg 0$, to construct a spectral section P between $\mathcal{E}(D)_{-\infty}^{\mu}$ and $\mathcal{E}(D)_{-\infty}^{\mu+\delta}$, it is sufficient to find a frame $\{f_1, \dots, f_r\}$ in $\mathcal{E}_0(D')_0^{\mu+\delta}$ such that

$$\mathcal{E}_0(D')_0^{\mu} \subset \operatorname{span}\{f_1, \dots, f_r\} \subset \mathcal{E}_0(D')_0^{\mu+\delta}, \tag{2.4.6}$$

because the direct sum $\mathcal{E}_0(D')_{-\infty}^{0} \oplus \operatorname{span}\{f_1, \dots, f_r\}$ is a spectral section between $\mathcal{E}_0(D)_{-\infty}^{\mu}$ and $\mathcal{E}_0(D)_{-\infty}^{\mu+\delta}$.

Put $d = \dim B$. Fix an integer N with $N \gg d$. By Theorem 2.4.3, there is $\delta_0 > 0$ such that

$$\dim(\mathcal{E}_0(D'_a))_{\mu}^{\mu+\delta_0} \geq N \tag{2.4.7}$$

for all $a \in B$ and $\mu \in \mathbb{R}$. By Lemma 2.4.4, we may assume that

$$\dim \mathcal{E}_0(D'_{a'})_0^{\mu-\delta_0} \leq \dim \mathcal{E}_0(D'_a)_0^{\mu} \leq \dim \mathcal{E}_0(D'_{a'})_0^{\mu+\delta_0} \tag{2.4.8}$$

for all $a, a' \in B$ and $\mu \in \mathbb{R}$ with $\mu > \delta_0$.

Fix a positive number δ with $\delta > 10\delta_0$. Take $\mu \in \mathbb{R}$ with $\mu \gg 0$. For $j \in \{0, 1, \dots, d\}$, choose positive numbers

$$\mu < a_j^- < b_j^- < c^- < c^+ < a_j^+ < b_j^+ < \mu + \delta$$

such that

$$\begin{aligned} b_{j+1}^- &< a_j^-, \quad & b_j^+ &< a_{j+1}^+, \\ b_j^- &< c^- - 2\delta_0, \quad & c^+ + 2\delta_0 &< a_j^+. \end{aligned}$$

Take a CW complex structure of B such that for each j-dimensional cell e there are real numbers $\mu^-(e), \mu^+(e)$ such that $\mu^-(e), \mu^+(e)$ are spectral gaps of D'_a for $a \in e$ with

$$a_j^- < \mu^-(e) < b_j^-, \quad a_j^+ < \mu^+(e) < b_j^+.$$

Choose a 0-dimensional cell e_0 (= 1 pt) and $\mu_0 \in (c^-, c^+)$, and then put $r := \dim \mathcal{E}_0(D'_{e_0})_0^{\mu_0}$.

Lemma 2.4.5. *For any cell e and $a \in e$, we have*

$$\dim \mathcal{E}_0(D'_a)_0^{\mu^-(e)} + N \leq r \leq \dim \mathcal{E}_0(D'_a)_0^{\mu^+(e)} - N.$$

Proof. Because $\mu_0 + 2\delta_0 < \mu^+(e)$, by (2.4.7) and (2.4.8), we have

$$\begin{aligned}\dim \mathcal{E}_0(D'_a)_0^{\mu^+(e)} &\geq \dim \mathcal{E}_0(D'_a)_0^{\mu_0+2\delta_0} \\ &= \dim \mathcal{E}_0(D'_a)_0^{\mu_0+\delta_0} + \dim \mathcal{E}_0(D'_a)_{\mu_0+\delta_0}^{\mu_0+2\delta_0} \\ &\geq \dim \mathcal{E}_0(D'_{e_0})_0^{\mu_0} + N \\ &= r + N.\end{aligned}$$

Hence

$$r \leq \dim \mathcal{E}_0(D'_a)_0^{\mu^+(e)} - N.$$

The proof of the inequality $\dim \mathcal{E}_0(D'_a)_0^{\mu^-(e)} + N \leq r$ is similar. ■

By Lemma 2.4.5, for each 0-dimensional cell e, we can take a frame (meaning a linearly independent collection) $\{f_1, \dots, f_r\}$ of $\mathcal{E}_0(D'_e)_0^{\mu^+(e)}$ such that

$$\mathcal{E}_0(D'_e)_0^{\mu^-(e)} \subset \langle f_1, \dots, f_r \rangle \subset \mathcal{E}_0(D'_e)_0^{\mu^+(e)}.$$

Assume that we have a frame $\{f_1, \dots, f_r\}$ in $\mathcal{E}_0(D')_0^{\infty}$ on the $(j-1)$-dimensional skeleton of B such that

$$\mathcal{E}_0(D'_a)_0^{\mu^-(e)} \subset \langle f_{1,a}, \dots, f_{r,a} \rangle \subset \mathcal{E}_0(D'_a)_0^{\mu^+(e)}$$

for each cell e with $\dim e \leq j-1$ and $a \in e$.

Take a cell e' of B with $\dim e' = j$. Note that $\mathcal{E}_0(D')_0^{\mu^+(e')}$, $\mathcal{E}_0(D')_0^{\mu^-(e')}$ are vector bundles over e'. We denote by $\mathcal{F}$ the bundle

$$\bigcup_{a \in e'} \{\text{frames of rank } r \text{ in } \mathcal{E}_0(D'_a)_0^{\mu^+(e')}\}$$

over e'.

Note that $\mu^+(e) \leq \mu^+(e')$ for any cell e with $\dim e \leq j-1$. Hence the frame $\{f_1, \dots, f_r\}$ defines a section of $\mathcal{F}$ on the boundary $\partial e'$.

We have a homeomorphism

$$\mathcal{F}_a \cong GL(m; \mathbb{C})/GL(m-r; \mathbb{C}),$$

where $a \in e'$, $\mathcal{F}_a$ is the fiber of $\mathcal{F}$ over a and $m = \dim \mathcal{E}_0(D'_a)_0^{\mu^+(e')}$. By Lemma 2.4.5,

$$m = \dim \mathcal{E}_0(D'_a)_0^{\mu^+(e')} \geq r + N.$$

Because $N \gg d$, we have

$$m, m-r \gg d.$$

By the homotopy exact sequence,

$$\pi_i(GL(m;\mathbb{C})/GL(m-r;\mathbb{C})) = 0$$

for $i = 0, 1, \dots, d$. Therefore we can extend $\{f_1, \dots, f_r\}$ to a frame in $\mathcal{E}_0(D')_0^{\mu^+(e')}$ over e'. We will denote the extended frame on e' by the same notation $\{f_1, \dots, f_r\}$. We will modify $\{f_1, \dots, f_r\}$ on the interior $\operatorname{Int} e'$ of e' to get a frame $\{f'_1, \dots, f'_r\}$ such that

$$\mathcal{E}_0(D')_0^{\mu^-(e')} \subset \langle f'_1, \dots, f'_r\rangle \subset \mathcal{E}_0(D')_0^{\mu^+(e')}$$

on e'. Since $\mu^-(e') \leq \mu^-(e)$, on $\partial e'$ we have

$$\mathcal{E}_0(D')_0^{\mu^-(e')} \subset \mathcal{E}_0(D')_0^{\mu^-(e)} \subset \operatorname{span}\{f_1, \dots, f_r\}.$$

As mentioned before, $\mathcal{E}_0(D')_0^{\mu^-(e')}$ and $\mathcal{E}_0(D')_0^{\mu^+(e')}$ are vector bundles over e'. Let

$$p\colon \mathcal{E}_0(D')_0^{\mu^+(e')}\big|_{e'} \to \mathcal{E}_0(D')_0^{\mu^-(e')}\big|_{e'}$$

be the orthogonal projection.

Lemma 2.4.6. *We can perturb $f_1, \dots, f_r$ slightly on* $\operatorname{Int} e'$ *such that*

$$\mathcal{E}_0(D')_0^{\mu^-(e')} = p(\langle f_1, \dots, f_r\rangle)$$

on e'. Here, $\operatorname{Int} e'$ *is the interior of e'.*

Proof. We may suppose that

$$\mathcal{E}_0(D')_0^{\mu^+(e')}\big|_{e'} = e' \times (\mathbb{C}^n \oplus \mathbb{C}^{n'}), \quad \mathcal{E}'_0(D')_0^{\mu^-(e')}\big|_{e'} = e' \times (\mathbb{C}^n \oplus \{0\}).$$

For each $a \in e'$, we can write

$$f_{j,a} = g_{j,a} \oplus g'_{j,a},$$

where

$$g_{j,a} \in \mathbb{C}^n, \quad g'_{j,a} \in \mathbb{C}^{n'}.$$

Note that

$$\mathbb{C}^n = p(\langle f_{1,a}, \dots, f_{r,a}\rangle)$$

if and only if the $(n \times r)$-matrix $(g_{1,a} \dots g_{r,a})$ is of rank n. Let M be the set of $(n \times r)$-complex matrices, which is naturally a smooth manifold of dimension $2nr$. We denote by R_l the set of $(n \times r)$-matrices of rank l. Then R_l is a smooth submanifold of M of codimension $2(n-l)(r-l)$. If $l \leq n-1$ we have

$$\operatorname{codim}_{\mathbb{R}}(R_l \subset M) = 2(n-l)(r-l) \geq 2(r-n+1) \geq 2(N+1) \gg d.$$

Here we have used

$$n = \dim \mathcal{E}_0(D'_a)_0^{\mu^-(e')} \leq r - N.$$

See Lemma 2.4.5. So we can slightly perturb $(g_1 \ldots g_r)$ on $\operatorname{Int} e'$ such that for all $a \in e'$ and $l \in \{0, 1, \ldots, n-1\}$,

$$(g_{1,a} \ldots g_{r,a}) \notin R_l.$$

Hence the rank of $(g_{1,a} \ldots g_{r,a})$ is n. Therefore $\mathbb{C}^n = p(\langle f_{1,a}, \ldots, f_{r,a}\rangle)$ for all $a \in e'$. We can assume that the perturbation is small enough such that after the perturbation, $f_1, \ldots, f_r$ is still linearly independent. ∎

By this lemma, we may suppose that

$$\mathcal{E}_0(D')_0^{\mu^-(e')} = p(\langle f_1, \ldots, f_r\rangle)$$

on e'. For $a \in e'$, define $F_a \colon \mathbb{C}^r \to \mathcal{E}_0(D'_a)_0^{\mu^+(e')}$ by

$$F_a(c_1, \ldots, c_r) = c_1 f_{1,a} + \cdots + c_r f_{r,a}.$$

We have

$$\mathcal{E}_0(D'_a)_0^{\mu^-(e')} = \operatorname{im}(p \circ F_a).$$

Put

$$K := \bigcup_{a \in e'} \ker(p \circ F_a).$$

Then K is a subbundle of the trivial bundle $\underline{\mathbb{C}}^r$ on e'. We have the orthogonal decomposition

$$\underline{\mathbb{C}}^r = K \oplus K^\perp.$$

We define

$$F' \colon \underline{\mathbb{C}}^r \to \mathcal{E}(D')_0^{\mu^+(e')}\big|_{e'}$$

by

$$F' = F|_K + p \circ F|_{K^\perp}.$$

Then

$$\mathcal{E}(D')_0^{\mu^-(e')}\big|_{e'} \subset \operatorname{im} F'.$$

Lemma 2.4.7. *The following statements hold:*

(1) $F = F'$ *on* $\partial e'$.

(2) *The map* F' *is injective on* e'.

Proof. (1) Take $a \in \partial e'$. It is sufficient to show that $F_a|_{K^\perp} = F'_a|_{K^\perp}$. Recall that

$$\mathcal{E}_0(D'_a)_0^{\mu^-(e')} \subset \operatorname{im} F_a.$$

Since $\operatorname{im} F_a|_{K_a} \subset (\mathcal{E}_0(D'_a)_0^{\mu^-(e')})^\perp$ and $\dim \mathcal{E}_0(D'_a)_0^{\mu^-(e')} = \dim K_a^\perp$, we have

$$\operatorname{im}(F_a|_{K^\perp}) = \mathcal{E}_0(D'_a)_0^{\mu^-(e')}.$$

Therefore, for $v \in K_a^\perp$, $F'_v(v) = pF_a(v) = F_a(v)$.

(2) Suppose that

$$F'(v, v') = 0$$

for $v \in K, v' \in K^\perp$. Then

$$F(v) + pF(v') = 0.$$

So we have

$$pF(v) + p^2F(v') = 0.$$

Since $v \in K = \ker p \circ F$ and $p^2 = p$,

$$pF(v') = 0.$$

Because $p \circ F$ is an isomorphism on $K^\perp$, we have

$$v' = 0.$$

Hence

$$F(v) = 0,$$

which implies that $v = 0$ because F is injective. ■

Put

$$f'_{1,a} := F'_a(e_1), \ldots, f'_{r,a} := F'_a(e_r)$$

for $a \in e'$. Here, $e_1, \ldots, e_r$ is the standard basis of $\mathbb{C}^r$. Then the frame $\{f'_1, \ldots, f'_r\}$ of $\mathcal{E}_0(D')_0^{\mu^+(e')}$ on e', which is an extension of the frame on $\partial e'$, has the property that

$$\mathcal{E}(D')_0^{\mu^-(e')} \subset \langle f'_1, \ldots, f'_r \rangle \subset \mathcal{E}(D')_0^{\mu^+(e')}.$$

We have obtained a frame $f_1, \ldots, f_r$ satisfying (2.4.6). Putting

$$P = \mathcal{E}_0(D')_{-\infty}^0 \oplus \langle f_1, \ldots, f_r \rangle,$$

we obtain a spectral section with

$$\mathcal{E}_0(D)_{-\infty}^{\mu} \subset P \subset \mathcal{E}_0(D)_{-\infty}^{\mu+\delta},$$

where $\delta > 0$ is a constant independent of μ.

Take another positive number $\tilde{\mu}$ with $\mu \ll \tilde{\mu}$. Doing this procedure one more time, we get a frame $\{\tilde{f}_1, \dots, \tilde{f}_s\}$ of $P^\perp \cap \mathcal{E}(D')_0^{\tilde{\mu}+\delta}$ such that

$$\mathcal{E}_0(D)_{-\infty}^{\tilde{\mu}} \subset P \oplus \langle \tilde{f}_1, \dots, \tilde{f}_s \rangle \subset \mathcal{E}_0(D)_{-\infty}^{\tilde{\mu}+\delta}.$$

Repeating this, we get a sequence of spectral sections satisfying the conditions of Theorem 2.4.1. ∎

We will state a Pin(2)-equivariant version of Theorem 2.4.1. If $\mathfrak{s}$ is a self-conjugate spin^c structure of Y, we have an action of Pin(2) on $\mathcal{E}_k$. The action is induced by the action of Pin(2) on $\mathcal{H}^1(Y) \times L_k^2(\mathbb{S})$, which is an extension of the S^1-action, defined by

$$j(a, \phi) = (-a, j\phi).$$

The Dirac operator D is Pin(2)-equivariant and we have the index

$$\operatorname{ind} D \in KQ^1(B).$$

Here, $KQ^1(B)$ is the quaternionic K-theory defined in [19], which is used in [33].

Theorem 2.4.8. *If $\mathfrak{s}$ is a self-conjugate* spin^c *structure of Y and* ind $D = 0$ *in $KQ^1(B)$, then we have a sequence P_n of* Pin(2)*-equivariant spectral sections having the properties of Theorem* 2.4.1.

Proof. We will show an outline of the proof. Since ind $D = 0$ in $KQ^1(B)$, it follows from the arguments in [33, Section 1] that the family D of Dirac operators is Pin(2)-equivariantly homotopic to a constant family. Hence we can apply the proof of [40, Proposition 1] to show that there exists a Pin(2)-equivariant spectral section P_0 of $-D$.

Choose a CW complex structure of B such that for each cell e, $(-1) \cdot e$ is also a cell. Note that

$$\pi_i(Sp(m)/Sp(m-r)) = 0$$

for $i = 1, \dots, d$, provided that $m, m-r \gg d$. Hence for $\mu \gg 0$, we can construct a Pin(2)-equivariant frame $f_1, \dots, f_r$ of $P_0^\perp$ with

$$\mathcal{E}_0(D')_0^{\mu} \subset \langle f_1, \dots, f_r \rangle \subset \mathcal{E}_0(D')_0^{\mu+\delta}$$

as in the proof of Theorem 2.4.1. Here, δ is the positive constant from the proof of Theorem 2.4.1. Then

$$P_0 \oplus \langle f_1, \dots, f_r \rangle$$

is a Pin(2)-equivariant spectral section between $\mathcal{E}_0(D)_{-\infty}^{\mu}$ and $\mathcal{E}_0(D)_{-\infty}^{\mu+\delta}$. Repeating this construction, we obtain the desired sequence P_n. ∎

2.5 Derivative of projections

Let $D\colon \mathcal{E}_k \to \mathcal{E}_{k-1}$ be the original Dirac operator. Recall that we have a canonical flat connection ∇ on $\mathcal{E}_k$. See Section 2.3. Note that for $a \in B$, $v \in T_a B = \mathcal{H}^1(Y)$, we have

$$\nabla_v D = \frac{d}{dt}\Big|_{t=0} D_{a+tv} = \frac{d}{dt}\Big|_{t=0} (D_a + t\rho(v)) = \rho(v).$$

Here, $\rho(v)$ is the Clifford multiplication. Since v is a harmonic (and hence smooth) 1-form, we have $\|v\|_k < \infty$ for any $k \geq 0$. Therefore $\nabla_v D$ is a bounded operator from $L^2_k(\mathbb{S})$ to $L^2_k(\mathbb{S})$ for each $k \geq 0$.

Take $\mu \in \mathbb{R}$. We write $\pi^{\mu}_{-\infty}$ for the L^2-projection on $\mathcal{E}_0(D)^{\mu}_{-\infty}$. Similarly, π^{μ}_{λ} is the L^2-projection on $\mathcal{E}_0(D)^{\mu}_{\lambda}$. We have the following proposition.

Proposition 2.5.1. *Fix $a \in B$. Let $\{e_i\}_{i=-\infty}^{\infty}$ be an L^2-orthonormal basis of $L^2(\mathbb{S})$ such that*

$$D_a e_i = \eta_i e_i.$$

Here, η_i are the eigenvalues of D_a. Take $\lambda, \mu \in \mathbb{R}$ with $\lambda < \mu$. Suppose that λ, μ are not eigenvalues of D_a. For $v \in T_a B = \mathcal{H}^1(Y)$,

$$\begin{aligned}&\langle(\nabla_v \pi^{\mu}_{\lambda})e_i, e_j\rangle_0 \\ &= \begin{cases} \dfrac{\langle\rho(v)e_i, e_j\rangle_0}{\eta_i - \eta_j} & \text{if } \eta_i < \lambda < \eta_j < \mu \text{ or } \lambda < \eta_j < \mu < \eta_i, \\ \dfrac{\langle\rho(v)e_i, e_j\rangle_0}{\eta_j - \eta_i} & \text{if } \eta_j < \lambda < \eta_i < \mu \text{ or } \lambda < \eta_i < \mu < \eta_j, \\ 0 & \text{otherwise,} \end{cases}\end{aligned} \tag{2.5.1}$$

and

$$\langle(\nabla_v \pi^{\mu}_{-\infty})e_i, e_j\rangle_0 = \begin{cases} \dfrac{\langle\rho(v)e_i, e_j\rangle_0}{\eta_i - \eta_j} & \text{if } \eta_j < \mu < \eta_i, \\ \dfrac{\langle\rho(v)e_i, e_j\rangle_0}{\eta_j - \eta_i} & \text{if } \eta_i < \mu < \eta_j, \\ 0 & \text{otherwise.} \end{cases} \tag{2.5.2}$$

Here, $\rho(v)$ is the Clifford multiplication by v.

Proof. Since the connection ∇ is induced by the trivial connection on $\mathcal{H}^1(Y) \times C^{\infty}(\mathbb{S})$, to compute $\nabla_v \pi^{\mu}_{\lambda}$, $\nabla_v \pi^{\mu}_{-\infty}$, we can do computations over $\mathcal{H}^1(Y)$ where we have the canonical trivialization, and the covariant derivative is equal to the usual exterior derivative.

Take a loop Γ^{μ}_{λ} in $\mathbb{C}$ defined by

$$\begin{aligned}\Gamma^{\mu}_{\lambda} = &\left\{x - i\varepsilon : \lambda \leq x \leq \mu\right\} \cup \left\{\mu + iy : -\varepsilon \leq y \leq \varepsilon\right\} \\ &\cup \left\{x + i\varepsilon : \lambda \leq x \leq \mu\right\} \cup \left\{\lambda + iy : -\varepsilon \leq y \leq \varepsilon\right\}\end{aligned}$$

for some $\varepsilon > 0$. We orient Γ_λ^μ counterclockwise. We will show that for $\phi \in C^\infty(\mathbb{S})$,

$$(\pi_a)_\lambda^\mu \phi = \frac{1}{2\pi i} \int_{\Gamma_\lambda^\mu} (z - D_a)^{-1} \phi \, dz.$$

See also [22, Chapter II, Section 4]. We can write

$$\phi = \sum_{i=-\infty}^{\infty} c_i e_i$$

for some $c_i \in \mathbb{C}$ with

$$\sum_{i=-\infty}^{\infty} |c_i|^2 (1 + |\eta_i|^{2k}) < \infty$$

for any $k \geq 0$. For $z \in \mathbb{C}$ which is not an eigenvalue of D_a, the operator $z - D_a$ is invertible and

$$(z - D_a)^{-1} \phi = \sum_{i=-\infty}^{\infty} \frac{c_i}{z - \eta_i} e_i. \tag{2.5.3}$$

Note that the sum in (2.5.3) converges uniformly on Γ_λ^μ in the L_k^2-norm for any $k \geq 0$ since

$$\left| \frac{c_i}{z - \eta_i} \right| \leq |c_i| \quad (z \in \Gamma_\lambda^\mu)$$

if $|i| \gg 0$. Hence, by the residue formula,

$$\begin{aligned} \frac{1}{2\pi i} \int_{\Gamma_\lambda^\mu} (z - D_a)^{-1} (\phi) \, dz &= \sum_{i=-\infty}^{\infty} \frac{1}{2\pi i} \left(\int_{\Gamma_\lambda^\mu} \frac{c_i}{z - \eta_i} \, dz \right) e_i \\ &= \sum_{\lambda < \eta_i < \mu} c_i e_i \\ &= (\pi_a)_\lambda^\mu \phi. \end{aligned}$$

Here we have used the fact that we are allowed to take the term-by-term integration because of the uniform convergence.

Take $v \in T_a B = \mathcal{H}^1(Y)$. Then, by the above formula for π_λ^μ, we have

$$\begin{aligned} (\nabla_v \pi_\lambda^\mu) e_i &= -\frac{1}{2\pi i} \int_{\Gamma_\lambda^\mu} (z - D_a)^{-1} (\nabla_v D)(z - D_a)^{-1} e_i \, dz \\ &= -\frac{1}{2\pi i} \int_{\Gamma_\lambda^\mu} (z - D_a)^{-1} \rho(v)(z - \eta_i)^{-1} e_i \, dz \\ &= -\frac{1}{2\pi i} \int_{\Gamma_\lambda^\mu} (z - \eta_i)^{-1} (z - D_a)^{-1} \rho(v) e_i \, dz. \end{aligned}$$

Therefore

$$\begin{aligned}\langle(\nabla_v \pi_\lambda^\mu)e_i, e_j\rangle_0 &= -\frac{1}{2\pi i}\int_{\Gamma_\lambda^\mu}(z-\eta_i)^{-1}\langle\rho(v)e_i, (\bar{z}-D_a)^{-1}e_j\rangle_0\, dz\\ &= -\frac{1}{2\pi i}\int_{\Gamma_\lambda^\mu}(z-\eta_i)^{-1}\langle\rho(v)e_i, (\bar{z}-\eta_j)^{-1}e_j\rangle_0\, dz\\ &= -\frac{\langle\rho(v)e_i, e_j\rangle_0}{2\pi i}\int_{\Gamma_\lambda^\mu}(z-\eta_i)^{-1}(z-\eta_j)^{-1}\, dz.\end{aligned}$$

From this, we obtain the formula (2.5.1) for $\langle(\nabla_v \pi_\lambda^\mu)e_i, e_j\rangle_0$.

Note that since $\rho(v)$ defines a bounded operator $L^2 \to L^2$, we can see that the operators $(T_a)_\lambda^\mu$, $(T_a)_{-\infty}^\mu$ defined by the right-hand side of (2.5.1) and (2.5.2) are bounded from L^2 to L^2. Moreover, for each compact set K in $\mathscr{H}^1(Y)$, $(T_a)_\lambda^\mu$ converges to $(T_a)_{-\infty}^\mu$ on K uniformly as $\lambda \to -\infty$. We have

$$\begin{aligned}\langle(\pi_{a+tv})_\lambda^\mu(e_i), e_j\rangle_0 - \langle(\pi_a)_\lambda^\mu(e_i), e_j\rangle_0 &= \int_0^t \frac{d}{ds}\langle\pi_{a+sv}e_i, e_j\rangle_0\, ds\\ &= \int_0^t \langle(\nabla_v\pi_{a+sv}e_i), e_j\rangle_0\, ds\\ &= \int_0^t \langle(T_{a+sv})_\lambda^\mu(e_i), e_j\rangle_0\, ds.\end{aligned}$$

Taking the limit as $\lambda \to -\infty$, we obtain

$$\langle(\pi_{a+tv})_{-\infty}^\mu(e_i), e_j\rangle_0 - \langle(\pi_a)_{-\infty}^\mu e_i\rangle_0 = \int_0^t \langle(T_{a+sv})_{-\infty}^\mu(e_i), e_j\rangle_0\, ds.$$

Therefore

$$\langle(\nabla_v\pi_{-\infty}^\mu)e_i, e_j\rangle_0 = \frac{d}{dt}\Big|_{t=0}\langle(\pi_{a+tv})_{-\infty}^\mu(e_i), e_j\rangle_0 = \langle(T_a)_{-\infty}^\mu(e_i), e_j\rangle_0.$$

We have obtained (2.5.2). ∎

Corollary 2.5.2. *Suppose that μ is not an eigenvalue of D_a. Then for each $v \in TB$ and nonnegative k,*

$$\nabla_v\pi_{-\infty}^\mu\colon L_k^2(\mathbb{S}) \to L_{k+1}^2(\mathbb{S})$$

is a bounded operator. Moreover, if $|\mu| \geq 2$, $\alpha < k$ and if there is no eigenvalue of D_a in the interval $[\mu - \mu^{-\alpha}, \mu + \mu^{-\alpha}]$, for $v \in T_aB$ with $\|v\| \leq 1$,

$$\|\nabla_v\pi_{-\infty}^\mu : L_k^2(\mathbb{S}) \to L_{k-\alpha}^2(\mathbb{S})\| \leq C.$$

Here, $C > 0$ is a constant independent of v, μ. Similar statements hold for $\nabla_v\pi_\lambda^\mu$, $\nabla_v\pi_\mu^\infty$.

Proof. Let e_i, η_i be as in Proposition 2.5.1. Take $v \in T_a B = \mathcal{H}^1(Y)$. Put

$$\rho_{ij} := \langle \rho(v) e_i, e_j \rangle_0.$$

Take $\phi = \sum_i c_i e_i \in C^\infty(\mathbb{S})$ with $\|\phi\|_k = 1$. Since $\rho(v)$ is a bounded operator from L^2_k to L^2_k we have

$$\|\rho(v)\phi\|_k^2 = \Big\| \sum_{i,j} c_i \rho_{ij} e_j \Big\|_k^2 = \sum_j \Big| \sum_i c_i \rho_{ij} \Big|^2 (1 + |\eta_j|^{2k}) \leq C_1,$$

where $C_1 > 0$ is a constant independent of ϕ.

By Proposition 2.5.1, we have

$$\begin{aligned}
&\|(\nabla_v \pi^\mu_{-\infty})\phi\|^2_{k+1} \\
&\quad = \Big\| \sum_{\eta_i < \mu < \eta_j} \frac{c_i \rho_{ij}}{\eta_j - \eta_i} e_j + \sum_{\eta_j < \mu < \eta_i} \frac{c_i \rho_{ij}}{\eta_i - \eta_j} e_j \Big\|^2_{k+1} \\
&\quad = \sum_{\mu < \eta_j} \Big| \sum_{\eta_i < \mu} \frac{c_i \rho_{ij}}{\eta_j - \eta_i} \Big|^2 (1 + |\eta_j|^{2k+2}) + \sum_{\eta_j < \mu} \Big| \sum_{\mu < \eta_i} \frac{c_i \rho_{ij}}{\eta_i - \eta_j} \Big|^2 (1 + |\eta_j|^{2k+2}).
\end{aligned}$$

Note that there is a constant $C_2 > 0$ independent of i, j such that

$$\frac{1 + |\eta_j|^{2k+2}}{|\eta_j - \eta_i|^2} \leq C_2 (1 + |\eta_j|^{2k})$$

for i, j with $\eta_i < \mu < \eta_j$ or $\eta_j < \mu < \eta_i$. Hence

$$\|(\nabla_v \pi^\mu_{-\infty})\phi\|^2_{k+1} \leq C_2 \sum_j \Big| \sum_i c_i \rho_{ij} \Big|^2 (1 + |\eta_j|^{2k}) \leq C_1 C_2.$$

Therefore $\nabla_v \pi^\mu_{-\infty}$ extends to a bounded operator $L^2_k \to L^2_{k+1}$.

Next assume that there is no eigenvalue of D_a in the interval $[\mu - \mu^{-\alpha}, \mu + \mu^{-\alpha}]$. Take $v \in T_a B$ with $\|v\| = 1$. It is easy to see that if $\eta_i < \mu < \eta_j$ or $\eta_j < \mu < \eta_i$ we have

$$\frac{1 + |\eta_j|^{2k-2\alpha}}{|\eta_i - \eta_j|^2} \leq C_3 (1 + |\eta_j|^{2k}),$$

where $C_3 > 0$ is independent of i, j. It follows from this and Proposition 2.5.1 that

$$\|\nabla_v \pi^\mu_{-\infty} \colon L^2_k \to L^2_{k-\alpha}\| \leq C_4,$$

where $C_4 > 0$ is a constant independent of μ and v. ∎

Lemma 2.5.3. *Fix positive numbers α, β with $\alpha + 3 < \beta$ and $a \in \mathcal{H}^1(Y)$. For $\mu \in \mathbb{R}$ with $|\mu| \gg 0$, there exists $\mu' \in (\mu - |\mu|^{-\alpha}, \mu + |\mu|^{-\alpha}]$ such that there is no eigenvalue of D_a in the interval $(\mu' - |\mu|^{-\beta}, \mu' + |\mu|^{-\beta}]$.*

Proof. Suppose that the statement is not true. Then there is a sequence μ_n with $|\mu_n| \to \infty$ such that for any $\mu' \in (\mu_n - |\mu_n|^{-\alpha}, \mu_n + |\mu_n|^{-\alpha}]$ there is an eigenvalue of D_a in $(\mu' - |\mu_n|^{-\beta}, \mu' + |\mu_n|^{-\beta}]$. Therefore, for each integer m with $1 \le m \le |\mu_n|^{\beta-\alpha}$, there is an eigenvalue of D_a in the interval $(\mu_n + (m-1)|\mu_n|^{-\beta}, \mu_n + m|\mu_n|^{-\beta}]$. This implies that

$$\dim(\mathcal{E}_0(D_a))_{\mu_n - |\mu_n|^{-\alpha}}^{\mu_n + |\mu_n|^{-\alpha}} \ge |\mu_n|^{\beta-\alpha} - 1.$$

On the other hand, by the Weyl law,

$$\dim(\mathcal{E}(D_a))_{\mu_n - |\mu_n|^{-\alpha}}^{\mu_n + |\mu_n|^{-\alpha}} \le C|\mu_n|^3.$$

We have obtained a contradiction. ∎

Corollary 2.5.4. *For $\mu \in \mathbb{R}$ with $|\mu| \gg 0$, there is $\mu' \in [\mu, \mu+1]$, such that for $v \in TB$ with $\|v\| = 1$,*

$$\|\nabla_v \pi_{-\infty}^{\mu'} : L_k^2(\mathbb{S}) \to L_{k-4}^2(\mathbb{S})\| \le C.$$

Here, $C > 0$ is a constant independent of v, μ. Similar statements hold for π_λ^∞, π_λ^μ.

Proof. This is a direct consequence of Corollary 2.5.2 and Lemma 2.5.3. ∎

Proposition 2.5.5. *Take a nonnegative real number m and a smooth spectral section P of $-D$ with*

$$(\mathcal{E}_0(D))_{-\infty}^{\mu_-} \subset P \subset (\mathcal{E}_0(D))_{-\infty}^{\mu_+}.$$

Let π_P be the L^2-projection onto P. Then for each $v \in TB$, $\nabla_v \pi_P$ is a bounded operator from $L_m^2(\mathbb{S})$ to $L_{m+1}^2(\mathbb{S})$.

Proof. We can take an open covering $\{U_i\}_{i=1}^N$ of B such that there are real numbers λ_i, ν_i with $\lambda_i < \mu_-$, $\mu_+ < \nu_i$, which are not eigenvalues of D_a for $a \in U_i$. Also we may assume that for each i, we have a trivialization

$$\mathcal{E}_0|_{U_i} \cong U_i \times L^2(\mathbb{S})$$

such that the flat connection ∇ is equal to the exterior derivative d through this trivialization. Also for each i, we have smooth L^2-orthonormal frames $f_{i,1}, \dots, f_{i,r_i}$ of the normal bundle of $(\mathcal{E}_0)_{-\infty}^{\lambda_i}|_{U_i}$ in $P|_{U_i}$. We can write

$$\pi_P = \pi_{-\infty}^{\lambda_i} + \sum_{l=1}^{r_i} f_{i,l}^* \otimes f_{i,l}$$

over U_i. We have

$$\nabla_v \pi_P = \nabla_v \pi_{-\infty}^{\lambda_i} + \sum_{l=1}^{r_i} (\nabla_v f_{i,l}^* \otimes f_{i,l} + f_{i,l}^* \otimes \nabla_v f_{i,l}).$$

By Corollary 2.5.2, $\nabla_v \pi_{-\infty}^{\lambda_i}$ is a bounded operator from L^2_m to L^2_{m+1}. Also we have

$$\nabla_v f_{i,l} = \nabla_v(\pi_{\lambda_i}^{\nu_i} f_{i,l}) = (\nabla_v \pi_{\lambda_i}^{\nu_i}) f_{i,l} + \pi_{\lambda_i}^{\nu_i}(\nabla_v f_{i,l}).$$

Since $f_{i,l}(b) \in C^\infty(\mathbb{S})$ for $b \in U_i$, and $\nabla_v \pi_{\lambda_i}^{\nu_i}$ is a bounded operator $L^2_m \to L^2_{m+1}$, we have

$$\nabla_v f_{i,l}(b) \in C^\infty(\mathbb{S})$$

for $b \in U_i$. Also we have

$$|f_{i,l}^*(\phi)| = |\langle f_{i,l}, \phi\rangle_0| \leq \|\phi\|_0$$

for $\phi \in C^\infty(\mathbb{S})$. Therefore

$$\sum_{l=1}^{r_i} f_{i,l}^* \otimes \nabla_v f_{i,l} \colon L^2_m \to L^2_{m+1}$$

is bounded.

Take $\phi \in C^\infty(\mathbb{S})$. We have

$$(\nabla_v f_{i,l}^*)(\phi) = \langle \phi, \nabla_v f_{i,l}\rangle_0.$$

Note that $\nabla_v f_{i,l}(b) \in C^\infty(\mathbb{S})$ for $b \in U_i$. Hence

$$\|(\nabla_v f_{i,l}^* \otimes f_{i,l})(\phi)\|_{m+1} = |(\nabla_v f_{i,l}^*)(\phi) \cdot f_{i,l}|_{m+1} \leq C\|\phi\|_0.$$

Therefore

$$\sum_{l=1}^{r_i} \nabla_v f_{i,l}^* \otimes f_{i,l} \colon L^2_m \to L^2_{m+1}$$

is bounded. ∎

Corollary 2.5.6. *Suppose that* ind $D = 0$ *in* $K^1(B)$ *and let* P_0 *be a spectral section of* $-D$*. Then there is a family of smoothing operators* $\mathbb{A}$ *acting on* $\mathcal{E}_0$ *such that the kernel of* $D' = D + \mathbb{A}$ *is trivial and*

$$P_0 = \mathcal{E}_0(D')^0_{-\infty}.$$

Moreover, for each positive number k *and* $v \in TB$,

$$\nabla_v D' \colon L^2_k(\mathbb{S}) \to L^2_k(\mathbb{S})$$

is bounded.

Proof. The operator $\mathbb{A}$ is obtained as follows. (See the proof of [40, Lemma 8].) We can take smooth spectral sections Q, R of D and a positive number s with

$$(\mathcal{E}_0)_{-\infty}^{-s} \subset P_0 \subset (\mathcal{E}_0)_{-\infty}^{s}, \quad (\mathcal{E}_0)_{-\infty}^{-2s} \subset Q \subset (\mathcal{E}_0)_{-\infty}^{-s}, \quad (\mathcal{E}_0)_{-\infty}^{s} \subset R \subset (\mathcal{E}_0)_{-\infty}^{2s}.$$

Put

$$D' = \pi_Q D \pi_Q - s\pi_{P_0}(1 - \pi_Q) + (1 - \pi_R) D (1 - \pi_R) + s(1 - \pi_{P_0})\pi_R,$$

where π_{P_0}, π_Q, π_R are the L^2-projections. Then $\ker D' = 0$. The operator $\mathbb{A}$ is given by

$$\mathbb{A} = D' - D.$$

The image of $\mathbb{A}$ is included in the subspace spanned by finitely many eigenvectors of D. By Proposition 2.5.10, $\nabla_v \pi_{P_0}$, $\nabla_v \pi_Q$, $\nabla_v \pi_R$ are bounded operators from $L_k^2(\mathbb{S})$ to $L_{k+1}^2(\mathbb{S})$. Note that $\nabla_v D$ is the Clifford multiplication of the harmonic 1-form v. Hence $\nabla_v D$ is a bounded operator $L_k^2(\mathbb{S}) \to L_k^2(\mathbb{S})$. Therefore $\nabla_v D'$ is a bounded operator from L_k^2 to L_k^2. ■

Proposition 2.5.7. *The statements of Proposition 2.5.1, Corollary 2.5.2 and Corollary 2.5.4 hold for the perturbed Dirac operator D', replacing $\rho(v)$ with $\nabla_v D'$.*

Proof. By Corollary 2.5.6, for any nonnegative number k,

$$\nabla_v D' \colon L_k^2(\mathbb{S}) \to L_k^2(\mathbb{S})$$

is bounded and we can do the same computations as those done for the original Dirac operator D. ■

Lemma 2.5.8. *For a positive integer k, a positive number l with $l \geq k - 1$ and $v \in T_a B$, the expression*

$$\nabla_v |D'|^k \colon L_l^2 \to L_{l-k+1}^2$$

is bounded.

Proof. Note that

$$|D'|^k = (D')^k (1 - \pi_{P_0}) + (-1)^k (D')^k \pi_{P_0}.$$

Here, π_{P_0} is the L^2-projection on P_0. We have

$$\nabla_v (D')^k = (\nabla_v D')(D')^{k-1} + D'(\nabla_v D')(D')^{k-2} + \cdots + (D')^{k-1} \nabla_v D',$$

which implies that $\nabla_v (D')^k$ is a bounded operator $L_l^2 \to L_{l-k+1}^2$ by Corollary 2.5.6. Also $\nabla_v \pi_{P_0}$ is a bounded operator $L_l^2 \to L_{l+1}^2$ by Proposition 2.5.5. ■

Remark 2.5.9. So far the authors have not been able to prove Lemma 2.5.8 in the case when k is not an integer, though there is an explicit formula

$$|D'|^k = \sum_j |\eta_j|^k \pi_j.$$

Here, π_j is the projection onto the jth eigenspace which can be written as

$$\pi_j = \frac{1}{2\pi i} \int_{\Gamma_j} (z - D)^{-1}\, dz.$$

Suppose that ind $D = 0$ in $K^1(B)$ and fix a spectral section P_0 and recall the definition of the $L^2_{k_+,k_-}$-inner product $\langle \cdot, \cdot \rangle_{k_+,k_-}$ defined by using the perturbed Dirac operator $D' = D + \mathbb{A}$ of Corollary 2.5.6. (See (2.3.5).) Let $\mathcal{E}_{k_+,k_-}$ be the completion of $\mathcal{E}_\infty$ with respect to $\langle \cdot, \cdot \rangle_{k_+,k_-}$.

We will prove a generalization of Proposition 2.5.5.

Proposition 2.5.10. *Take nonnegative half-integers k_+, k_- and a smooth spectral section P of $-D$ with*

$$(\mathcal{E}_0)^{\mu_-}_{-\infty} \subset P \subset (\mathcal{E}_0)^{\mu_+}_{-\infty}.$$

Let π_P be the $L^2_{k_+,k_-}$-projection on P. Then for each nonnegative real number m, $v \in TB$, $\nabla_v \pi_P$ is a bounded operator from $L^2_m(\mathbb{S})$ to $L^2_{m+1}(\mathbb{S})$.

Proof. Let U_i, λ_i, ν_i be as in the proof of Proposition 2.5.5 and $f_{i,1}, \dots, f_{i,r_i}$ are smooth $L^2_{k_+,k_-}$-orthonormal frames of the normal bundle of $(\mathcal{E}_0)^{\lambda_i}_{-\infty}|_{U_i}$ in P. We can write

$$\pi_P = \pi^{\lambda_i}_{-\infty} + \sum_{l=1}^{r_i} f^*_{i,l} \otimes f_{i,l}$$

on U_i. Here,

$$f^*_{i,l}(\phi) = \langle \pi_{P_0}\phi, |D'|^{2k_-} f_{i,l} \rangle_0 + \langle (1 - \pi_{P_0})\phi, |D'|^{2k_+} f_{i,l} \rangle_0,$$

P_0 is the fixed spectral section used to define the $L^2_{k_+,k_-}$-norm, and π_{P_0} is the L^2-projection onto P_0. We have

$$\nabla_v \pi_P = \nabla_v \pi^{\lambda_i}_{-\infty} + \sum_{l=1}^{r_i} (\nabla_v f^*_{i,l} \otimes f_{i,l} + f^*_{i,l} \otimes \nabla_v f_{i,l}).$$

As stated in the proof of Proposition 2.5.5, $\nabla_v \pi^{\lambda_i}$ and $f^*_{i,l} \otimes \nabla_v f_{i,l}$ are bounded operators from L^2_m to L^2_{m+1}.

For $\phi \in C^\infty(\mathbb{S})$,

$$(\nabla_v f_{i,l}^*)(\phi) = \langle (\nabla_v \pi_{P_0})\phi, |D'|^{2k_-} f_{i,+}\rangle_0 + \langle \pi_{P_0}\phi, (\nabla_v |D'|^{2k_-}) f_{i,l}\rangle_0$$
$$+ \langle \pi_{P_0}\phi, |D'|^{2k_-}(\nabla_v f_{i,l})\rangle_0 - \langle (\nabla_v \pi_{P_0})\phi, |D'|^{2k_+} f_{i,;}\rangle_0$$
$$+ \langle (1-\pi_{P_0})\phi, (\nabla_v |D'|^{2k_+}) f_{i,l}\rangle_0$$
$$+ \langle (1-\pi_{P_0})\phi, |D'|^{2k_+}(\nabla_v f_{i,l})\rangle_0.$$

Note that $2k_\pm$ are nonnegative integers. By Proposition 2.5.5 and Lemma 2.5.8,

$$\|(\nabla_v f_{i,l}^* \otimes f_{i,l})(\phi)\|_{m+1} = \|(\nabla_v f_{i,l}^*)(\phi)\cdot f_{i,l}\|_{m+1} \le C\|\phi\|_0.$$

Hence $\nabla_v f_{i,l}^* \otimes f_{i,l}$ are bounded operators from L^2_m to L^2_{m+1}. ∎

Lemma 2.5.11. *Let ∇ be a connection on $\mathcal{E}_{k_+,k_-}$ (which is not necessarily the flat connection defined in Section 2.3). Let F be a subbundle in $\mathcal{E}_{k_+,k_-}$ of finite rank and $\pi_F\colon \mathcal{E}_{k_+,k_-} \to F$ be the $L^2_{k_+,k_-}$-projection. For $a \in B$, $\phi, \psi \in F_a$ and $v \in T_a B$, we have*

$$\langle (\nabla_v \pi_F)\phi, \psi\rangle_{k_+,k_-} = 0.$$

Similarly, for $\phi', \psi' \in F_a^\perp$, we have

$$\langle (\nabla_v \pi_F)\phi', \psi'\rangle_{k_+,k_-} = 0.$$

Proof. Since

$$\pi_F \pi_F = \pi_F,$$

we have

$$(\nabla_v \pi_F)\pi_F + \pi_F(\nabla_v \pi_F) = \nabla_v \pi_F.$$

Hence

$$(\nabla_v \pi_F)\phi + \pi_F(\nabla_v \pi_F)\phi = (\nabla_v \pi_F)\phi.$$

Here we have used $\pi_F\phi = \phi$. Therefore

$$\pi_F(\nabla_v \pi_F)\phi = 0,$$

which implies that

$$\langle (\nabla_v \pi_F)\phi, \psi\rangle_{k_+,k_-} = 0.$$

The proof of the other equality is similar. ∎

2.6 Weighted Sobolev space

Assume that ind $D = 0$ and fix a spectral section P_0 of $-D$. Let $D' = D + \mathbb{A}$ be the perturbed Dirac operator as in Corollary 2.5.6.

From now on, for $k > 0$, we consider the norm defined by

$$\|\phi\|_k = \| |D'|^k \phi\|_0.$$

Note that this norm is equivalent to the original L^2_k-norm since $\ker D' = 0$. That is, there is a constant $C > 1$ such that

$$C^{-1}\|(1 + |D|^k)\phi\|_0 \leq \| |D'|^k \phi\|_0 \leq C\|(1 + |D|^k)\phi\|_0.$$

Hence we can apply Corollary 2.5.2, Corollary 2.5.4, Proposition 2.5.7 to the Sobolev norms with respect to D'.

Let P_n, Q_n be spectral sections of $-D$, D with

$$(\mathcal{E}_0(D))_{-\infty}^{\mu_{n,-}} \subset P_n \subset (\mathcal{E}_0(D))_{-\infty}^{\mu_{n,+}},$$
$$(\mathcal{E}_0(D))_{\lambda_{n,+}}^{\infty} \subset Q_n \subset (\mathcal{E}_0(D))_{\lambda_{n,-}}^{\infty}.$$

We may suppose that

$$\mu_{n,-} + 10 < \mu_{n,+} < \mu_{n+1,-} - 10,$$
$$\lambda_{n+1,+} + 10 < \lambda_{n,-} < \lambda_{n,+} - 10,$$
$$\mu_{n,+} - \mu_{n,-} < \delta, \quad \lambda_{n,+} - \lambda_{n,-} < \delta$$

for some positive number δ independent of n. See Theorem 2.4.1. By the definition of $D' = D + \mathbb{A}$ in the proof of Corollary 2.5.6, we have

$$\mathcal{E}_0(D)_{-\infty}^{\mu_{n,\pm}} = \mathcal{E}_0(D')_{-\infty}^{\mu_{n,\pm}},$$
$$\mathcal{E}_0(D)_{\lambda_{n,\pm}}^{\infty} = \mathcal{E}_0(D')_{\lambda_{n,\pm}}^{\infty}$$

for $n \gg 0$. Fix half-integers $k_+, k_- > 5$. Put $\ell = \min\{k_+, k_-\}$. Let π_{P_n}, π_{Q_n} be the $L^2_{k_+,k_-}$-projections on P_n, Q_n. By Proposition 2.5.10, we can assume that for each n, there is $C_n > 0$ such that for $v \in TB$ with $\|v\| \leq 1$,

$$\|\nabla_v \pi_{P_n} \colon L^2_{k_+,k_-} \to L^2_{\ell+1}\| \leq C_n, \quad \|\nabla_v \pi_{Q_n} \colon L^2_{k_+,k_-} \to L^2_{\ell+1}\| \leq C_n. \quad (2.6.1)$$

Define a finite-dimensional subbundle F_n of $\mathcal{E}_\infty$ by

$$F_n = P_n \cap Q_n \subset (\mathcal{E}_0)_{\lambda_{n,-}}^{\mu_{n,+}}.$$

We will next introduce weighted Sobolev spaces. Take positive numbers ε_n with

$$C_n \varepsilon_n \leq \frac{1}{n}, \quad (2.6.2)$$

where C_n are the constants from (2.6.1). Fix a smooth function

$$w\colon \mathbb{R} \to \mathbb{R}$$

with

$$\begin{aligned} &0 < w(x) \leq 1 && \text{for all } x \in \mathbb{R}, \\ &w(x) = \varepsilon_n && \text{if } x \in [\lambda_{n,-} - 3, \lambda_{n,+} + 3] \cup [\mu_{n,-} - 3, \mu_{n,+} + 3] \text{ for some } n. \end{aligned}$$

Take $a \in \mathcal{H}^1(Y)$. Let $\{e_j\}_j$ be an orthonormal basis of $L^2(\mathbb{S})$ with

$$D'_a e_j = \eta_j e_j,$$

where η_j are the eigenvalues of D'_a.

For a positive number k and $\phi = \sum_j c_j e_j \in C^\infty(\mathbb{S})$, we define a weighted Sobolev norm $\|\phi\|_{a,k,w}$ by

$$\|\phi\|_{a,k,w} := \left(\sum_j |c_j|^2 |\eta_j|^{2k} w(\eta_j)^2 \right)^{\frac{1}{2}}.$$

Denote by $L^2_{a,k,w}(\mathbb{S})$ the completion of $C^\infty(\mathbb{S})$ with respect to $\|\cdot\|_{a,k,w}$. The family $\{\|\cdot\|_{a,k,w}\}_{a \in \mathcal{H}^1(Y)}$ of norms induces a fiberwise norm $\|\cdot\|_{k,w}$ on $\mathcal{E}_\infty$. We denote the completion of $\mathcal{E}_\infty$ with respect to $\|\cdot\|_{k,w}$ by $\mathcal{E}_{k,w}$. Note that

$$\|\phi\|_{k,w} \leq \|\phi\|_k.$$

Proposition 2.6.1. *Let k_+, k_- be half-integers with $k_+, k_- > 5$ and put $\ell = \min\{k_+, k_-\}$. Then*

$$\sup_{v \in B(TB;1)} \|\nabla_v \pi_{P_n} \colon L^2_{k_+,k_-} \to L^2_{\ell-5,w}\| \to 0.$$

A similar statement holds for π_{Q_n}.

Proof. For $\lambda, \mu \in \mathbb{R}$, let π^μ_λ be the L^2-projection to $(\mathcal{E}_0(D'))^\mu_\lambda$. Take $a \in B$ and $v \in T_a B$ with $\|v\| \leq 1$. By Corollary 2.5.4 and Proposition 2.5.7, for $n \gg 0$, we can take

$$\nu_{n,-} \in [\mu_{n,-} - 2, \mu_{n,-} - 1], \quad \nu_{n,+} \in [\mu_{n,+} + 1, \mu_{n,+} + 2]$$

such that

$$\begin{aligned} \|\nabla_v \pi^{\nu_{n,-}}_{-\infty} \colon L^2_{\ell-1} \to L^2_{\ell-5}\| \leq C, \\ \|(\nabla_v \pi^{\nu_{n,+}}_{\nu_{n,-}}) \colon L^2_{\ell-1} \to L^2_{\ell-5}\| \leq C, \end{aligned}$$

where $C > 0$ is a constant independent of n. Note that

$$\begin{aligned}\pi_{P_n} &= \mathrm{id}_{\mathcal{E}_0} \circ \pi_{P_n} \\ &= (\pi_{-\infty}^{\nu_{n,-}} + \pi_{\nu_{n,-}}^{\nu_{n,+}} + \pi_{\nu_{n,+}}^{\infty}) \circ \pi_{P_n} \\ &= \pi_{-\infty}^{\nu_{n,-}} + \pi_{\nu_{n,-}}^{\nu_{n,+}} \circ \pi_{P_n}.\end{aligned}$$

Hence

$$\nabla_v \pi_{P_n} = \nabla_v \pi_{-\infty}^{\nu_{n,-}} + (\nabla_v \pi_{\nu_{n,-}}^{\nu_{n,+}})\pi_{P_n} + \pi_{\nu_{n,-}}^{\nu_{n,+}}(\nabla_v \pi_{P_n}). \tag{2.6.3}$$

For $\varepsilon > 0$, take a positive number β with $\beta > \frac{1}{\varepsilon}$. Then for any $\phi \in \mathcal{E}_{k_+,k_-}$ with $\|\phi\|_{k_+,k_-} \leq 1$, we have

$$\|\pi_\beta^\infty \phi\|_{\ell-1} < \varepsilon.$$

By Proposition 2.5.1 and Corollary 2.5.4, for $n \gg 0$ with $\beta < \nu_{n,-}$,

$$\begin{aligned}\|(\nabla_v \pi_{-\infty}^{\nu_{n,-}})\phi\|_{\ell-5} &= \|(\nabla_v \pi_{-\infty}^{\nu_{n,-}})(\pi_{-\infty}^{\beta}\phi + \pi_\beta^\infty \phi)\|_{\ell-5} \\ &\leq C'\Big(\frac{1}{|\beta - \nu_{n,-}|} + \varepsilon\Big).\end{aligned} \tag{2.6.4}$$

Here, $C' > 0$ is independent of n. Similarly,

$$\|(\nabla_v \pi_{\nu_{n,-}}^{\nu_{n,+}})\pi_{P_n}\phi\|_{\ell-5} \leq C''\Big(\frac{1}{\min\{|\beta - \nu_{n,+}|, |\beta - \nu_{n,-}|\}} + \varepsilon\Big) \tag{2.6.5}$$

for $n \gg 0$, where $C'' > 0$ is a constant independent of n. By the definition of the weighted Sobolev norm $\|\cdot\|_{\ell,w}$ and (2.6.2),

$$\|\pi_{\nu_{n,-}}^{\nu_{n,+}}(\nabla_v \pi_{P_n})\phi\|_{\ell,w} \leq C_n \varepsilon_n \|\phi\|_{k_+,k_-} \leq \frac{1}{n}. \tag{2.6.6}$$

The statement follows from (2.6.3), (2.6.4), (2.6.5), (2.6.6). ∎

Lemma 2.6.2. *Let K be a compact set in $\mathcal{H}^1(Y)$. There is a norm $\|\cdot\|_{K,k,w}$ on $C^\infty(\mathbb{S})$ such that for any $a \in K$ and $\phi \in C^\infty(\mathbb{S})$ we have*

$$\|\phi\|_{K,k,w} \leq \|\phi\|_{a,k,w}.$$

Let $L^2_{K,k,w}$ be the completion of $C^\infty(\mathbb{S})$ with respect to $\|\cdot\|_{K,k,w}$. For $l \geq k$, the natural map $L^2_l \to L^2_{K,k,w}$ is injective.

Proof. Take a compact set K in $\mathcal{H}^1(Y)$ and fix $a_0 \in K$. Choose $a \in K$. Put

$$\begin{aligned} a_t &= (1-t)a_0 + ta, \\ r &= \|a_0 - a\|, \\ \delta &:= \max\big\{\|\nabla_v D' \colon L^2 \to L^2\| : t \in [0,1],\ v \in T_{a_t}\mathcal{H}^1(Y),\ \|v\| = 1\big\}.\end{aligned}$$

Let $\tilde{\mathcal{E}}_0$ be the trivial bundle $\mathcal{H}^1(Y) \times L^2(\mathbb{S})$ over $\mathcal{H}^1(Y)$, which is the pullback of $\mathcal{E}_0$ by the projection $\mathcal{H}^1(Y) \to B$. Also take a sequence $\{\lambda_l\}_{l=-\infty}^{\infty}$ of real numbers with

$$\lambda_l + r\delta \ll \lambda_{l+1}.$$

We will prove that for each l, there is a constant $c_l(a) > 0$ such that for $\phi \in \tilde{\mathcal{E}}_0(D'_{a_0})_{\lambda_l}^{\lambda_{l+1}}$, we have

$$c_l(a)\|\phi\|_0 \leq \|(\pi_a)_{\lambda_l - r\delta}^{\lambda_{l+1}+r\delta}\phi\|_0. \tag{2.6.7}$$

Fix an integer l. We consider the following set:

$$I = \big\{t \in [0,1] : \forall s \in [0,1],\ s \leq t,\ \exists c(s) > 0,\ \forall \phi \in \tilde{\mathcal{E}}_0(D'_{a_0})_{\lambda_l}^{\lambda_{l+1}},$$
$$c(s)\|\phi\|_0 \leq \|(\pi_{a_s})_{\lambda_l - sr\delta}^{\lambda_{l+1}+sr\delta}\phi\|_0\big\}.$$

Note that $0 \in I$. To prove (2.6.7), it is sufficient to show that $\sup I = 1$. Put $t_0 = \sup I$ and assume that $t_0 < 1$.

Then take $t_+ \in (t_0, 1]$ with $|t_+ - t_0|$ sufficiently small. For $t \in [t_0, t_+]$, let

$$\nu_1(t), \dots, \nu_m(t)$$

be the eigenvalues of D'_{a_t} which are continuous in t such that

$$\lambda_l - t_0 r\delta < \nu_1(t_0), \nu_2(t_0), \dots, \nu_m(t_0) \leq \lambda_{l+1} + t_0 r\delta,$$
$$\dim \tilde{\mathcal{E}}_0(D'_{a_t})_{\lambda_l - t_0 r\delta}^{\lambda_{l+1} t_0 r\delta} = m.$$

Take real numbers λ_-, λ_+ sufficiently close to $\lambda_l - t_0 r\delta$, $\lambda_{l+1} + t_0 r\delta$, which are not eigenvalues of D'_{a_t} for $t \in [t_0, t_+]$, such that

$$\tilde{\mathcal{E}}_0(D'_{a_{t_0}})_{\lambda_-}^{\lambda_+} = \tilde{\mathcal{E}}_0(D'_{a_{t_0}})_{\lambda_l - t_0 r\delta}^{\lambda_{l+1}+t_0 r\delta}.$$

By [22, Theorem 4.10, p. 291], for $t \in [t_0, t_+]$,

$$\lambda_l - tr\delta < \nu_1(t), \dots, \nu_m(t) \leq \lambda_{l+1} + tr\delta$$

which implies that

$$\tilde{\mathcal{E}}_0(D'_{a_t})_{\lambda_-}^{\lambda_+} = \tilde{\mathcal{E}}_0(D'_{a_t})_{\lambda_l - tr\delta}^{\lambda_{l+1}+tr\delta}.$$

So we have

$$\|(\pi_{a_t})_{\lambda_-}^{\lambda_+}\phi\|_0 = \|(\pi_{a_t})_{\lambda_l - tr\delta}^{\lambda_{l+1}+tr\delta}\phi\|_0.$$

From the equality

$$\frac{d}{dt}\|(\pi_{a_t})_{\lambda_-}^{\lambda_+}\phi\|_0^2 = 2\,\mathrm{Re}\langle(\nabla_v(\pi_{a_t})_{\lambda_-}^{\lambda_+})\phi, \phi\rangle_0,$$

for $t \in [t_0, t_+]$ and $\phi \in \tilde{\mathcal{E}}_0(D'_{a_{t_-}})^{\lambda_{l+1}+t_- r\delta}_{\lambda_l - t_- r\delta}$, we have

$$\{1 - 2M(t - t_0)\}\|\phi\|_0 \le \|(\pi_{a_t})^{\lambda_+}_{\lambda_-}\phi\|_0 = \|(\pi_{a_t})^{\lambda_{l+1}+tr\delta}_{\lambda_l - tr\delta}\phi\|_0,$$

where

$$M = \max\bigl\{\|\nabla_v(\pi_t)^{\lambda_+}_{\lambda_-} : L^2 \to L^2\| : t \in [t_0, t_+]\bigr\}$$

and $v = a - a_0$. Taking t_+ sufficiently close to t_0, we have

$$2M|t_+ - t_0| < 1.$$

This implies that

$$t_+ \in I$$

and we get a contradiction. We have obtained (2.6.7).

Take a sufficiently small open neighborhood $U_{l,a}$ of a in $\mathcal{H}^1(Y)$. Then for all $a' \in U_{l,a}$ we have

$$\frac{1}{2}c_l(a)\|\phi\|_0 \le \|(\pi_{a'})^{\lambda_{l+1}+r\delta+1}_{\lambda_l - r\delta - 1}\phi\|_0$$

for $\phi \in \tilde{\mathcal{E}}_0(D'_{a_0})^{\lambda_{l+1}}_{\lambda_l}$. Since K is compact, there exist $a_{l,1}, \dots, a_{l,N_l} \in K$ such that

$$K \subset U_{l,a_1} \cup \cdots \cup U_{l,a_{N_l}}.$$

Take a small positive number $\varepsilon > 0$ such that there are no eigenvalues of D'_a in $[-\varepsilon, \varepsilon]$ for $a \in K$. Put

$$\begin{aligned} c_l &= \min\bigl\{c_l(a_{l,1}), \dots, c_l(a_{l,N_l})\bigr\}, \\ \underline{w}(l) &:= \min\bigl\{|x|^k w(x) : x \notin [-\varepsilon, \varepsilon],\ x \in [\lambda_{l-1}, \lambda_{l+2}]\bigr\}. \end{aligned}$$

For $\phi \in C^\infty(\mathbb{S})$, define

$$\|\phi\|_{K,k,w} = \left\{\sum_l \Bigl(\frac{1}{10}c_l \underline{w}(l)\|(\pi_{a_0})^{\lambda_{l+1}}_{\lambda_l}\phi\|_0\Bigr)^2\right\}^{\frac{1}{2}}. \tag{2.6.8}$$

Then

$$\|\phi\|_{K,k,w} \le \|\phi\|_{a,k,w}$$

for all $a \in K$ and $\phi \in C^\infty(\mathbb{S})$. From definition (2.6.8) of $\|\cdot\|_{K,k,w}$, we have that the natural map $L^2_l \to L^2_{K,k,w}$ is injective for $l \ge k$. ∎

Proposition 2.6.3. *Let W be a closed, oriented, smooth manifold and E be a vector bundle on W. Let k be a positive number with $k \ge 1$, I be a compact interval in $\mathbb{R}$ and $\|\cdot\|$ be any norm on $C^\infty(E)$ such that $\|\phi\| \le \|\phi\|_{k-1}$ for all $\phi \in C^\infty(E)$. Assume that the natural map $L^2_l(E) \to \overline{C^\infty(E)}$ is injective for $l \ge k-1$. Here, $\overline{C^\infty(E)}$ is* *the completion with respect to the norm $\|\cdot\|$. We consider $L^2_l(E)$ to be a subspace of $\overline{C^\infty(E)}$ through this map.*

Suppose that we have a sequence $\gamma_n\colon I \to C^\infty(E)$ such that γ_n are equicontinuous in $\|\cdot\|$ and uniformly bounded in L^2_k. Then after passing to a subsequence, γ_n converges uniformly in L^2_{k-1} to a continuous

$$\gamma\colon I \to L^2_{k-1}(E).$$

Proof. Let $q_1, q_2, \dots,$ be the rational numbers in I. Since γ_n are uniformly bounded in L^2_k, it follows from the Rellich lemma and the diagonal argument that there is a subsequence $n(i)$ such that $\gamma_{n(i)}(q_m)$ converges in L^2_{k-1} (and hence in $\|\cdot\|$) as $i \to \infty$ for each m. Since γ_n are equicontinuous in $\|\cdot\|$, for any $\varepsilon > 0$ and $t \in I$, we can find q_m which is independent of i, with

$$\|\gamma_{n(i)}(t) - \gamma_{n(i)}(q_m)\| < \varepsilon.$$

So we have, for any t,

$$\begin{aligned}
&\|\gamma_{n(i)}(t) - \gamma_{n(j)}(t)\| \\
&\qquad \le \|\gamma_{n(i)}(t) - \gamma_{n(i)}(q_m)\| + \|\gamma_{n(i)}(q_m) - \gamma_{n(j)}(q_m)\| + \|\gamma_{n(j)}(q_m) - \gamma_{n(j)}(t)\| \\
&\qquad \le \|\gamma_{n(i)}(q_m) - \gamma_{n(j)}(q_m)\| + 2\varepsilon.
\end{aligned}$$

This implies that for each $t \in I$, $\gamma_{n(i)}(t)$ is a Cauchy sequence in $\|\cdot\|$, and hence $\gamma_{n(i)}$ has a pointwise limit $\gamma\colon I \to \overline{C^\infty(E)}$, where $\overline{C^\infty(E)}$ is the completion with respect to $\|\cdot\|$.

Since γ_n are equicontinuous in $\|\cdot\|$, for any $\varepsilon > 0$ there is $\delta > 0$ such that for $t, t' \in I$ with $|t - t'| < \delta$ we have $\|\gamma_n(t) - \gamma_n(t')\| < \varepsilon$. Taking the limit, we have $\|\gamma(t) - \gamma(t')\| \le \varepsilon$. We can choose finitely many rational numbers $q_1, \dots, q_N$ in I such that for all $t \in I$ there is q_l with $l \in \{1, \dots, N\}$ such that $|t - q_l| < \delta$. If i_0 is large enough, for $i > i_0$ we have $\|\gamma_{n(i)}(q_m) - \gamma(q_m)\| < \varepsilon$ for all $m \in \{1, \dots, N\}$. Therefore, for $i > i_0$,

$$\begin{aligned}
\|\gamma_{n(i)}(t) - \gamma(t)\| &\le \|\gamma_{n(i)}(t) - \gamma_{n(i)}(q_l)\| + \|\gamma_{n(i)}(q_l) - \gamma(q_l)\| + \|\gamma(q_l) - \gamma(t)\| \\
&< 3\varepsilon.
\end{aligned}$$

Hence $\gamma_{n(i)}$ converges uniformly to γ in $\|\cdot\|$.

We first show that the limit γ defined above in fact lies in $L^2_{k-\frac{1}{2}}$. Indeed, for any fixed t_∞ and any sequence $t_i \to t_\infty$ in I, we have that $\gamma_{n(i)}(t_i)$ converges, in $(k - \frac{1}{2})$-norm, after extracting a subsequence, to some δ. However, as above, $\gamma_{n(i)}(t_i)$ also converges in $\|\cdot\|$-norm to $\gamma(t_\infty)$. Recall that $L^2_{k-\frac{1}{2}}$ is a subspace of $\overline{C^\infty(E)}$, so $\delta \in \overline{C^\infty(E)}$, and we have

$$\begin{aligned}
\|\gamma(t_\infty) - \delta\| &\le \|\gamma(t_\infty) - \gamma_{n(i)}(t_i)\| + \|\gamma_{n(i)}(t_i) - \delta\| \\
&\le \|\gamma(t_\infty) - \gamma_{n(i)}(t_i)\| + \|\gamma_{n(i)}(t_i) - \delta\|_{k-\frac{1}{2}}.
\end{aligned}$$

It follows that $\delta = \gamma(t_\infty)$. This establishes that γ is defined as a function $I \to L^2_{k-\frac{1}{2}}$, but not that it is continuous, nor that the $\{\gamma_{n(i)}\}$ converges pointwise in $(k-\frac{1}{2})$-norm. Note that, since $\|\gamma_n(t)\|_k \le C$ for a positive constant C independent of n, t by assumption, we have $\|\gamma(t)\|_{k-\frac{1}{2}} \le C$ for all $t \in I$.

Assume that $\gamma_{n(i)}$ does not converge uniformly in L^2_{k-1}. Then after passing to a subsequence, there is $\varepsilon_0 > 0$ such that for any i we have $t_i \in I$ with

$$\|\gamma_{n(i)}(t_i) - \gamma(t_i)\|_{k-1} \ge \varepsilon_0.$$

After passing to a subsequence, t_i converges to some $t_\infty \in I$. Then $\gamma_{n(i)}(t_i)$ converges to $\gamma(t_\infty)$ in $\|\cdot\|$. Since $\gamma_{n(i)}(t_i)$ are uniformly bounded in L^2_k, by the Rellich lemma, after passing to a subsequence $\gamma_{n(i)}(t_i)$ converges to some δ in L^2_{k-1}; by the argument to show that $\gamma(t_\infty) \in L^2_{k-\frac{1}{2}}$ above, we see that $\delta = \gamma(t_\infty)$. Similarly, since $\|\gamma(t_i)\|_{k-\frac{1}{2}} \le C$ for all i, after passing to a subsequence, $\gamma(t_i)$ converges to some δ' in L^2_{k-1}. Since $\gamma(t_i) \to \gamma(t_\infty)$ in $\overline{C^\infty(E)}$, the previous argument gives that $\delta' = \gamma(t_\infty)$.

Therefore, after passing to a subsequence,

$$\|\gamma_{n(i)}(t_i) - \gamma(t_i)\|_{k-1} \to 0$$

as $i \to \infty$. This is a contradiction. Thus $\gamma_{n(i)}$ converges to γ in L^2_{k-1} uniformly. Since the convergence is uniform in L^2_{k-1}, γ is continuous in L^2_{k-1}. ∎

2.7 Proof of Theorem 2.3.3

Take half-integers k_+, k_- with $k_+, k_- > 5$ and with $|k_+ - k_-| \le \frac{1}{2}$. We put $\ell = \min\{k_+, k_-\}$ and

$$A_n := (B_{k_+}(F_n^+; R) \times_B B_{k_-}(F_n^-; R)) \times_B (B_{k_+}(W_n^+; R) \times_B B_{k_-}(W_n^-; R)).$$

We want to prove that A_n are isolating neighborhoods for $\varphi_{n,k_+,k_-} = \varphi_n$ for n large. If this is not true, after passing to a subsequence,

$$\mathrm{inv}\, A_n \cap \partial A_n \ne \emptyset$$

for all n. Then we can take

$$y_{n,0} = (\phi_{n,0}, \omega_{n,0}) \in \mathrm{inv}\, A_n \cap \partial A_n.$$

After passing to a subsequence, we may suppose that one of the following cases holds for all n:

(i) $\phi^+_{n,0} \in S_{k_+}(F_n^+; R)$

(ii) $\phi^-_{n,0} \in S_{k_-}(F_n^-; R)$,

(iii) $\omega^+_{n,0} \in S_{k_+}(W_n^+; R)$,

(iv) $\omega^-_{n,0} \in S_{k_-}(W_n^-; R)$.

Let $\gamma_n = (\phi_n, \omega_n): \mathbb{R} \to F_n \oplus W_n$ be the solution to (2.3.10) with $\gamma_n(0) = y_{n,0}$:

$$\begin{aligned} \Big(\frac{d\phi_n}{dt}(t)\Big)_V &= -(\nabla_{X_H}\pi_{F_n})\phi_n(t) - \pi_{F_n}\big(D\phi_n(t) + c_1(\gamma_n(t))\big), \\ \Big(\frac{d\phi_n}{dt}(t)\Big)_H &= -X_H(\phi_n(t)), \\ \frac{d\omega_n}{dt}(t) &= - * d\omega_n(t) - \pi_{W_n} c_2(\gamma_n(t)). \end{aligned} \tag{2.7.1}$$

We have

$$\|\phi_n^+(t)\|_{k_+} \le R, \quad \|\phi_n^-(t)\|_{k_-} \le R, \quad \|\omega_n^+(t)\|_{k_+} \le R, \quad \|\omega_n^-(t)\|_{k_-} \le R \tag{2.7.2}$$

for all $t \in \mathbb{R}$. By the Sobolev multiplication theorem,

$$\begin{aligned} \|c_1(\gamma_n(t))\|_\ell &\le C\|\gamma_n(t)\|_\ell^2 \le CR^2, \\ \|c_2(\gamma_n(t))\|_\ell &\le C\|\gamma_n(t)\|_\ell^2 \le CR^2, \\ \|X_H(\phi(t))\|_\ell &\le C\|\gamma_n(t)\|_\ell^2 \le CR^2. \end{aligned}$$

Let $\Delta \subset \mathcal{H}^1(Y)$ be a fundamental domain of the action of $H^1(Y;\mathbb{Z})$ on $\mathcal{H}^1(Y)$, which is a bounded set. By the path lifting property of the covering space $\mathcal{H}^1(Y) \times L^2_{k_+,k_-}(\mathbb{S}) \to \mathcal{E}_{k_+,k_-}$, we have a lift

$$\tilde{\gamma}_n = (\tilde{\phi}_n, \omega_n): \mathbb{R} \to \mathcal{H}^1(Y) \times L^2_{k_+,k_-}(\mathbb{S}) \times L^2_{k_+,k_-}(\operatorname{im} d^*)$$

of γ_n with

$$p_{\mathcal{H}}(\tilde{\gamma}_n(0)) \in \Delta. \tag{2.7.3}$$

By (2.7.1), we have

$$\Big\| \Big(\frac{d\phi_n}{dt}(t)\Big)_H \Big\| \le CR^2. \tag{2.7.4}$$

Fix $T > 0$. It follows from (2.7.3) and (2.7.4) that we can take a compact set K_T of $\mathcal{H}^1(Y)$ such that for any n and $t \in [-T, T]$ we have

$$p_{\mathcal{H}}(\tilde{\gamma}_n(t)) \in K_T.$$

Note that $\frac{d\tilde{\phi}_n}{dt}$ is uniformly bounded on $[-T, T]$ in $\|\cdot\|_{K_T,\ell-5,w}$ by (2.7.1), Proposition 2.6.1 and Lemma 2.6.2, which implies that $\tilde{\phi}_n$ are equicontinuous in $L^2_{K_T,\ell-5,w}$ on $[-T, T]$. The ω_n are also equicontinuous in $L^2_{\ell-1}$. By Proposition 2.6.3, after passing to a subsequence, $\tilde{\gamma}_n|_{[-T,T]}$ converges to a map

$$\tilde{\gamma}^{(T)} = (\tilde{\phi}^{(T)}, \omega^{(T)}): [-T, T] \to \mathcal{H}^1(Y) \times L^2_{\ell-1}(\mathbb{S}) \times L^2_{\ell-1}(\operatorname{im} d^*)$$

uniformly in $L^2_{\ell-1}$. By the diagonal argument, we can show that there is a continuous map

$$\tilde{\gamma} = (\tilde{\phi}, \omega) : \mathbb{R} \to \mathcal{H}^1(Y) \times L^2_{\ell-1}(\mathbb{S}) \times L^2_{\ell-1}(\operatorname{im} d^*)$$

such that, after passing to a subsequence, $\tilde{\gamma}_n$ converges to $\tilde{\gamma}$ uniformly in $L^2_{\ell-1}$ on each compact set in $\mathbb{R}$.

Lemma 2.7.1. *The limit $\tilde{\gamma}$ is a solution to the Seiberg–Witten equations over $Y \times \mathbb{R}$.*

Proof. Fix $T > 0$. For $t \in [-T, T]$, we have

$$\begin{aligned}
&\tilde{\phi}_n(t) - \tilde{\phi}_n(0) \\
&\quad = \int_0^t \frac{d\tilde{\phi}_n}{ds}(s)\, ds \\
&\quad = -\int_0^t (\nabla_{X_H} \pi_{\tilde{F}_n})\tilde{\phi}_n(s) + \pi_{\tilde{F}_n}\big(D\tilde{\phi}_n(t) + c_1(\tilde{\gamma}_n(t))\big) + X_H(\phi_n(s))\, ds. \quad (2.7.5)
\end{aligned}$$

We have that $p_{\mathcal{H}}(\tilde{\gamma}_n(t)) \in K_T$ for any n and $t \in [-T, T]$. Note that we have no estimate on $(\nabla_{X_H} \pi_{F_n})\tilde{\phi}_n$ in *any* L^2_j-norm and that we just have control on it in the auxiliary space $L^2_{K_T, \ell-5, w}$. By Proposition 2.6.1 and Lemma 2.6.2,

$$(\nabla_{X_H} \pi_{\tilde{F}_n})\tilde{\phi}_n(s) \to 0$$

uniformly in $L^2_{K_T, \ell-5, w}$ as $n \to \infty$. Recall that $\tilde{\phi}_n$, ω_n converge in $L^2_{\ell-1}$ uniformly on $[-T, T]$. It follows from Proposition 2.4.2 and the inequality

$$\begin{aligned}
\|\pi_{F_n} D\tilde{\phi}_n - D\tilde{\phi}\|_{\ell-2} &= \|\pi_{F_n} D\tilde{\phi}_n - D\tilde{\phi}_n + D\tilde{\phi}_n - D\tilde{\phi}\|_{\ell-2} \\
&\le \|[\pi_{F_n}, D]\tilde{\phi}_n\|_{\ell-2} + \|D\tilde{\phi}_n - D\tilde{\phi}\|_{\ell-2}
\end{aligned}$$

that $\pi_{F_n} D\tilde{\phi}_n$ converges to $D\tilde{\phi}$ uniformly in $L^2_{\ell-2}$ on $[-T, T]$.

Taking the limit with $n \to \infty$ in (2.7.5), we obtain

$$\tilde{\phi}(t) - \tilde{\phi}(0) = -\int_0^t \big(D\tilde{\gamma}(t) + c_1(\tilde{\gamma}(t))\big) + X_H(\tilde{\phi}(s))\, ds.$$

Hence, by the fundamental theorem of calculus,

$$\frac{d\tilde{\phi}}{dt}(t) = -\big(D\tilde{\phi}(t) + c_1(\tilde{\gamma}(t))\big) - X_H(\tilde{\phi}(t)).$$

A priori, the left-hand side $\frac{d\tilde{\phi}}{dt}(t)$ only lives in the auxiliary space $L^2_{K_T, \ell-5, w}$. However, since $L^2_{\ell-2}$ is a subspace of $L^2_{K_T, \ell-2, w}$ and the right-hand side is in $L^2_{\ell-2}$, $\frac{d\tilde{\phi}}{dt}(t)$ is in $L^2_{\ell-2}$ and both sides are equal to each other as elements of $L^2_{\ell-2}$.

Similarly, we can show that

$$\frac{d\omega}{dt}(t) = - * d\omega(t) - c_2(\tilde{\gamma}(t)).$$

Therefore $\tilde{\gamma}$ is a solution to the Seiberg–Witten equations (2.3.4) and the ordinary theory of elliptic regularity shows that $\tilde{\gamma}$ is in C^∞ as a section on any compact set in $Y \times (-T, T)$. ■

Composing $\tilde{\gamma}\colon \mathbb{R} \to \mathcal{H}^1(Y) \times L^2_{\ell-1}(\mathbb{S}) \times L^2_{\ell-1}(\operatorname{im} d^*)$ with the projection

$$\mathcal{H}^1(Y) \times L^2_{\ell-1}(\mathbb{S}) \times L^2_{\ell-1}(\operatorname{im} d^*) \to \mathcal{E}_{\ell-1} \oplus \mathcal{W}_{\ell-1},$$

we get a Seiberg–Witten trajectory

$$\gamma\colon \mathbb{R} \to \mathcal{E}_{\ell-1} \oplus \mathcal{W}_{\ell-1}.$$

Since $\|\gamma(t)\|_{\ell-1} \leq R$ for all $t \in \mathbb{R}$, γ has finite energy. By Proposition 2.3.2,

$$\|\gamma(t)\|_{k_+,k_-} \leq R_{k_+,k_-}, \tag{2.7.6}$$

for all $t \in \mathbb{R}$.

Assume that case (i) holds for all n. We have

$$\|\phi_n^+(0)\|_{k_+} = R.$$

Lemma 2.7.2. *There is a constant $C > 0$ such that for all n,*

$$\|\phi_n^+(0)\|_{k_+ + \frac{1}{2}} < C.$$

Proof. Note that

$$\frac{d}{dt}\Big|_{t=0} \|\phi_n^+(t)\|^2_{k_+} = 0.$$

Let us consider the case when $k_+ \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$.

Let π^+ be the $L^2_{k_+,k_-}$-projection onto $\mathcal{E}^+_{k_+,k_-}$. (That is, $\pi^+ = 1 - \pi_{P_0}$.) Then we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big|_{t=0} \|\phi_n^+(t)\|^2_{k_+} &= \frac{1}{2}\frac{d}{dt}\Big|_{t=0} \langle |D'|^{k_+ + \frac{1}{2}} \pi^+ \phi_n(t), |D'|^{k_+ - \frac{1}{2}} \pi^+ \phi_n(t)\rangle_0 \\ &= \langle (\nabla_{X_H} |D'|^{k_+ + \frac{1}{2}}) \phi_n^+(0), |D'|^{k_+ - \frac{1}{2}} \phi_n^+(0)\rangle_0 \\ &\quad + \langle |D'|^{k_+ + \frac{1}{2}} \phi_n^+(0), (\nabla_{X_H} |D'|^{k_+ - \frac{1}{2}}) \phi_n^+(0)\rangle_0 \\ &\quad + \operatorname{Re}\langle (\nabla_{X_H} \pi^+) \phi_n(0), \phi_n^+(0)\rangle_{k_+} + \operatorname{Re}\Big\langle \frac{d\phi_n}{dt}(0), \phi_n^+(0)\Big\rangle_{k_+}.\end{aligned}$$

Note that $k_+ + \frac{1}{2}$ and $k_+ - \frac{1}{2}$ are integers. By Lemma 2.5.8,

$$\begin{aligned}
\left|\langle(\nabla_{X_H}|D'|^{k_++\frac{1}{2}})\phi_n^+(0), |D'|^{k_+-\frac{1}{2}}\phi_n^+(0)\rangle_0\right| &\le C\|\phi_n^+(0)\|_{k_+-\frac{1}{2}}^2 \le CR^2,\\
\left|\langle|D'|^{k_++\frac{1}{2}}\phi_n^+(0), (\nabla_{X_H}|D'|^{k_+-\frac{1}{2}})\phi_n^+(0)\rangle_0\right| &\le C\|\phi_n^+(0)\|_{k_++\frac{1}{2}}\|\phi_n^+(0)\|_{k_+-\frac{1}{2}}\\
&\le CR\|\phi_n^+(0)\|_{k_++\frac{1}{2}}.
\end{aligned}$$

By Proposition 2.5.10,

$$\begin{aligned}
\left|\langle(\nabla_{X_H}\pi^+)\phi_n(0), \phi_n^+(0)\rangle_{k_+}\right| &\le \|(\nabla_{X_H}\pi^+)\phi_n(0)\|_{k_+}\|\phi_n^+(0)\|_{k_+}\\
&\le C\|\phi_n(0)\|_{k_+-1}\|\phi_n^+(0)\|_{k_+}\\
&\le C\|\phi_n(0)\|_{\ell}\|\phi_n^+(0)\|_{k_+}\\
&\le CR^2.
\end{aligned}$$

We have

$$\begin{aligned}
&\left\langle\frac{d\phi_n}{dt}(0), \phi_n^+(0)\right\rangle_{k_+}\\
&\qquad = -\left\langle(\nabla_{X_H}\pi_{F_n})\phi_n(0) + \pi_{F_n}\left(D'\phi_n(0) - \mathbb{A}\phi_n(0) + c_1(\gamma_n(0))\right), \phi_n^+(0)\right\rangle_{k_+}.
\end{aligned}$$

By Lemma 2.5.11,

$$\langle(\nabla_{X_H}\pi_{F_n})\phi_n(0), \phi_n^+(0)\rangle_{k_+} = \langle(\nabla_{X_H}\pi_{F_n})\phi_n(0), \phi_n^+(0)\rangle_{k_+,k_-} = 0.$$

We have

$$\begin{aligned}
\langle\pi_{F_n}D'\phi_n(0), \phi_n^+(0)\rangle_{k_+} &= \langle D'\phi_n(0), \pi_{F_n}\phi_n^+(0)\rangle_{k_+}\\
&= \langle D'\phi_n(0), \phi_n^+(0)\rangle_{k_+}\\
&= \|\phi_n^+(0)\|_{k_++\frac{1}{2}}^2.
\end{aligned}$$

Since $\mathbb{A}$ is a smoothing operator,

$$|\langle\pi_{F_n}\mathbb{A}\phi_n(0), \phi_n^+(0)\rangle_{k_+}| \le C\|\phi_n(0)\|_0\|\phi_n(0)\|_{k_+} \le CR^2.$$

Since D' is self-adjoint,

$$\begin{aligned}
|\langle\pi_{F_n}c_1(\gamma_n(0)), \phi_n^+(0)\rangle_{k_+}| &= |\langle c_1(\gamma_n(0)), \phi_n^+(0)\rangle_{k_+}|\\
&= |\langle|D'|^{k_+}c_1(\gamma_n(0)), |D'|^{k_+}\phi_n^+(0)\rangle_0|\\
&= |\langle|D'|^{k_+-\frac{1}{2}}c_1(\gamma_n(0)), |D'|^{k_++\frac{1}{2}}\phi_n^+(0)\rangle_0|\\
&\le \|c_1(\gamma_n(0))\|_{k_+-\frac{1}{2}}\|\phi_n^+(0)\|_{k_++\frac{1}{2}}\\
&\le C\|c_1(\gamma_n(0))\|_{\ell}\|\phi_n^+(0)\|_{k_++\frac{1}{2}} \quad (\ell = \min\{k_+, k_-\})\\
&\le CR^2\|\phi_n^+(0)\|_{k_++\frac{1}{2}}.
\end{aligned}$$

Therefore

$$0 = \frac{1}{2}\frac{d}{dt}\Big|_{t=0} \|\phi_n^+(t)\|_{k_+}^2 \le -\|\phi_n^+(0)\|_{k_+ + \frac{1}{2}}^2 + CR^2\|\phi_n^+(0)\|_{k_+ + \frac{1}{2}} + CR^2.$$

This inequality implies that the sequence $\|\phi_n^+(0)\|_{k_+ + \frac{1}{2}}$ is bounded.

The proof in the case $k_+ \in \mathbb{Z}$ is similar. ∎

It follows from Lemma 2.7.2 and the Rellich lemma that after passing to a subsequence, $\phi_n^+(0)$ converges to $\phi^+(0)$ in $L^2_{k_+}$ strongly. By the assumption, $\|\phi_n^+(0)\|_{k_+} = R$ for all n. Hence,

$$\|\gamma(0)\|_{k_+,k_-} \ge \|\phi^+(0)\|_{k_+,k_-} = R.$$

This contradicts (2.7.6).

Let us consider case (ii). In this case, we have

$$\|\phi_n^-(0)\|_{k_-} = R.$$

Lemma 2.7.3. *There is a constant $C > 0$ such that for all n,*

$$\|\phi_n^-(0)\|_{k_- + \frac{1}{2}} < C.$$

Proof. Note that

$$\langle D'\phi_n(0), \phi_n^-(0)\rangle_{k_-} = -\|\phi_n^-(0)\|_{k_- + \frac{1}{2}}^2.$$

As in the proof of Lemma 2.7.2, we can show that

$$0 = \frac{d}{dt}\Big|_{t=0} \|\phi_n^-(t)\|_{k_-}^2 \ge \|\phi_n^-(0)\|_{k_- + \frac{1}{2}}^2 - CR^2\|\phi_n^+(0)\|_{k_- + \frac{1}{2}} - CR^2.$$

This implies that the sequence $\|\phi_n^-(0)\|_{k_- + \frac{1}{2}}$ is bounded. ∎

By the Rellich lemma, $\phi_n^-(0)$ converges to $\phi^-(0)$ in $L^2_{k_-}$ strongly. Hence

$$\|\gamma(0)\|_{k_+,k_-} \ge \|\phi^-(0)\|_{k_-} = R.$$

We get a contradiction.

In the other cases (iii), (iv) where $y_{n,0}$ is in the other components of ∂A_n, we similarly have a contradiction.

Definition 2.7.4. For this definition we refer to some notions from parameterized homotopy theory and parameterized Conley index theory; refer to Sections A.1 and A.2, respectively. For notation as in Theorem 2.3.3, let $\mathcal{SWF}_{[n]}(Y, \mathfrak{s})$ be the parameterized Conley index of the flow φ_{n,k_+,k_-} on the isolated invariant set A_n. We call $\mathcal{SWF}_{[n]}(Y, \mathfrak{s})$ the *pre-Seiberg–Witten Floer invariant* of $(Y, \mathfrak{s})$ (for short, the pre-SWF invariant of $(Y, \mathfrak{s})$). The object $\mathcal{SWF}_{[n]}(Y, \mathfrak{s})$ is an (equivariant) topological space, depending on a number of choices (which are not all reflected in its notation). First, $\mathcal{SWF}_{[n]}(Y, \mathfrak{s})$ depends on the choice of an index pair, but its (equivariant, parameterized) homotopy type is independent of the choice of index pair – we will abuse notation and also write $\mathcal{SWF}_{[n]}(Y, \mathfrak{s})$ for its (equivariant, parameterized) homotopy type. It also depends on a choice of metric on Y, as well as spectral sections P_n, Q_n and subspaces $W_n^{\pm}$, as in the preliminaries to Theorem 2.3.3.

The projection used in the parameterized Conley index is from the ex-space $B_{n,R}$ over $\mathrm{Pic}(Y)$, as explained in the discussion after Theorem 2.3.3.

We write $\mathcal{SWF}^{u}_{[n]}(Y, \mathfrak{s})$ to refer to the Conley index with trivial parameterization. By Lemma A.2.7, $\nu_! \mathcal{SWF}_{[n]}(Y, \mathfrak{s}) = \mathcal{SWF}^{u}_{[n]}(Y, \mathfrak{s})$, where $\nu\colon B \to *$ is the map collapsing the Picard torus to a point, and $\nu_!$ is as defined in Appendix A.1.

If $\mathfrak{s}$ is a self-conjugate spin^c structure, the bundle $L^2_k(\mathbb{S}) \times \mathcal{H}^1(Y) \times L^2_k(\mathrm{im}\, d^*)$ admits a Pin(2)-action extending the S^1-action on spinors, by

$$j(\phi, v, \omega) = (j\phi, -v, -\omega).$$

In the event that the spectral sections P_n, Q_n are preserved by the Pin(2)-action, then the approximate flow on $F_n \oplus W_n$ will be Pin(2)-equivariant, and we define $\mathcal{SWF}^{\mathrm{Pin}(2)}_{[n]}(Y, \mathfrak{s})$ to be the Pin(2)-equivariant parameterized Conley index, so that its underlying S^1-space is $\mathcal{SWF}_{[n]}(Y, \mathfrak{s})$. We similarly define $\mathcal{SWF}^{u,\mathrm{Pin}(2)}_{[n]}(Y, \mathfrak{s})$ (and we will occasionally write $\mathcal{SWF}^{u,S^1}_{[n]}(Y, \mathfrak{s})$ to distinguish what equivariance is meant). See Theorem 2.4.8 for the existence of Pin(2)-equivariant spectral sections.